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# Gevrey singularities for nonlinear wave equations

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## 1. INTRODUCTION

We consider the following semilinear wave equations,

$$\Box u=f(u) \quad \text{in } \Omega\subset R_t\times R_x^2, \tag{1}$$

where $u$ is a real valued function, $\Box=\partial^2/\partial t^2-\triangle$ with $\triangle=\sum_{j=1}^2\partial^2/\partial x_j^2$, $\Omega$ is a bounded domain which contains the origin and $f(u)$ is a polynomial of $u$ with $f(0)=0$.

We study the interaction of Gevrey singularities for this equation. We assume that solutions that we study here are all in $H^s(\Omega)$ with $s>3/2$ where $H^s(\Omega)$ is a Sobolev space of order $s$ in $\Omega$. In 1982, J. Rauch and M. Reed [4] have made an example in which three singularities produce new singularities. In 1984, J. M. Bony [2] and R. Melrose and N. Ritter [3] have had a general result for $C^\infty$singularity independently. We put $\Sigma_j=\{(t,x)\in R^3; t=\omega_j\cdot x\}$ $(j=1,2,3)$ with $\omega_j\in S^1$. Their result for the equation (1) is as follows.

**Theorem 1.1 (J. M. Bony [2], R. Melrose and N. Ritter [3]).** *If $u$ is conormal with respect to $\Sigma_1\cup\Sigma_2\cup\Sigma_3$ in $\Omega_-=\Omega\cup\{t<0\}$, then the solution $u$ is $C^\infty$ in $K\backslash(\Sigma_1\cup\Sigma_2\cup\Sigma_3\cup\{t^2=|x|^2\})$ where $K$ is a domain of determine with respect to $\Omega_-$.*

In this talk, we shall make the Gevrey version of the above result.

**Definition 1.1 (Gevrey conormal distribution).** For $s>3/2, \sigma\leq 1$, we call that $u\in H^s(\Sigma, G^{(\sigma)};\Omega)$, if and only if for any compact set $K\subset\Omega$ and for any vector fields $V_1,\dots,V_l$ with analytic coefficients and any integer $l$ which are tangent to $\Sigma$, there exist constants $C, A>0$ such that

$$\|V_1^{\alpha_1}\cdots V_l^{\alpha_l}u\|_{H^s(K)}\leq CA^{|\alpha|}(|\alpha|!)^\sigma \tag{2}$$

for any integers $\alpha_1,\dots,\alpha_l$.

**Theorem 1.2.** *Suppose that $u$ is in $H^s(\Omega)$ for some $s>5/2$, $u$ satisfies the equation (1) and $u\in H^s(\Sigma_1, G^{(\sigma)};\Omega_-)$. Then we have*

$$u\in H^s(\Sigma_1, G^{(\sigma)};K), \tag{3}$$

*where* $\Omega_- = \Omega \cap \{(t,x); t < 0\}$, $K$ *is the domain of dertermine with respect to* $\Omega_-$.

**Theorem 1.3.** *Suppose that* $u$ *is in* $H^s(\Omega)$ *for some* $s > 5/2$, $u$ *satisfies the equation* (1) *and* $u \in H^s(\Sigma_1 \cup \Sigma_2, G^{(\sigma)}; \Omega_-)$. *Then we have*

$$u \in H^s(\Sigma_1 \cup \Sigma_2, G^{(\sigma)}; K), \tag{4}$$

*where* $\Omega_- = \Omega \cap \{(t,x); t < 0\}$, $K$ *is the domain of dertermine with respect to* $\Omega_-$.

**Theorem 1.4 (Main result).** *Suppose that* $u \in H^s(\Omega)$ $(s > 5/2)$, $u$ *satisfies the equation* (1) *and*

$$u \in H^s(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3, G^{(\sigma)}; \Omega_-). \tag{5}$$

*Then* $u$ *is a Gevrey class function of order* $\sigma$ *in* $K \backslash \Sigma_1 \cup S_2 \cup \Sigma_3 \cup \Gamma_+$, *where* $\Gamma_+ = \{t^2 = |x|^2, t > 0\}$, $\Omega_- = \Omega \cap \{t < 0\}$ *and* $K$ *is a domain of determine with respect to* $\Omega_-$.

