

1. Classical results from complex geometry

$T_{\mathbb{C}}$: a complex 2-torus, $-1_{T_{\mathbb{C}}} : T_{\mathbb{C}} \rightarrow T_{\mathbb{C}}; x \mapsto -x$.
 $X = \text{Km}(T_{\mathbb{C}}) :=$ the minimal resolution of $T_{\mathbb{C}}/ -1_{T_{\mathbb{C}}}$.
 ω : a holomorphic 2-form on X .
 $\text{NS}(X) = \{x \in H^2(X, \mathbb{Z}) \mid \omega \cup x = 0\}$.

$U(n)$: the lattice with a Gram matrix $\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$.

Theorem (Classical result for Kummer surfaces $\text{Km}(T_{\mathbb{C}})$)

$\exists q_* : H^2(T_{\mathbb{C}}, \mathbb{Z}) \rightarrow H^2(\text{Km}(T_{\mathbb{C}}), \mathbb{Z})$, $q_*(D) \cup q_*(D) = 2D \cup D$,
 $q_*(H^2(T_{\mathbb{C}}, \mathbb{Z})) \simeq U(2)^{\oplus 3}$ and $q_*(\text{NS}(T_{\mathbb{C}})^{\perp}) \simeq \text{NS}(\text{Km}(T_{\mathbb{C}}))^{\perp}$.

The main result is about a tropical analog of the above theorem for tropical Kummer surfaces.

complex Kummer surfaces	tropical Kummer surfaces
$\wedge_{i=1}^2 \Omega_{\text{Km}(T_{\mathbb{C}})} \simeq \mathcal{O}_{\text{Km}(T_{\mathbb{C}})}$	$\iota_*(\wedge_{i=1}^2 \mathcal{T}_{\text{Km}(T)_0}^{\vee}) \simeq \mathbb{R}_{\text{Km}(T)}$
$h^i(\text{Km}(T_{\mathbb{C}}), \mathcal{O}_{\text{Km}(T_{\mathbb{C}})})$	$h^i(\text{Km}(T), \mathbb{R}_{\text{Km}(T)})$
$\mathcal{O}_{\text{Km}(T_{\mathbb{C}})}^{\times}$	$\iota_* \text{Aff}_{\mathbb{Z}, \text{Km}(T)_0}$
ω (holomorphic 2-form)	$c_{\text{Km}(T)}$ (radiance obstruction)
$\text{NS}(\text{Km}(T_{\mathbb{C}})) = \{\omega \cup d = 0\}$	$\text{Im } c_1 = \{c_{\text{Km}(T)} \cup d = 0\}$
$\dim \Gamma(\check{X}, L) = \int_{\check{X}} \frac{c_1(L)^2}{2} + 2$	$ \text{Km}(T)(\mathbb{Z}) = \frac{c_{\text{Km}(T)} \cup c_{\text{Km}(T)}}{2} + 2$
$q_*(H^2(T_{\mathbb{C}}, \mathbb{Z})) \simeq U(2)^{\oplus 3}$	$q_*(H^1(T, \mathbb{Z}_{\text{Km}(T)})) \simeq U(2)^{\oplus 2}$
$q_*(\text{NS}(T_{\mathbb{C}})^{\perp}) \simeq \text{NS}(\text{Km}(T_{\mathbb{C}}))^{\perp}$	$q_*(\text{Im } c_1, T)^{\perp} = (\text{Im } c_1, \text{Km}(T))^{\perp}$

Table: Comparison table for complex and tropical Kummer surfaces

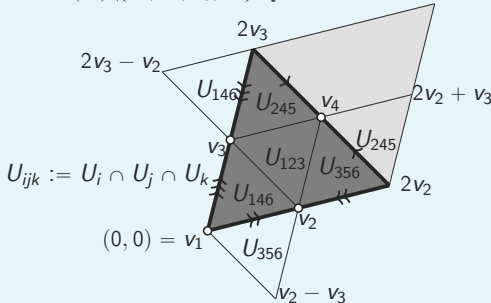
2. TASS and tropical Kummer surfaces

Definition (cf. Gross-Siebert'03, Kontsevich-Soibelman'04)

- (1) B_0 is an n -dim'l tropical (resp. integral) affine manifold if B_0 is a manifold equipped with an atlas $\{(U_i, \psi_i)\}$
s.t. $\psi_i \circ \psi_j^{-1}|_{U_i \cap U_j} \in \text{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ (resp. $\text{GL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$).
(2) An n -dim'l topological mfd B equipped with an inclusion $\iota : B_0 \hookrightarrow B$ of tropical (resp. integral) affine manifold is a tropical (resp. integral) affine manifold with singularities if $\text{codim } B \setminus B_0 \geq 2$.

