

Kernel functions and symbols of pseudodifferential operators of infinite order with an apparent parameter

By

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Abstract

We introduce a new representation of pseudodifferential operators of infinite order (or holomorphic microlocal operators) and symbol class.

Introduction

In this report, we shall announce our recent results about a complete symbol theory for the sheaf $\mathcal{E}_X^{\mathbb{R}}$ of pseudodifferential operators in the complex analytic category. The foundation of the symbol theory of $\mathcal{E}_X^{\mathbb{R}}$ at the present stage (see [1], [4]) is quite unsatisfactory. There are the following two issues:

- (i) Kamimoto and Kataoka have pointed out in their work [5] that the space of the kernel functions which comes from standard Čech representation of cohomology groups is not closed under composition of kernel functions defined by naive integration employed in [1], [4].
- (ii) the relation between the action of operators by integration of kernel functions and canonical action through cohomological definition was not clarified (see [7], [8]).

For (i), Kamimoto and Kataoka [5], [6] give a possible solution by introducing the notion of formal kernels. On the other hand, we shall start our study from other point of view; we use a new isomorphism of cohomology groups (see Proposition 1.2), we establish a

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new symbol theory, and solve (i) and (ii) above. Details in this article will be published in our forthcoming paper [2].

§ 1. Local Cohomology Groups on a Vector Space

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, of real numbers and of complex numbers respectively. Further set $\mathbb{N} := \{m \in \mathbb{Z}; m \geq 1\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{C}^\times := \{c \in \mathbb{C}; c \neq 0\}$. Let X be a finite dimensional $\mathbb{R}$ -vector space. Let a, b, r be real numbers such that $0 < b - a < \pi$ and $r > 0$. We define an open proper sector $S \subset \mathbb{C}$ by

$$S := \{\eta \in \mathbb{C}; a < \arg \eta < b, 0 < |\eta| < r\},$$

Let (x, η) be coordinates of $X \times \mathbb{C}$, and $\pi_\eta: X \times \mathbb{C} \ni (x, \eta) \mapsto x \in X$ the canonical projection. Let $G \subset X$ be a closed subset (not necessarily convex) and $U \subset X$ an open neighborhood of the origin. In this section we give another representation of local cohomology groups $H_{G \cap U}^k(U; \mathcal{F})$ for a sheaf $\mathcal{F}$ on X . For this purpose, we need some preparations. Let Z be a closed subset in X and $\varphi: X \times [0, 1] \rightarrow X$ a continuous deformation mapping which satisfies the following conditions:

- (i) $\varphi(x, 1) = x$ for any $x \in X$ and $\varphi(z, s) = z$ for any $z \in Z$ and $s \in [0, 1]$.
- (ii) $\varphi(\varphi(x, s), 0) = \varphi(x, 0)$ for any $s \in [0, 1]$ and $x \in X$.
- (iii) We set

$$\rho_\varphi(x, s) := |\varphi(x, s) - \varphi(x, 0)|.$$

Then $\rho_\varphi(x, s)$ is a strictly increasing function of s outside Z , i.e. if $s_1 < s_2$, we have $\rho_\varphi(x, s_1) < \rho_\varphi(x, s_2)$ for any $x \in X \setminus Z$.

We set, for short

$$\rho_\varphi(x) := \rho_\varphi(x, 1) = |\varphi(x, 1) - \varphi(x, 0)| = |x - \varphi(x, 0)|.$$

Here we remark

$$\rho_\varphi(\varphi(x, s)) = |\varphi(x, s) - \varphi(\varphi(x, s), 0)| = |\varphi(x, s) - \varphi(x, 0)| = \rho_\varphi(x, s).$$

Let $\varrho > 0$ be a positive constant. We define the subsets in $X \times \mathbb{C}$ by

$$\begin{aligned} \widehat{U} &:= \{(x, \eta) \in U \times S; \rho_\varphi(x) < \varrho|\eta|\}, \\ \widehat{G} &:= \{(\varphi(x, s), \eta) \in X \times \mathbb{C}; x \in G, \rho_\varphi(x) \leq \varrho|\eta|, 0 \leq s \leq 1\}. \end{aligned}$$

Note that $\widehat{G} \cap \widehat{U}$ is a closed subset in $\widehat{U}$.