**Corollary 1.1.** *Suppose that* $u \in H^s(\Omega)$ $(s > 5/2)$, $u$ *satisfies the equation* (1) *and*

$$u \in H^s(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3, G^{(1)}; \Omega_-). \tag{6}$$

*Then* $u$ *is real analytic in* $K \backslash \Sigma_1 \cup S_2 \cup \Sigma_3 \cup \Gamma_+$, *where* $\Gamma_+ = \{t^2 = |x|^2, t > 0\}$, $\Omega_- = \Omega \cap \{t < 0\}$ *and* $K$ *is a domain of determine with respect to* $\Omega_-$.

## 2. Preliminaries

Let $K$ be a relatively compact set in $R^3 = R_t \times R_x^2$ such that each subset $K \cap \{(t,x); s \le t \le T\}$ is a domain of determine with respect to $K \cap \{(t,x); t \le s\}$ for $S \le s \le T$. For $m > 5/2$ and $f \in H^m(K)$, we put

$$E_m(t)[f] = \|f(t)\|_{H^{m-1/2}(K(t))} + \|\partial_t f(t)\|_{H^{m-3/2}(K(t))} \tag{7}$$

with $K(s) = K \cap \{(t,x); t = s\}$.

**Proposition 2.1 (Energy estimate).** *For* $f \in H^m(K)$, *we have*

$$E_m(t_2)[f] \le E_m(t_1)[f] + C(T) \int_{t_1}^{t_2} \|\Box f\|_{H^{m-3/2}(K(s))} ds \tag{8}$$

*for* $S \le t_1 < t_2 \le T$.

**Proposition 2.2.** *For* $u, v \in H^m(K)$, *we have*

$$E_m(t)[uv] \le C(n) E_m[u] E_m[v]. \tag{9}$$

Let $Q$ be a analytic vector field on $K$. We define a quantity $\|f(t)\|_{G_A^s(Q;E_m)}$ by

$$\|f(t)\|_{G_A^s(Q;E_m)} = \sum_{l=0}^{\infty} \frac{A^l}{l!^s} E_m(t)[P^l] \tag{10}$$

and we put $\|f(t)\|_{X(Q)} = \|f\|_{G_A^s(Q;E_m)}$ and $|\!|\!| f |\!|\!|_{Y_A([t_1,t_2];Q)} |\!|\!| f |\!|\!|_{Y([t_1,t_2];Q)} = \sup_{t_1 \le t \le t_2} \|f(t)\|_{X(Q)}$ for abbreviation.

**Proposition 2.3.** *For $c \in R$,*

$$\|f\|_{G_A^s(Q+c;E_m)} \le e^{|c|A} \|f\|_{X(Q)}. \tag{11}$$

**Proposition 2.4.**

$$\|uv\|_{X(Q)} \le C(n)\|u\|_{X(Q)}\|v\|_{X(Q)}.$$

For $\Sigma_1$ and $\Sigma_2$, we put $\tilde{\omega}_j = (1, -\omega_j)$, $\tilde{\omega}_j^* = (1, \omega)$, $\nabla = (\partial_t, \partial_{x_1}, \partial_{x_2})$ and put

$$X_1 = (\tilde{\omega}_1 \times \tilde{\omega}_2) \cdot \nabla \tag{12}$$

$$X_2 = (t - \omega_2 \cdot x)\tilde{\omega}_1^* \cdot \nabla \tag{13}$$

$$X_3 = \tilde{\omega}_2^* \cdot \nabla \tag{14}$$

$$X_4 = (t - \omega_1 \cdot x)\tilde{\omega}_2^* \cdot \nabla \tag{15}$$

**Proposition 2.5.** *We have*

$$[X_j, X_k] = 0 \quad \textit{for} \quad 1 \le j, k \le 4, \tag{16}$$

$$[\Box, X_1] = [\Box, X_3] = 0, \tag{17}$$

$$[\Box, X_2] = [\Box, X_3] = C_1\Box + C_2 X_1^2, \tag{18}$$

*for some $C_1$ and $C_2$.*

**Proposition 2.6.** (1) *$X_1, X_2$ and $X_3$ are all tangent to $\Sigma_1$.*
(2) *$X_1, X_2$ and $X_4$ are all tangent to $\Sigma_1 \cup \Sigma_2$.*