Example

- (1) $B = B_0 = \mathbb{R}^n/\Lambda$: tropical tori [Mikhalkin-Zharkov'06].
(2) $T = \mathbb{R}^2/\Lambda$: a tropical 2-torus, $-1_T : T \rightarrow T; x \mapsto -x$,
 $\text{Km}(T) := T / -1_T (\simeq S^2)$: the **tropical Kummer surface**.
 $\text{Km}(T)_0 := \text{Km}(T) \setminus \{v_1, v_2, v_3, v_4\}$. [Foster-Rabinoff-Shokrieh-Soto]



The light gray region is a fundamental domain of T .
The dark gray region is that of $\text{Km}(T)$.

A tropical affine surface with singularities (**TASS** for short) is a 2-dim'l tropical affine manifold with singularities satisfying the following condition;

Condition (cf. Kontsevich-Soibelman'04)

- $\forall x \in B \setminus B_0$, $\exists \{V_i\}_{i \in I}$ s.t. (i) $|I| < \infty$, (ii) $\bigcap_{i \in I} V_i = \{x\}$
(iii) $\bigcup_{i \in I} V_i \ni x$: open, (iv) $\exists u_i, v_i \in \mathbb{R}^2$,
 $U_i \setminus \{x\} \simeq \{ru_i + sv_i \in \mathbb{R}^2 \mid 0 \leq r, s \text{ and } r+s < 1\} \setminus \{(0,0)\}$,
(v) $\forall i \neq j \in I$, $(V_i \cap V_j) \setminus \{x\} \simeq (0, r)$ or $\emptyset$.

Example

$B_0 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R} \hookrightarrow \mathbb{C}P^1 = B$ does not satisfy the above Condition.

3. Main result

B : a TASS, $\{(U_i, \psi_i)\}$: an affine atlas of B_0 .
 Aff_{B_0} : the sheaf of locally affine functions.
 $\text{Aff}_{\mathbb{Z}, B_0}$: the sheaf of locally affine functions with integer slopes.
 $\mathcal{T}_{B_0}$: the locally constant sheaf associated with the local trivialization
 $\theta_i : TU_i \rightarrow U_i \times \mathbb{R}^2; (x, v) \mapsto (x, d\psi_{i,x}(v))$.
 $\mathcal{T}_{\mathbb{Z}, B_0}$: the subsheaf of $\mathcal{T}_{B_0}$ consisting of integer valued vectors.
 $\mathcal{T}_{\mathbb{Z}, B_0}^{\vee} := \text{Hom}_{\mathbb{Z}_{B_0}}(\mathcal{T}_{\mathbb{Z}, B_0}, \mathbb{Z}_{B_0})$, $\mathcal{T}_{B_0}^{\vee} := \text{Hom}_{\mathbb{R}_{B_0}}(\mathcal{T}_{B_0}, \mathbb{R}_{B_0})$.

$$0 \rightarrow \mathbb{R}_B \rightarrow \iota_* \text{Aff}_{\mathbb{Z}, B_0} \rightarrow \iota_* \mathcal{T}_{\mathbb{Z}, B_0}^{\vee} \rightarrow 0 : \text{exact}$$

$c_B \in H^1(B, \text{Hom}_{\mathbb{Z}_B}(\iota_* \mathcal{T}_{\mathbb{Z}, B_0}^{\vee}, \mathbb{R}_B)) \subset \text{Ext}^1(\iota_* \mathcal{T}_{\mathbb{Z}, B_0}^{\vee}, \mathbb{R}_B)$
: the extension class of the above exact sequence.

$\text{Hom}_{\mathbb{Z}_B}(\iota_* \mathcal{T}_{\mathbb{Z}, B_0}^{\vee}, \mathbb{R}_B) \simeq \iota_* \mathcal{T}_{B_0}$,

c_B is called the **radiance obstruction** of B .

