

A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function

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Abstract

For $p > 1$ and $d \geq 1$ J. Kinnunen proved that if f is a function on the Sobolev space $W^{1,p}(\mathbf{R}^d)$, then the Hardy-Littlewood maximal function $\mathcal{M}f$ has the first order weak partial derivatives which belong to $L^p(\mathbf{R}^d)$ and whose L^p -norm are controlled by those of f . We improve Kinnunen's result to $p = 1$ and $d = 1$ by making an expression of the maximal function by the one-sided maximal functions. We also study some properties of the one-sided maximal functions.

1 Introduction

For f a locally integrable function on $\mathbf{R}^d$, $d \geq 1$, define the Hardy-Littlewood maximal function $\mathcal{M}f$ as

$$(\mathcal{M}f)(x) = \sup_Q \frac{1}{|Q|} \int_Q |f| dy,$$

where the supremum is taken over all cubes Q containing $x \in \mathbf{R}^d$. Here, $|Q|$ denotes the volume of the cube Q . The maximal function is a basic tool in real analysis. The well-known theorem of Hardy, Littlewood and Wiener asserts that if $f \in L^p(\mathbf{R}^d)$, with $1 < p \leq \infty$, then $\mathcal{M}f \in L^p(\mathbf{R}^d)$ and

$$\|\mathcal{M}f\|_p \leq A_p \|f\|_p, \quad (1)$$

where the constant A_p depends only on p and the dimension d . Moreover, one knows that the constant A_p satisfies

$$A_p = O(1/(p-1)), \text{ as } p \rightarrow 1. \quad (2)$$

(See [St].) Recall that the Sobolev space $W^{1,p}(\mathbf{R}^d)$, $1 \leq p \leq \infty$, consists of functions f in $L^p(\mathbf{R}^d)$, whose first order weak partial derivatives $D_i f$, $i = 1, 2, \dots, d$, belong to $L^p(\mathbf{R}^d)$.

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In [K]¹ J. Kinnunen showed that if $f \in W^{1,p}(\mathbb{R}^d)$, with $1 < p < \infty$ and $d \geq 1$, then $\mathcal{M}f \in W^{1,p}(\mathbb{R}^d)$ and

$$|(D_i \mathcal{M}f)(x)| \leq (\mathcal{M}D_i f)(x), \quad i = 1, 2, \dots, d, \quad (3)$$

almost everywhere $x \in \mathbb{R}^d$. This implies by (1) that

$$\|D_i \mathcal{M}f\|_p \leq A_p \|D_i f\|_p \quad i = 1, 2, \dots, d. \quad (4)$$

Kinnunen's methods used to prove (3) mainly depend on the L^p -boundedness of $\mathcal{M}$ and the fact that the first order Sobolev space is a lattice (see [GT]). So, one cannot directly extend the result of (4) to the case $p = 1$ because then we do not have the Hardy-Littlewood-Wiener theorem available. But, according to the embedding theorem of Sobolev, $W^{1,1}(\mathbb{R}^d)$ can be continuously embedded in $L^{d/(d-1)}(\mathbb{R}^d)$. (See [St].) Therefore, by the Hardy-Littlewood-Wiener theorem we see that $\mathcal{M}f \in L^{d/(d-1)}(\mathbb{R}^d)$, if $f \in W^{1,1}(\mathbb{R}^d)$. In particular, $\mathcal{M}f$ becomes a locally integrable function and hence is differentiable in distribution sense.

Now, we have the following problems.

- (I) If $f \in W^{1,1}(\mathbb{R}^d)$, are the derivatives of $\mathcal{M}f$ function or not?
- (II) If the derivatives of $\mathcal{M}f$ are functions, is it possible to show some norm estimates?

The purpose of this note is to investigate the above problems for the one-dimensional case. As yet we have not been able to prove the corresponding results in the higher dimensions. Our result is the following.

THEOREM 1 *If $f \in W^{1,1}(\mathbb{R})$, then the derivative of $\mathcal{M}f$ becomes an integrable function and*

$$\|(\mathcal{M}f)'\|_1 \leq 2\|f'\|_1. \quad (5)$$

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2 The one-sided maximal functions

A crucial point in our argument is to consider one-sided maximal functions. In this section we shall discuss our problem for one-sided maximal functions.