Example 1.1. Let $X = \mathbb{C}^n \supset Z := \{z \in X; z_1 = 0\}$, and we set $\varphi(z, s) := (sz_1, z_2, \dots, z_n)$. Then $\rho_\varphi(z, s) = |sz_1|$ and $\rho_\varphi(z) = |z_1|$. Let $\kappa := (r, r', \varrho, \theta) \in \mathbb{R}^4$ be a 4-tuple of positive constants with

$$(1.1) \quad 0 < \theta < \frac{\pi}{2}, \quad 0 < \varrho < 1, \quad 0 < r < \varrho r'.$$

We set

$$(1.2) \quad \begin{aligned} S_\kappa &:= S_{r, \theta/4} = \{\eta \in \mathbb{C}; |\arg \eta| < \frac{\theta}{4}, 0 < |\eta| < r\}, \\ U_\kappa &:= \bigcap_{i=2}^n \{z \in X; |z_1| < \varrho r, |z_i| < r'\}, \\ G_\kappa &:= \bigcap_{i=2}^d \{z \in X; |\arg z_1| \leq \frac{\pi}{2} - \theta, \varrho^2 |z_i| \leq |z_1|\}. \end{aligned}$$

Then we have

$$(1.3) \quad \begin{aligned} \widehat{U}_\kappa &= \{(z, \eta) \in U_\kappa \times S_\kappa; |z_1| < \varrho |\eta|\}, \\ \widehat{G}_\kappa &= \{(\varphi(z, s), \eta) \in X \times \mathbb{C}; z \in G_\kappa, \rho_\varphi(z) \leq \varrho |\eta|, 0 \leq s \leq 1\} \\ &= \bigcap_{i=2}^n \{(sz_1, z_2, \dots, z_n, \eta) \in X \times \mathbb{C}; |\arg z_1| \leq \frac{\pi}{2} - \theta, \varrho^2 |z_i| \leq |z_1|, \\ &\quad |z_1| \leq \varrho |\eta|, 0 \leq s \leq 1\} \\ &= \bigcap_{i=2}^n \{(z, \eta) \in X \times \mathbb{C}; |z_1| \leq \varrho |\eta|, |\arg z_1| \leq \frac{\pi}{2} - \theta, \varrho |z_i| \leq |\eta|\}. \end{aligned}$$

We can have the following:

Proposition 1.2. *Let $\mathcal{F}$ be a complex of Abelian sheaves on X . Assume that U satisfies that $\sup_{x \in U} \rho_\varphi(x) < \varrho r$. Then there exists the following quasi-isomorphism:*

$$R\Gamma_{G \cap U}(U; \mathcal{F}) \simeq R\Gamma_{\widehat{G} \cap \widehat{U}}(\widehat{U}; \pi_\eta^{-1} \mathcal{F}).$$

§ 2. Holomorphic Microfunctions with an Apparent Parameter

Let X be an n -dimensional $\mathbb{C}$ -vector space with the coordinates $z = (z_1, \dots, z_n)$, and Y the closed complex submanifold of X defined by $\{z' = 0\}$ where $z = (z', z'')$ with $z' := (z_1, \dots, z_d)$ for some $1 \leq d \leq n$. In what follows, we denote an object defined on the space $X \times \mathbb{C}$ by a symbol with $\widehat{\cdot}$ like $\widehat{U}_\kappa$ etc. For any $z \in \mathbb{C}^n$, we set $\|z\| := \max_{1 \leq i \leq n} \{|z_i|\}$. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X , and $\mathcal{C}_{Y|X}^\mathbb{R}$ the sheaf of real holomorphic microfunctions along Y on the conormal bundle T_Y^*X to Y

