

On representations of $\mathbb{R}$ -elementary groups at 2

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G (finite group) の $\mathbb{R}$ (real number field) 上の表現を考えたい。ここでは, Brauer-Witt induction theorem ([4], p31) に
よる立場に立ち, $\mathbb{R}$ -elementary groups at 2 の $\mathbb{R}$ 上の表現を考
える。この小文の目的は次の問題の 1 つの解答を得る事であ
る。

Problem. Let G be an $\mathbb{R}$ -elementary group at 2. If G
has an irr. char. of Schur index 2 over $\mathbb{R}$. Then G ?

1. 2-groups

2-groups については, Witt-Roquette theorem ([2], p73) に,
次の Lemma を混ぜて, ちょっとかき回してみると, 問題の解
答を得る事ができる。(Witt-Roquette + Lemma $\Rightarrow$ Theorem)

Lemma. G : a finite group, H : a subgroup of G .

- (1). ζ ; a linear char. of H . $\chi = \zeta^G$. Then χ is not irr. iff $\exists x \in G - H$ such that $xhx^{-1}h^{-1} \in \text{Ker } \zeta$ for $\forall h \in H^x \cap H$.
- (2). ζ ; a char. of H . $\chi = \zeta^G$. Then $\text{Ker } \chi = \bigcap_{g \in G} (\text{Ker } \zeta)^g$.

Λ is the ordinary quaternion algebra over $\mathbb{Q}$ or $\mathbb{R}$.

Theorem. F : a field of characteristic 0. P : 2-group.

Then, $\exists \chi$: faith. irr. char. of P such that $m_F(\chi) = 2$

iff

- (i). $P \geq Q > A > K$ such that $Q \triangleright K$, A/K : cyclic and $|Q:A| = 2$,
- (ii). Q/K : generalized quaternion of order $2^{n+1} \geq 8$,
- (iii). $x \in P - A \Rightarrow \exists a \in A^x \cap A$ such that $xa x^{-1}a^{-1} \notin K$.
- (iv). $\bigcap_{x \in P} K^x = 1$.
- (v). $F(\varepsilon_{2^n} + \varepsilon_{2^n}^{-1}) \otimes_{\mathbb{Q}} \Lambda$: a division alg., where ε_{2^n} is a primitive 2^n -th root of unity.

Remark. 条件 (v) には、Fein, Gordon and Smith [1] を参照されたい。特に $F = \mathbb{R}$ の時は、 $\mathbb{R}(\varepsilon_{2^n} + \varepsilon_{2^n}^{-1}) \otimes_{\mathbb{Q}} \Lambda = \mathbb{H}$ (Hamilton's quaternion field) である。

Example. W is a generalized quaternion group.

$$P \equiv W \setminus \underbrace{Z_2 \setminus Z_2 \setminus \cdots \setminus Z_2}_{r-1}. \quad (P \text{ is a Sylow } 2\text{-group of } Sp_{2r}(8))$$

$\Rightarrow \exists \chi$: faith. irr. char. of P of Schur index 2 over $\mathbb{R}$.

$M_{2^{r-1}}(\mathbb{H})$ is a simple component of $\mathbb{R}P$.

2. $\mathbb{R}$ -elementary groups at 2.

この節では次の Frobenius-Schur theorem を $\mathbb{R}$ -elementary group at 2 の場合に, 群の言葉に書きなおす.

Frobenius-Schur theorem. ([2], p. 21). G : finite group.

χ : irr. char. of G . $\nu(\chi) = |G|^{-1} \sum_{g \in G} \chi(g^2)$. Then

(i) $\nu(\chi) = 1$ iff $\mathbb{R}(\chi) = \mathbb{R}$ and $m_{\mathbb{R}}(\chi) = 1$

(ii) $\nu(\chi) = -1$ iff $\mathbb{R}(\chi) = \mathbb{R}$ and $m_{\mathbb{R}}(\chi) = 2$

(iii) $\nu(\chi) = 0$ iff $\mathbb{R}(\chi) = \mathbb{C}$.