For f a locally integrable function on the line define the one-sided maximal functions $\mathcal{M}_l f$ and $\mathcal{M}_r f$ as

$$(\mathcal{M}_l f)(x) = \sup_{s>0} \frac{1}{s} \int_{x-s}^x |f| dy,$$

¹Roughly speaking, the Hardy-Littlewood maximal function may be distinguished into two types. The first is defined as the supremum taken over all balls centered at x , and the second is defined as the supremum taken over all cubes containing x . Kinnunen proved his results for the first type maximal function. But, one sees that the corresponding results hold for the second one by the slight modification of the argument.

and

$$(\mathcal{M}_r f)(x) = \sup_{t>0} \frac{1}{t} \int_x^{x+t} |f| dy.$$

We will refer $\mathcal{M}_l f$ and $\mathcal{M}_r f$ to the left maximal function and the right maximal function respectively.

The first lemma represents an expression of the one-dimensional Hardy-Littlewood maximal function by the one-sided maximal functions.

LEMMA 2 *Let f be a locally integrable function on the line. Then we can express $\mathcal{M}f$ by $\mathcal{M}_l f$ and $\mathcal{M}_r f$ as*

$$(\mathcal{M}f)(x) = \max\{(\mathcal{M}_l f)(x), (\mathcal{M}_r f)(x)\}.$$

Proof. It follows from the definitions that

$$\max\{(\mathcal{M}_l f)(x), (\mathcal{M}_r f)(x)\} \leq (\mathcal{M}f)(x). \quad (6)$$

To prove the reverse inequality we see that

$$\begin{aligned} \frac{1}{s+t} \int_{x-s}^{x+t} |f| dy &= \frac{s}{s+t} \frac{1}{s} \int_{x-s}^x |f| dy + \frac{t}{s+t} \frac{1}{t} \int_x^{x+t} |f| dy \\ &\leq \frac{s}{s+t} (\mathcal{M}_l f)(x) + \frac{t}{s+t} (\mathcal{M}_r f)(x) \leq \max\{(\mathcal{M}_l f)(x), (\mathcal{M}_r f)(x)\}. \end{aligned} \quad (7)$$

Taking the supremum on the left hand side of (7) over $s, t > 0$, we obtain

$$(\mathcal{M}f)(x) \leq \max\{(\mathcal{M}_l f)(x), (\mathcal{M}_r f)(x)\}. \quad (8)$$

(6) and (8) imply the desired relation. ■

Next, we shall prove some elementary propositions. We will state only the case $\mathcal{M}_l$, but the corresponding results hold for the case $\mathcal{M}_r$ as well. In the following we assume that the function f is continuous. With this assumption, we note that

$$\lim_{s \rightarrow 0} \frac{1}{s} \int_{x-s}^x |f| dy = |f(x)|, \quad (9)$$

and

$$(\mathcal{M}_l f)(x) \geq |f(x)|$$

for every $x \in \mathbf{R}$.

PROPOSITION 3 *Let $f \in C(\mathbf{R})$. Assume that there exists a point $x_0 \in \mathbf{R}$ such that*

$$(\mathcal{M}_l f)(x_0) = \infty.$$

Then we have

$$\mathcal{M}_l f \equiv \infty.$$

Proof. By the assumption there exists a sequence $\{s_n\}$ of positive numbers, which converges to ∞ , such that

$$\frac{1}{s_n} \int_{x_0-s_n}^{x_0} |f| dy > n. \quad (10)$$

Fix a point x on the line and take s_n sufficiently large, then by (10) we see that

$$\begin{aligned} n &< \frac{1}{s_n} \int_{x_0-s_n}^{x_0} |f| dy = \frac{s_n + x - x_0}{s_n} \frac{1}{s_n + x - x_0} \int_{x_0-s_n}^x |f| dy - \frac{1}{s_n} \int_{x_0}^x |f| dy \\ &\leq \left(1 + \frac{x - x_0}{s_n}\right) (\mathcal{M}_I f)(x) - \frac{\int_{x_0}^x |f| dy}{s_n}. \end{aligned}$$

Letting n tend to ∞ , we obtain

$$(\mathcal{M}_I f)(x) = \infty. \quad \blacksquare$$

COROLLARY 4 Let $f \in C(\mathbf{R})$. Assume that there exists a point $x_0 \in \mathbf{R}$ such that

$$(\mathcal{M}_I f)(x_0) < \infty.$$

Then we have

$$(\mathcal{M}_I f)(x) < \infty$$

for every $x \in \mathbf{R}$.