**Proposition 2.7.** (1) *$X_1, X_2$ and $X_3$ are linearly independent in $\Sigma_1^c$.*
(2) *$X_1, X_2$ and $X_4$ are linearly independent in $(\Sigma_1 \cup \Sigma_2)^c$.*

## 3. Lemmas

In this section, we prepare several lemmas which are used to prove the theorems. We put $P = t\partial_t + x \cdot \partial_x$. Let $K'$ be a relatively compact open set in $K$ satisfying the same condition $K$ of the section 2. We consider the following linearized equation,

$$\begin{cases} \Box v = F(w), \\ v = u(-\epsilon, x) \quad \partial_t v = \partial_t u \quad \text{for} \quad t = -\epsilon, \end{cases}$$

where we take $\epsilon$ is so small that $K'(-\epsilon)$ determines $K' \cap \{-\epsilon < t < T\}$. Let $S$ denote the mapping that corresponds $w$ to $v$. We put $u_0 = S[0]$ and $u_n = Su_{n-1}$. Since $u_0$

is a solution to the homogenous linear wave equation, there exists a constant $A$ such that $|||u_0|||_{Y_A([-\epsilon,T];P)} < \infty$. We put $B_0 = \max(|||u_0|||_{Y([-\epsilon,T];P)}, 2\|u(t_1)\|_{X(P)}$.

**Lemma 3.1.** *If $u$ satisfies the assumption of Theorem 1.2 or 1.3 or 1.4, we have*

$$|||u|||_{Y([t_1,t_2];P)} \leq \|u(t_1)\|_{X(P)}, \tag{19}$$

*for $-\epsilon \leq t_1 < t_2 \leq T$ with $t_2 - t_1 < 1/(2C(T)F(C(n))G(B_0))$.*

*Proof.* Using Propositions 2.1, 2.3 and 2.4, we have the lemma. □

**Lemma 3.2 (the Energy estimate).** *If $u$ satisfies the assumption of Thoeorem 1.2 or 1.3 or 1.4, we have*

$$\sup_{0\leq t\leq T} |||u(t)|||_{X(P)} \leq |||\phi|||_{G^s_A(x\cdot\nabla;E_m)}. \tag{20}$$

*Proof.* Using the lemma 3.1 several times, we have the lemma. □

## 4. Proof of Theorem 1.1 and 1.2

First we prove Theorem 1.1. From Propositions 2.6 and 2.7 it suffices to show that for every compact set $K' \subset K$ there exist constants $C_1$ and $A_1$ such that

$$\|X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}u\|_{H^m(K')} \leq C_1A_1^{|\alpha|}|\alpha|^\sigma, \tag{21}$$

for all non negative integers $\alpha_1, \alpha_2$ and $\alpha_3$ with $|\alpha| = \alpha_1 + \alpha_2\alpha_3$. We can prove the above by the same argument as in the proof of Lemma 3.2.

## 5. Regularity in the interior of the cone

Let $\sigma$ be a real number greater than or equal to 1. We put $P = t\partial_t + x\cdot\partial_x$. The following lemma is a key kemma to prove Theorem 1.4.

**Lemma 5.1 (Key lemma).** *Suppose that*

$$\|P^l u\|_{H^s(K)} \leq C_1A_1^l(l!)^\sigma \quad \text{for } \forall l \in N\cup\{0\} \tag{22}$$

*and $u$ satisfies the equation (1). Then $u$ is a Gevrey class function of order $\sigma$ in $\Gamma_+$, where $\Gamma_+ = \{(t,x) \in R^3; t^2 > |x|^2, t > 0\}$.*

*Proof.* For simplicity, we prove only the case $f(u) = u^m$. Let $B \subset \Gamma_+$ be a relatively compact ball. It suffices to show that $u$ is a Gevrey class function of order $\sigma$ in each $B \subset \Gamma_+$. We put $M = \Box^2 + P^4$.

Let $\chi(x)$ be a $C^\infty$ function in $B$ such that $0 < \chi$ in $B$ and $\chi(x) = \mathrm{dist}(x, \partial B)$ near $\partial B$. We put $\psi(x) = \chi(x)^N$ and we take $N$ sufficiently large that $\|\partial^\beta(\psi u)\|_B \leq c\|M\psi u\|_B$ for $|\beta| \leq 4$.