(see [9], [11]). Let $z_0^* = (0; 1, 0, \dots, 0) \in T_Y^*X$. Let $\kappa = (r, r', \varrho, \theta) \in \mathbb{R}^4$ be as in (1.1), and recall the sets of (1.2) and (1.3) in Example 1.1. By the definition of $\mathcal{C}_{Y|X}^{\mathbb{R}}$, we have

$$\mathcal{C}_{Y|X, z_0^*}^{\mathbb{R}} = \varinjlim_{\kappa \rightarrow \mathbf{0}} H_{G_{\kappa} \cap U_{\kappa}}^d(U_{\kappa}; \mathcal{O}_X).$$

Here $\mathbf{0} := (0, 0, 0, 0)$. We can apply the result in §1 to this case. Assume that U_{κ} is sufficiently small so that the assumption of Proposition 1.2 is satisfied for $\varphi(z, s) = (sz_1, z_2, \dots, z_n)$. Set $\pi_{\eta}: X \times \mathbb{C} \ni (z, \eta) \mapsto z \in X$. Thus, from the exact sequence

$$0 \rightarrow \pi_{\eta}^{-1} \mathcal{O}_X \rightarrow \mathcal{O}_{X \times \mathbb{C}} \xrightarrow{\partial_{\eta}} \mathcal{O}_{X \times \mathbb{C}} \rightarrow 0,$$

we obtain the following distinguished triangle:

$$\begin{array}{c} R\Gamma_{\hat{G}_{\kappa} \cap \hat{U}_{\kappa}}(\hat{U}_{\kappa}; \pi_{\eta}^{-1} \mathcal{O}_X) \rightarrow R\Gamma_{\hat{G}_{\kappa} \cap \hat{U}_{\kappa}}(\hat{U}_{\kappa}; \mathcal{O}_{X \times \mathbb{C}}) \xrightarrow{\partial_{\eta}} R\Gamma_{\hat{G}_{\kappa} \cap \hat{U}_{\kappa}}(\hat{U}_{\kappa}; \mathcal{O}_{X \times \mathbb{C}}) \xrightarrow{+1} \\ \uparrow \wr \\ R\Gamma_{G_{\kappa} \cap U_{\kappa}}(U_{\kappa}; \mathcal{O}_X). \end{array}$$

Further we can prove:

Proposition 2.1. *If $k \neq d$, then*

$$H_{\hat{G}_{\kappa} \cap \hat{U}_{\kappa}}^k(\hat{U}_{\kappa}; \mathcal{O}_{X \times \mathbb{C}}) = H_{G_{\kappa} \cap U_{\kappa}}^k(U_{\kappa}; \mathcal{O}_X) = 0.$$

Definition 2.2. By Proposition 2.1, we define

$$\begin{aligned} \hat{C}_{Y|X}^{\mathbb{R}}(\kappa) &:= H_{\hat{G}_{\kappa} \cap \hat{U}_{\kappa}}^d(\hat{U}_{\kappa}; \mathcal{O}_{X \times \mathbb{C}}), \\ C_{Y|X}^{\mathbb{R}}(\kappa) &:= \text{Ker}(\partial_{\eta}: \hat{C}_{Y|X}^{\mathbb{R}}(\kappa) \rightarrow \hat{C}_{Y|X}^{\mathbb{R}}(\kappa)). \end{aligned}$$

Summing up, we obtain:

Theorem 2.3. *There exist isomorphisms*

$$\begin{array}{ccc} H_{G_{\kappa} \cap U_{\kappa}}^d(U_{\kappa}; \mathcal{O}_X) & \xrightarrow{\sim} & C_{Y|X}^{\mathbb{R}}(\kappa) \\ \downarrow & & \downarrow \\ \mathcal{C}_{Y|X, z_0^*}^{\mathbb{R}} & \xrightarrow{\sim} & \varinjlim_{\kappa \rightarrow \mathbf{0}} C_{Y|X}^{\mathbb{R}}(\kappa). \end{array}$$