Next, if $f \in C(\mathbf{R})$ and $\mathcal{M}_I f < \infty$, then the lower semi-continuity of $\mathcal{M}_I f$ implies that the set E :

$$E = \{x \in \mathbf{R} \mid (\mathcal{M}_I f)(x) > |f(x)|\}$$

is open. Hence, E can be written as $E = \sum_j I_j$, where I_j denotes an open interval.

PROPOSITION 5 Let $f \in C(\mathbf{R})$, $\mathcal{M}_I f < \infty$ and $E = \sum_j I_j$. Then $(\mathcal{M}_I f)(x)$ becomes a non-increasing function of x on each I_j .

Proof. Fix $x \in I_j$ and set $\epsilon = (\mathcal{M}_I f(x) - |f(x)|)/2 > 0$. By the continuity of $|f|$ there exists a $\delta > 0$ such that $|x - y| \leq \delta$ implies

$$|f(y)| < |f(x)| + \epsilon. \quad (11)$$

(11) yields

$$(\mathcal{M}_I f)(x) = \sup_{s > \delta} \frac{1}{s} \int_{x-s}^x |f| dy. \quad (12)$$

For $x - h \in I_j \cap (x - \delta, x)$, and $s > \delta$ it follows from (11) that

$$\begin{aligned} \frac{1}{s} \int_{x-s}^x |f| dy &= \frac{s-h}{s} \cdot \frac{1}{s-h} \int_{x-s}^{x-h} |f| dy + \frac{h}{s} \cdot \frac{1}{h} \int_{x-h}^x |f| dy \\ &\leq \max\{(\mathcal{M}_I f)(x-h), |f(x)| + \epsilon\}. \end{aligned} \quad (13)$$

Taking the supremum on the left hand side of (13) over $s > \delta$, we have

$$(\mathcal{M}_I f)(x) \leq \max\{(\mathcal{M}_I f)(x-h), |f(x)| + \epsilon\}$$

by (12). Since, the relation: $(\mathcal{M}_I f)(x) \leq |f(x)| + \epsilon$ and the choice of ϵ give

$$(\mathcal{M}_I f)(x) \leq |f(x)|,$$

which contradicts $x \in I_j$, we obtain

$$(\mathcal{M}_I f)(x) \leq (\mathcal{M}_I f)(x - h). \quad \blacksquare$$

PROPOSITION 6 *Let the assumptions and notation be the same as those of Proposition 5. Then $(\mathcal{M}_I f)(x)$ becomes a locally Lipschitz function of x on each I_j . In particular, $\mathcal{M}_I f$ becomes an absolutely continuous function on each I_j .*

Proof. Take $K = [\alpha, \beta] \subset I_j$. By the lower semi-continuity of $\mathcal{M}_I f$ and the continuity of $|f|$ there exists an $\epsilon > 0$ such that $x \in K$ implies

$$(\mathcal{M}_I f)(x) > |f(x)| + \epsilon. \quad (14)$$

By the uniform continuity of $|f|$ there then exists a $\delta > 0$ such that $x \in K$ and $|y - x| \leq \delta$ imply

$$|f(y)| < |f(x)| + \frac{\epsilon}{2}. \quad (15)$$

(14) and (15) yield that $x \in K$ implies

$$(\mathcal{M}_I f)(x) = \sup_{s > \delta} \frac{1}{s} \int_{x-s}^x |f| dy. \quad (16)$$