We show that

$$\|\psi^{|\alpha|}\partial^\alpha P^l u\| \leq C_2A_2^{|\alpha|+l}((|\alpha|+l)!)^\sigma \tag{23}$$

for some $C_2 > 0$ and $A_2 > 0$ for all $\alpha \geq 0$ and all $l \geq 0$. We show (23) by induction with respect to $\alpha$.

When $|\alpha| = 0$, (23) is nothing but the assumption (22). We assume that (23) is valid until $|\alpha| = m$.

First we prove the case $0 \leq m \leq 3$. For $|\alpha| = m+1$, we have

$$\|\psi^{|\alpha|}\partial^\alpha P^l u\|_B \leq \|\partial^\alpha \psi^{|\alpha|} P^l u\|_B + \|[\psi^{|\alpha|}, \partial^\alpha] P^l u\|_B. \tag{24}$$

The second term of the right hand side is estimated by

$$\sum_{\alpha' < \alpha} C_3 \|\psi^{|\alpha'|}\partial^{\alpha'} P^l u\| \leq C_4 C_2 A_2^{m+l}((m+l)!)^\sigma \tag{25}$$

$$\leq \frac{1}{2} C_2 A_2^{m+l+1}((m+l)!)^\sigma \tag{26}$$

if we take $A_2 \geq 2C_4$. Since $|\alpha| = m+1 \leq 4$, the first term is estimated by

$$\|M\psi^{|\alpha|}P^l u\|_B \leq \|\psi^{|\alpha|} M P^l u\|_B + \|[M, \psi^{|\alpha|}] P^l u\|_B. \tag{27}$$

The second term of the right hand side of the above inequality can be estimated by

$$C_5 \sum_{|\alpha'| \leq 3} \|\psi^{|\alpha'|}\partial^{\alpha'} P^l u\|_B \leq \frac{1}{4} C_2 A_2^{m+l+1}((m+l+1)!)^\sigma \tag{28}$$

if we take $A_2$ sufficiently large. The first term is estimated by

$$\|\psi^{|\alpha|}\Box^2 P^l u\|_B + \|\psi^{|\alpha|} P^{l+4} u\|_B. \tag{29}$$

The second term of the right hand side of the above can estimated by

$$\begin{aligned} C_6\|P^{l+4}u\|_B &\leq C_6 C_2 A_2^{l+4}((l+4)!)^\sigma \\ &\leq \frac{1}{8} C_2 A_2^{l+m+1}((l+m+1)!)^\sigma, \end{aligned}$$

if we take $C_2$ and $A_2$ sufficiently large. The first term of the right hand side of the above is estimated by

$$\begin{aligned} \|\psi^{|\alpha|}(P+4)^l \Box^2 u\|_B &= \|\psi^{|\alpha|}(P+4)^l \Box(u^m)\|_B \\ &\leq m\|\psi^{|\alpha|}(P+4)^l u^{2m-1}\| + \binom{m}{2} \sum_{j=0}^{2} \|\psi^{|\alpha|}(P+4)^l u^{m-2}(\partial_j u)^2\|_B. \end{aligned}$$

The second term of the right hand side of the above is esitmated by

$$\begin{aligned} \frac{m(m-1)}{2} \sum_{j=0}^{2} \sum_{\alpha_1+\cdots+\alpha_m=\alpha} \frac{\alpha!}{\alpha_1!\cdots\alpha_m!} \sum_{l_1+\cdots+l_m=l} \frac{l!}{l_1!\cdots l_m!} \|\psi^{|\alpha_1|}(P+4)^{l_1} u\|_B \times \\ \|\psi^{|\alpha_2|}P^{l_2}u\|_B \cdots \|\psi^{|\alpha_{m-2}|}P^{l_{m-2}}u\|_B \|\psi^{|\alpha_{m-1}|}P^{l_{m-1}}\partial_j u\|_B \|\psi^{|\alpha_m|}P^{l_m}\partial_j u\|_B. \end{aligned} \tag{30}$$

This can be estimated by $\frac{1}{16}C_2A_2^{l+m+1}((l+m+1)!)^\sigma$ if we take $C_2$ and $A_2$ sufficiently large. The first term can be also estimated by $\frac{1}{16}C_2A_2^{l+m+1}((L+m+1)!)^\sigma$.