We now consider a Čech representation of $C_{Y|X}^{\mathbb{R}}(\kappa)$. We set

$$\begin{aligned} V_{\kappa}^{(1)} &:= \{z \in U_{\kappa}; \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta\}, \\ V_{\kappa}^{(i)} &:= \{z \in U_{\kappa}; \varrho^2 |z_i| > |z_1|\} \quad (2 \leq i \leq d), \end{aligned}$$

$$\begin{aligned}\widehat{V}_{\kappa}^{(1)} &:= \{(z, \eta) \in \widehat{U}_{\kappa}; \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta\}, \\ \widehat{V}_{\kappa}^{(i)} &:= \{(z, \eta) \in \widehat{U}_{\kappa}; \varrho|z_i| > |\eta|\} \quad (2 \leq i \leq d).\end{aligned}$$

Let $\mathcal{P}_d$ be the set of all the subset of $\{1, \dots, d\}$ and $\mathcal{P}_d^{\vee} \subset \mathcal{P}_d$ consisting of $\alpha \in \mathcal{P}_d$ with $\#\alpha = d - 1$ ($\#\alpha$ denotes the number of elements in α). For $\alpha \in \mathcal{P}_d$, we define

$$\widehat{V}_{\kappa}^{(\alpha)} := \bigcap_{i \in \alpha} \widehat{V}_{\kappa}^{(i)}, \quad V_{\kappa}^{(\alpha)} := \bigcap_{i \in \alpha} V_{\kappa}^{(i)}, \quad \widehat{V}_{\kappa}^{(*)} := \widehat{V}_{\kappa}^{\{1, \dots, d\}} = \bigcap_{i=1}^d \widehat{V}_{\kappa}^{(i)}.$$

As each $\widehat{V}_{\kappa}^{(\alpha)}$ (resp. $V_{\kappa}^{(\alpha)}$) is pseudoconvex, we have

$$H_{G_{\kappa} \cap U_{\kappa}}^d(U_{\kappa}; \mathcal{O}_X) = \frac{\Gamma(V_{\kappa}^{(*)}; \mathcal{O}_X)}{\sum_{\alpha \in \mathcal{P}_d^{\vee}} \Gamma(V_{\kappa}^{(\alpha)}; \mathcal{O}_X)} \simeq \{u \in \frac{\Gamma(\widehat{V}_{\kappa}^{(*)}; \mathcal{O}_{X \times \mathbb{C}})}{\sum_{\alpha \in \mathcal{P}_d^{\vee}} \Gamma(\widehat{V}_{\kappa}^{(\alpha)}; \mathcal{O}_{X \times \mathbb{C}})}; \partial_{\eta} u = 0\} = C_{Y|X}^{\mathbb{R}}(\kappa).$$

§ 3. Cohomological Representation of $\mathcal{E}_X^{\mathbb{R}}$ with an Apparent Parameter

We inherit the same notation from the previous section. Let (z, w) be coordinates of $X \times X$. Let $\Delta \subset X \times X$ be the diagonal set. We identify X with Δ , and

$$T^*X = \{(z; \zeta)\} \simeq \{(z, z; \zeta, -\zeta)\} = T_{\Delta}^*(X \times X).$$

Let $\mathcal{E}_X^{\mathbb{R}}$ denote the sheaf of pseudodifferential operators on the cotangent bundle T^*X of X . Let Ω_X be the sheaf of holomorphic n -forms on X , and set $p_2: T^*(X \times X) \ni (z, w; \zeta, \tilde{\zeta}) \mapsto w \in X$. Then by the definition, $\mathcal{E}_X^{\mathbb{R}} = \mathcal{E}_{X|X \times X}^{\mathbb{R}} \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$. We fix $z_0^* = (z_0; \zeta_0) := (0; 1, 0, \dots, 0) \in T^*X$. Let $\kappa = (r, r', \varrho, \theta) \in \mathbb{R}^4$ be as in (1.1). We set