For $x, x + h \in K$, $h > 0$, and $s > \delta$ it follows from the previous proposition that

$$\begin{aligned} \frac{1}{s} \int_{x-s}^x |f| dy - \frac{1}{s+h} \int_{x-s}^{x+h} |f| dy &\leq \frac{1}{s} \int_{x-s}^x |f| dy - \frac{1}{s+h} \int_{x-s}^x |f| dy \\ &= \frac{1}{s+h} \cdot \frac{1}{s} \int_{x-s}^x |f| dy \cdot h \leq \frac{(\mathcal{M}_I f)(x)}{\delta} \cdot h \leq \frac{(\mathcal{M}_I f)(\alpha)}{\delta} \cdot h. \end{aligned} \quad (17)$$

Moving $s > \delta$ on the left hand side of (17) freely, we obtain

$$0 \leq (\mathcal{M}_I f)(x) - (\mathcal{M}_I f)(x + h) \leq Ch. \quad \blacksquare$$

PROPOSITION 7 *Let the assumptions and notation be the same as those of Proposition 5. Then $(\mathcal{M}_I f)(x)$ is continuous at the boundary of E .*

Proof. Fix a point a at the boundary of E . Then we have

$$(\mathcal{M}_I f)(a) = |f(a)|. \quad (18)$$

Clearly, we see

$$\liminf_{x \rightarrow a} (\mathcal{M}_I f)(x) \geq |f(a)| \quad (19)$$

by the lower semi-continuity of $\mathcal{M}_I f$.

Now, we assume that

$$\limsup_{x \rightarrow a} (\mathcal{M}_I f)(x) - |f(a)| = \epsilon > 0. \quad (20)$$

With this assumption, by the continuity of $|f|$ there exists a $\delta > 0$ such that $|y - a| \leq 2\delta$ implies

$$|f(y)| < |f(a)| + \frac{\epsilon}{2},$$

and hence we can select by (20) the sequences $\{x_n\}$ and $\{s_n\}$, where $\{x_n\}$ converges to a and $s_n > \delta$, such that

$$\frac{1}{s_n} \int_{x_n-s_n}^{x_n} |f| dy > |f(a)| + \frac{\epsilon}{2}. \quad (21)$$

Taking η_n so that

$$(1 + \eta_n)s_n = a - x_n + s_n, \quad (22)$$

we have

$$|\eta_n| \leq \frac{|x_n - a|}{\delta} \quad (23)$$

by the fact that $s_n > \delta$. It follows from (21) and (22) that

$$\begin{aligned} |f(a)| + \frac{\epsilon}{2} &< \frac{1}{s_n} \int_{x_n-s_n}^{x_n} |f| dy \\ &= \frac{1 + \eta_n}{a - x_n + s_n} \left(\int_{x_n-s_n}^a |f| dy - \int_{x_n}^a |f| dy \right) \\ &\leq (1 - \eta_n)(\mathcal{M}_l f)(a) + \frac{1 + \eta_n}{a - x_n + s_n} \left| \int_{x_n}^a |f| dy \right|. \end{aligned}$$

Letting n tend to ∞ , we obtain

$$|f(a)| + \frac{\epsilon}{2} \leq (\mathcal{M}_l f)(a) \quad (24)$$

by (23) and the fact that $s_n > \delta$. (24) contradicts (18) and hence we obtain

$$\lim_{x \rightarrow a} (\mathcal{M}_l f)(x) = |f(a)|$$

by (20) and (19). ■

The next theorem is the key of our argument, and will be proved by using the above propositions.

THEOREM 8 *If $f \in W^{1,1}(\mathbf{R})$, then the distribution derivatives of $\mathcal{M}_l f$ and $\mathcal{M}_r f$ become integrable functions and*

$$\|(\mathcal{M}_l f)'\|_1 \leq \|f'\|_1, \quad \|(\mathcal{M}_r f)'\|_1 \leq \|f'\|_1. \quad (25)$$

Proof. It suffices to prove only the case $\mathcal{M}_l f$. We note that if $f \in W^{1,1}(\mathbf{R})$, then $|f| \in W^{1,1}(\mathbf{R})$ and

$$\| |f|' \|_1 = \|f'\|_1. \quad (26)$$

(See [GT].) By the fact that $|f| \in W^{1,1}(\mathbf{R})$ we note further that $|f|$ becomes a continuous function and hence the assumptions of Propositions 5–7 are satisfied.