$c_1 : H^1(B, \iota_* \text{Aff}_{\mathbb{Z}, B_0}) \rightarrow H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}, B_0}^{\vee})$, $\text{Im } c_1 = \{c_B \cup d = 0\}$.

B is a *special TASS* if $\forall \psi_i \circ \psi_j^{-1} \in \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{R}^2$.

$\cup : H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}, B_0})^{\otimes 2} \rightarrow H^2(B, \iota_*(\wedge_{i=1}^2 \mathcal{T}_{\mathbb{Z}, B_0})) \simeq \mathbb{Z}$.

Lemma (cf. Degtyarev'12)

If B is a compact special TASS, then $H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}, B_0})_{\text{free}}$ is an even lattice.

$B(\mathbb{Z})$: the set of integer points of an integral TASS B .

Theorem (T.)

- $q : T \rightarrow T / -1_T = \text{Km}(T)$
(1) $\exists q_* : H^1(T, \mathcal{T}_{\mathbb{Z}, T}) \xrightarrow{\sim} H^1(\text{Km}(T), \iota_* \mathcal{T}_{\mathbb{Z}, \text{Km}(T)_0})$,
 $q_*(D) \cup q_*(D) = 2D \cup D$, $(q_* \otimes_{\mathbb{Z}} \mathbb{R})(c_T) = 2c_{\text{Km}(T)}$,
 $q_*(\text{Im } c_1, T)^{\perp} = (\text{Im } c_1, \text{Km}(T))^{\perp}$.
In particular, $H^1(\text{Km}(T), \iota_* \mathcal{T}_{\mathbb{Z}, \text{Km}(T)_0}) \simeq U(2)^{\oplus 2}$.
(2) If $c_{\text{Km}(T)} \in H^1(\text{Km}(T), \iota_* \mathcal{T}_{\mathbb{Z}, \text{Km}(T)_0})$, then
 $|\text{Km}(T)(\mathbb{Z})| = \frac{c_{\text{Km}(T)} \cup c_{\text{Km}(T)}}{2} + 2 = \text{vol}(\text{Km}(T)) + \chi_{\text{top}}(\text{Km}(T))$.

Remark

- (i) (2) is true for every compact special TASS admitting an integer affine triangulation of B (cf. Kontsevich-Soibelman).
(ii) L : an ample line bundle on an complex K3 surface X .
 $\dim \Gamma(X, L) = \int_X \frac{c_1(L)^2}{2} + 2$.
(iii) X_{Σ} : a projective toric variety, D : an ample torus-invariant divisor on X_{Σ} , P_D : the Newton polytope of D .
 $|P_D(\mathbb{Z})| = \dim \Gamma(X_{\Sigma}, \mathcal{O}_{X_{\Sigma}}(D))$ (e.g. Pick's formula)

4. TASS with boundary and wallpaper groups

Theorem (Classical result for Abelian surfaces)

$T_{\mathbb{C}}$: an Abelian surface, G : a finite subgroup of $\text{Aut}(T_{\mathbb{C}})$.
The minimal model of $T_{\mathbb{C}}/G$ is either Abelian, bielliptic, K3, Enriques, ruled, or rational.

There exists its tropical analog as follows;

$T = \mathbb{R}^2/\Lambda$: a tropical 2-torus

G : a finite subgroup of the affine automorphisms group of T . T/G has the canonical structure of TASS with boundary.

Example (Known facts)

17 flat (parabolic) 2-orbifolds $\mathbb{R}^2/\mathcal{C} \iff 17$ wallpaper groups $\mathcal{C}$.
flat tori $\iff$ principally polarized tropical abelian varieties

flat 2-orbifolds	tropical —
T (2-torus)	Abelian surface
K (Klein bottle)	bielliptic surface
$S^2(2, 2, 2, 2) = T / -1_T = \text{Km}(T)$	Kummer surface
$S^2(3, 3, 3)$, $S^2(2, 4, 4)$, $S^2(2, 3, 6)$	Kummer type K3 surface
$\mathbb{R}P^2(2, 2)$	Enriques surface
$[0, 1] \times S^1$ (Cylinder)	$\mathbb{T}P^1 \times S^1$
$(S^1)^2 / \mathcal{G}_2$ (Möbius band)	$\mathbb{T}P^1$ -bundle
$D^2(a, b, c; p, q, r)$ (closed disk)	rational surface

Table: flat 2-orbifolds v.s. TASS with boundary