$$\begin{aligned}\widehat{U}_{\Delta, \kappa} &:= \bigcap_{i=2}^n \{(z, w, \eta) \in X \times X \times \mathbb{C}; \|z\| < r', \eta \in S_{\kappa}, |z_1 - w_1| < \varrho|\eta|, |z_i - w_i| < r'\}, \\ \widehat{V}_{\Delta, \kappa}^{(1)} &:= \{(z, w, \eta) \in \widehat{U}_{\Delta, \kappa}; \frac{\pi}{2} - \theta < \arg(z_1 - w_1) < \frac{3\pi}{2} + \theta\}, \\ \widehat{V}_{\Delta, \kappa}^{(i)} &:= \{(z, w, \eta) \in \widehat{U}_{\Delta, \kappa}; \varrho|z_i - w_i| > |\eta|\} \quad (2 \leq i \leq n), \\ U_{\Delta, \kappa} &:= \bigcap_{i=2}^n \{(z, w) \in X \times X; \|z\| < r', |z_1 - w_1| < \varrho r, |z_i - w_i| < r'\}, \\ G_{\Delta, \kappa} &:= \bigcap_{i=2}^n \{(z, w) \in X \times X; |\arg(z_1 - w_1)| \leq \frac{\pi}{2} - \theta, \varrho^2|z_i - w_i| \leq |z_1 - w_1|\}, \\ V_{\Delta, \kappa}^{(1)} &:= \{(z, w) \in U_{\Delta, \kappa}; \frac{\pi}{2} - \theta < \arg(z_1 - w_1) < \frac{3\pi}{2} + \theta\}, \\ V_{\Delta, \kappa}^{(i)} &:= \{(z, w) \in U_{\Delta, \kappa}; \varrho^2|z_i - w_i| > |z_1 - w_1|\} \quad (2 \leq i \leq n).\end{aligned}$$

We also set

$$E_X^{\mathbb{R}}(\kappa) := C_{X|X \times X}^{\mathbb{R}}(\kappa) \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X.$$

Further set for short:

$$\begin{aligned} N_X^{\mathbb{R}}(\kappa) &:= \sum_{\alpha \in \mathcal{P}_n^\vee} \Gamma(V_{\Delta, \kappa}^{(\alpha)}; \mathcal{O}_{X \times X}^{(0, n)}) \subset K_X^{\mathbb{R}}(\kappa) := \Gamma(V_{\Delta, \kappa}^{(*)}; \mathcal{O}_{X \times X}^{(0, n)}), \\ \hat{N}_X^{\mathbb{R}}(\kappa) &:= \sum_{\alpha \in \mathcal{P}_n^\vee} \Gamma(\hat{V}_{\Delta, \kappa}^{(\alpha)}; \mathcal{O}_{X \times X \times \mathbb{C}}^{(0, n, 0)}) \subset \hat{K}_X^{\mathbb{R}}(\kappa) := \{\psi \in \Gamma(\hat{V}_{\Delta, \kappa}^{(*)}; \mathcal{O}_{X \times X \times \mathbb{C}}^{(0, n, 0)}); \partial_\eta \psi \in \hat{N}_X^{\mathbb{R}}(\kappa)\}. \end{aligned}$$

Here $\mathcal{O}_{X \times X \times \mathbb{C}}^{(0, n, 0)} := \mathcal{O}_{X \times X \times \mathbb{C}} \otimes_{q_2^{-1}\mathcal{O}_X} q_2^{-1}\Omega_X$ with $q_2: X \times X \times \mathbb{C} \ni (z, w, \eta) \mapsto w \in X$, and $\mathcal{O}_{X \times X}^{(0, n)}$ is defined in a similar way. Then by Theorem 2.3, we obtain