First, we shall prove that $\mathcal{M}_l f$ have a weak derivative. Recall that

$$E = \{x \in \mathbf{R} \mid (\mathcal{M}_l f)(x) > |f(x)|\}$$

$$E = \sum_j I_j = \sum_j (\alpha_j, \beta_j).$$

Set $F = \mathbf{R} \setminus E$. From Propositions 5, 6 $\mathcal{M}_I f$ has a weak derivative $v \leq 0$ on each I_j . For a test function $\phi \in \mathcal{D}(\mathbf{R})$ we see then that

$$\int_{I_j} \mathcal{M}_I f \phi' dy = [|f(\beta_j)|\phi(\beta_j) - |f(\alpha_j)|\phi(\alpha_j)] - \int_{I_j} v \phi dy \quad (27)$$

by Proposition 7. It follows from (27) that

$$\begin{aligned} \int_{\mathbf{R}} \mathcal{M}_I f \phi' dy &= \int_{E+F} \mathcal{M}_I f \phi' dy \\ &= \sum_j [|f(\beta_j)|\phi(\beta_j) - |f(\alpha_j)|\phi(\alpha_j)] - \int_E v \phi dy + \int_F |f| \phi' dy \\ &= \int_E |f| \phi' dy + \int_E |f|' \phi dy - \int_E v \phi dy + \int_F |f| \phi' dy \\ &= \int_{\mathbf{R}} |f| \phi' dy + \int_E |f|' \phi dy - \int_E v \phi dy = - \int_{\mathbf{R}} (\chi_E v + \chi_F |f|') \phi dy. \end{aligned}$$

Here, χ_E and χ_F denote the indicator functions of the sets E and F respectively. This relation implies that $\mathcal{M}_I f$ has a weak derivative $(\mathcal{M}_I f)' = \chi_E v + \chi_F |f|'$.

Lastly, we shall prove (25). For each finite interval I_j , by the fact that $v \leq 0$ and Proposition 7 we have

$$\begin{aligned} \int_{I_j} |v| dy &= (\mathcal{M}_I f)(\alpha_j) - (\mathcal{M}_I f)(\beta_j) \\ &= |f(\alpha_j)| - |f(\beta_j)| = \int_{I_j} |f|' dy \leq \int_{I_j} \|f|\| dy. \end{aligned} \quad (28)$$

If there exists an infinite interval I_{j_1} such that

$$I_{j_1} = (-\infty, \beta_{j_1}),$$

then from Proposition 5 and the definition of I_{j_1} we see that

$$(\mathcal{M}_I f)(x) \geq (\mathcal{M}_I f)(\beta_{j_1}) > 0, \quad \forall x \in I_{j_1}. \quad (29)$$

(29) contradicts the weak type (1, 1) inequality for $\mathcal{M}_I f$. (See [St].)

If there exists an infinite interval I_{j_2} such that

$$I_{j_2} = (\alpha_{j_2}, \infty),$$

then we have

$$\begin{aligned} \int_{\alpha_{j_2}}^r |v| dy &= (\mathcal{M}_I f)(\alpha_{j_2}) - (\mathcal{M}_I f)(r) \\ &\leq |f(\alpha_{j_2})| - |f(r)| = \int_{\alpha_{j_2}}^r |f|' dy \leq \int_{\alpha_{j_2}}^r \|f|\| dy \end{aligned} \quad (30)$$

for $\alpha_{j_1} < r$. From (28) and (30) we obtain

$$\|(\mathcal{M}_I f)'\|_1 \leq \| |f|' \|_1 = \|f'\|_1$$

by (26). Thus, we have proved the theorem. ■

3 Proof of Theorem 1

The proof of Theorem 1 follows now easily from Theorem 8. We shall need only the following lemma.

LEMMA 9 *Let f and g be the (real valued) integrable functions on the line. Set*

$$F(x) = \int_{-\infty}^x f \, dy,$$

$$G(x) = \int_{-\infty}^x g \, dy,$$

and

$$H(x) = \max\{F(x), G(x)\}.$$

Then the weak derivative of H becomes an integrable function and

$$\|H'\|_1 \leq \|f\|_1 + \|g\|_1.$$

This lemma can be proved easily. (cf. [GT], Lemma 7.6.)

Theorem 1 can be now proved by Lemmas 2, 9 and Theorem 8.

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