Next we prove the case $m \geq 4$. For $|\alpha| = m-3$ and $|\beta| = 4$, we have

$$\|\psi^{m+1}\partial^{\alpha+\beta}P^l u\|_B \leq \|\partial^\beta\psi^{m+1}\partial^\alpha P^l u\|_B + \|[\psi^{m+1}, \partial^\beta]\partial^\alpha P^l u\|_B. \tag{31}$$

The second term of the right hand side of the above can be estimated by

$$C_7 \sum_{\beta'<\beta} \|\psi^{m-3+|\beta'|}\partial^{\beta'}P^l u\|_B \leq C_7C_2A_2^{m+l}((m+l)!)^\sigma \tag{32}$$

$$\leq \frac{1}{2}C_2A_2^{m+l+1}((m+l+1)!)^\sigma \tag{33}$$

if we take $A_2 \geq 2C_7$. Since $|\beta| = 4$, the first term is extimated by

$$\|M\psi^{m+1}\partial^\alpha P^l u\|_B \leq \|\psi^{m+1}M\partial^\alpha P^l u\|_B + \|[M, \psi^{m+1}]\partial^\alpha P^l u\|_B. \tag{34}$$

Using the same argument as in the case $m \leq 3$, we can estimate the right hand side of the above by $\frac{1}{2}C_2A_2^{m+l+1}((m+l+1)!)^\sigma$. But we note that we do not change $C_2$ at each step of induction in the case $m \geq 4$ not as in the case $m \leq 3$.

□

## 6. Proof of Main result

We devide $K\backslash\Sigma_1 \cup S_2 \cup \Sigma_3 \cup \Gamma_+$ into 4 parts, $\bigcup_{i=1}^4 \mathcal{O}_i$ with

(35)
$$\mathcal{O}_1 = \{(t,x) \in R^3; t-\omega_1\cdot x > 0, t-\omega_2\cdot x < 0, t-\omega_3\cdot x < 0\} \cup \cdots$$

(36)
$$\mathcal{O}_2 = \{(t,x) \in R^3; t-\omega_1\cdot x > 0, t-\omega_2\cdot x > 0, t-\omega_3\cdot x < 0\} \cup \cdots$$

(37)
$$\mathcal{O}_3 = \{(t,x) \in R^3; t-\omega_1\cdot x > 0, t-\omega_2\cdot x > 0, t-\omega_3\cdot x > 0, t^2-|x|^2 < 0\}$$

(38)
$$\mathcal{O}_4 = \{(t,x) \in R^3; t-\omega_1\cdot x > 0, t-\omega_2\cdot x > 0, t-\omega_3\cdot x > 0, t^2-|x|^2 > 0\}.$$

For $x$ in $\mathcal{O}_1\cup\mathcal{O}_2\cup\mathcal{O}_3$, the backward light cone $\Gamma_x^-$ from $x$ does not contain the origin. So we can prove that $u$ is in $G^{(\sigma)}$ in this area by the same argument as in the proof of Theorems 1.2 and 1.3.

To prove that $u$ is in $G^{(\sigma)}$ in $\mathcal{O}_4$, we use the operator $P = t\partial_t + x\cdot\partial_x$. Using this operator, M. Beals[1] has given another proof of the theorem 1.1 of Bony and Melrose–Ritter. Note that for all relatively compact open set $L \subset \Omega_-$, there exist constants $C, A_1 > 0$ such that

$$\|P^k u\|_{H^s(L)} \leq CA_1^k(k!)^\sigma \quad \text{for} \forall k, \tag{39}$$

from the assumptions of Theorem 1.2. Since $[\Box, P] = 2\Box$,

$$\Box(Pu) = P\Box u + [\Box, P]u = (P+2)f(u). \tag{40}$$

So we have

$$\Box(P^k u) = (P+2)^k f(u). \tag{41}$$

Using the energy inequality 3.2, we have for all relatively compact open set $L \subset K$, there exist constants $C, A > 0$ such that

$$\|P^k u\|_{H^s(L)} \leq CA^k (k!)^\sigma \quad \text{for } \forall k. \tag{42}$$

From Lemma 5.1, we have that $u$ is in $G^{(\sigma)}$ in $\mathcal{O}_4$.

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