$$\begin{array}{ccc} H_{G_{\Delta, \kappa} \cap U_{\Delta, \kappa}}^n(U_{\Delta, \kappa}; \mathcal{O}_{X \times X}^{(0, n)}) & \xrightarrow{\sim} & E_X^{\mathbb{R}}(\kappa) \\ \parallel & & \parallel \\ K_X^{\mathbb{R}}(\kappa)/N_X^{\mathbb{R}}(\kappa) & \xrightarrow{\sim} & \hat{K}_X^{\mathbb{R}}(\kappa)/\hat{N}_X^{\mathbb{R}}(\kappa) \\ \downarrow & & \downarrow \\ \mathcal{E}_{X, z_0}^{\mathbb{R}} & \xrightarrow{\sim} & \varinjlim_{\kappa \rightarrow 0} E_X^{\mathbb{R}}(\kappa). \end{array}$$

Let $(z, \eta) \in X \times \mathbb{C}$. Set $\beta_0 := \frac{\varrho}{2} e^{-\sqrt{-1}(\pi+\theta)/2}$ and $\beta_1 := \frac{\varrho}{2} e^{\sqrt{-1}(\pi+\theta)/2}$, and we define, for a sufficiently small $\varepsilon > 0$, the path $\gamma_i(z, \eta; \varrho, \theta)$ in $\mathbb{C}_{w_i}$ by

$$\begin{aligned} \gamma_1(z, \eta; \varrho, \theta) &:= \{w_1 = z_1 + t\beta_0\eta; 1 \geq t \geq \varepsilon\} \vee \{w_1 = z_1 + \varepsilon\beta_0\eta e^{\sqrt{-1}(\pi+\theta)t}; 0 \leq t \leq 1\} \\ &\quad \vee \{w_1 = z_1 + t\beta_1\eta; \varepsilon \leq t \leq 1\}, \\ \gamma_i(z, \eta; \varrho) &:= \{w_i = z_i + \left(\frac{|\eta|}{\varrho} + \varepsilon\right) e^{2\pi\sqrt{-1}t}; 0 \leq t \leq 1\} \quad (2 \leq i \leq n). \end{aligned}$$

Note that $\gamma_1(z, \eta; \varrho, \theta)$ joins the two points $z_1 + \beta_0\eta$ and $z_1 + \beta_1\eta$, which depend on the variables z_1 and η holomorphically. Define the real n -dimensional chain in X made from these paths by

$$\gamma(z, \eta; \varrho, \theta) := \gamma_1(z, \eta; \varrho, \theta) \times \gamma_2(z, \eta; \varrho) \times \cdots \times \gamma_n(z, \eta; \varrho) \subset X.$$

Theorem 3.1. *The bi-linear morphism*

$$\begin{array}{ccc} \mu: E_X^{\mathbb{R}}(\kappa) \otimes_{\mathbb{C}} C_{Y|X}^{\mathbb{R}}(\kappa) & \longrightarrow & C_{Y|X}^{\mathbb{R}}(\tilde{\kappa}) \\ \Psi & & \Psi \\ [\psi(z, w, \eta) dw] \otimes [u(z, \eta)] & \longmapsto & \left[\int_{\gamma(z, \eta; \varrho, \theta)} \psi(z, w, \eta) u(w, \eta) dw \right] \end{array}$$

is well defined. Here $\tilde{\kappa} = (\tilde{r}, \tilde{r}', \tilde{\varrho}, \tilde{\theta})$ is a 4-tuple of positive constants satisfying

$$0 < \tilde{r} < r, \quad 0 < \tilde{r}' < \frac{r'}{2}, \quad 0 < \tilde{\theta} < \frac{\theta}{4}, \quad 0 < \tilde{\varrho} < \frac{\varrho}{2} \sin \frac{\theta}{4}.$$