H is an $\mathbb{R}$ -elementary group at 2, i.e. $H = P \langle a \rangle$ (semi-direct product), P : 2-group, $\langle a \rangle$: 2'-group, $x \in P \Rightarrow xax^{-1} = a$ or a^{-1} , とする. $A \equiv \langle a \rangle$, $Q \equiv C_P(a)$ とおく. 次を仮定する.

Assumption. $A \neq 1$, $P \neq Q$.

$\{1, g\}$; a set of representatives of P/Q , χ ; irr. char. of H such that $\text{Ker } \chi \cap A = 1$, とすると, 次の事はすぐわかる.

Lemma. $\exists \mu$; irr. char. of Q , $\exists \tau$: faith linear char. of A such that $\mu^{\tau} = \bar{\mu}$, $(\mu\tau)^P = \chi$.

Lemma. (i) $\nu(\chi) = -1$ iff $|P|^{-1} \sum_{h \in Q} (\mu + \mu^{\tau})(gh)^2 = -1$.

(ii). If $|P|^{-1} \sum_{h \in Q} (\mu + \mu^{\tau})(gh)^2 = -1$, then

(1). μ^P : irr. $\Rightarrow \nu(\mu) = 0$, $\nu(\mu^P) = -1$.

(2). μ^P : non-irr. $\Rightarrow \exists \zeta, \zeta'$; irr. char. of P such that $\zeta \neq \zeta'$ and $\mu^P = \zeta + \zeta'$. Further.

(a). $\nu(\mu) = \nu(\zeta) = \nu(\zeta') = -1$

or. (b). $\nu(\mu) = 1$ and $\nu(\zeta) = \nu(\zeta') = 0$.

$R \pm$ Schur index ν の char. をもつか否かという問題は 2-group に関する議論に移る事になる。上の Lemma の (ii) のとれどいれの場合について考察する事により次の Theorem を得る。

Theorem. $\exists \chi$; irr. char. of H such that $\nu(\chi) = -1$ and $\text{Ker } \chi \cap A = 1$, iff one of the following conditions (A),

(B), (C) are satisfied.

(A). $\exists K \leq Q$ such that $K \triangleleft P$ and P/K is a cyclic group of order 4.

(B). $\exists K \triangleleft \exists T \leq Q$ which satisfy the following conditions.

(i). T/K : cyclic

(ii). $x \in Q - T \Rightarrow \exists a \in T^x \cap T$ such that $xa x^{-1}a^{-1} \notin K$.

(iii). $\exists y \in P - Q$ such that $yby^{-1}b^{-1} \in K$ for $\forall b \in T^y \cap T$.

(iv). $X = \{h_1, \dots, h_m\}$; a set of representatives of Q/T .

$\alpha = \# \{ (h_i, h) \in X \times Q \mid h_i(gh)^2 h_i^{-1} \in T - K, h_i(gh)^4 h_i^{-1} \in K \}$, $\beta = \# \{ (h_i, h) \in X \times Q \mid h_i(gh)^2 h_i^{-1} \in K \}$. Then $|Q|^{-1}(\beta - \alpha) = -1$.

(C). $\exists K \leq \exists T \leq \exists S \leq P$ which satisfy the following conditions

(i). $S \triangleright K$, S/K ; a generalized quaternion group,

T/K ; a cyclic subgroup of S/K of index 2.

(ii). $x \in P - T \Rightarrow \exists a \in T^x \cap T$ such that $xa x^{-1}a^{-1} \notin K$.

(iii). One of the following conditions are satisfied.

(a). $K \leq T \cap Q$ and $(S \cap Q)/K$ is a cyclic subgroup of S/K of index 2.

(b). $K \leq T \cap Q$ and $(S \cap Q)/K$ is non abelian.

or . $K \not\leq T \cap Q$. $X = \{h_1, \dots, h_m\}$; a set of representatives of $Q/(T \cap Q)$. $\gamma = \# \{ (h_i, h) \in X \times Q \mid h_i(gh)^2 h_i^{-1} \in T - K, h_i(gh)^4 h_i^{-1} \in K \}$, $\delta = \# \{ (h_i, h) \in X \times Q \mid h_i(gh)^2 h_i^{-1} \in K \}$. Then $|P|^{-1}(\delta - \gamma) = -1$ and

$$|T: T \cap Q| = 2.$$

References

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