Moreover,

$$\mathcal{E}_{X, z_0^*}^{\mathbb{R}} \otimes_{\mathbb{C}} \mathcal{C}_{Y|X, z_0^*}^{\mathbb{R}} = \varinjlim_{\kappa \rightarrow \mathbf{0}} (E_X^{\mathbb{R}}(\kappa) \otimes_{\mathbb{C}} C_{Y|X}^{\mathbb{R}}(\kappa)) \xrightarrow{\mu} \varinjlim_{\kappa \rightarrow \mathbf{0}} C_{Y|X}^{\mathbb{R}}(\kappa) = \mathcal{C}_{Y|X, z_0^*}^{\mathbb{R}}$$

coincides with the cohomological action of $\mathcal{E}_{X, z_0^*}^{\mathbb{R}}$ on $\mathcal{C}_{Y|X, z_0^*}^{\mathbb{R}}$.

As a corollary of the theorem, we have the result on the composition on $E_X^{\mathbb{R}}(\kappa)$:

Corollary 3.2. *Let $\tilde{\kappa} = (\tilde{r}, \tilde{r}', \tilde{\varrho}, \tilde{\theta}) \in \mathbb{R}^4$ satisfying*

$$0 < \tilde{r} < r, \quad 0 < \tilde{r}' < \frac{r'}{8}, \quad 0 < \tilde{\theta} < \frac{\theta}{4}, \quad 0 < \tilde{\varrho} < \frac{\varrho}{2} \sin \frac{\theta}{4},$$

and the corresponding conditions to (1.1). Then there exists the bi-linear morphism

$$\begin{aligned} \mu: E_X^{\mathbb{R}}(\kappa) \otimes_{\mathbb{C}} E_X^{\mathbb{R}}(\kappa) &\longrightarrow E_X^{\mathbb{R}}(\tilde{\kappa}) \\ \downarrow &\quad \quad \quad \downarrow \\ [\psi_1(z, w, \eta) dw] \otimes [\psi_2(z, w, \eta) dw] &\mapsto \left[\left(\int_{\gamma(z, \eta; \varrho, \theta)} \psi_1(z, \tilde{w}, \eta) \psi_2(\tilde{w}, w, \eta) d\tilde{w} \right) dw \right]. \end{aligned}$$

Moreover the multiplication of the ring $\mathcal{E}_{X, z_0^*}^{\mathbb{R}}$ coincides with

$$\mathcal{E}_{X, z_0^*}^{\mathbb{R}} \otimes_{\mathbb{C}} \mathcal{E}_{X, z_0^*}^{\mathbb{R}} = \varinjlim_{\kappa \rightarrow \mathbf{0}} (E_X^{\mathbb{R}}(\kappa) \otimes_{\mathbb{C}} E_X^{\mathbb{R}}(\kappa)) \xrightarrow{\mu} \varinjlim_{\kappa \rightarrow \mathbf{0}} E_X^{\mathbb{R}}(\kappa) = \mathcal{E}_{X, z_0^*}^{\mathbb{R}}.$$

Remark 3.3. For any $\psi_1(z, w, \eta), \psi_2(z, w, \eta) \in \varinjlim_{\kappa \rightarrow \mathbf{0}} \hat{K}_X^{\mathbb{R}}(\kappa)$, we can show

$$\int_{\gamma(z, \eta; \varrho, \theta)} \psi_1(z, \tilde{w}, \eta) \psi_2(\tilde{w}, w, \eta) d\tilde{w} \in \varinjlim_{\kappa \rightarrow \mathbf{0}} \hat{K}_X^{\mathbb{R}}(\kappa).$$

Further if either ψ_1 or ψ_2 belongs to $\varinjlim_{\kappa \rightarrow \mathbf{0}} \hat{N}_X^{\mathbb{R}}(\kappa)$, then

$$\int_{\gamma(z, \eta; \varrho, \theta)} \psi_1(z, \tilde{w}, \eta) \psi_2(\tilde{w}, w, \eta) d\tilde{w} \in \varinjlim_{\kappa \rightarrow \mathbf{0}} \hat{N}_X^{\mathbb{R}}(\kappa).$$

However in general, for $\psi_1(z, w), \psi_2(z, w) \in \varinjlim_{\kappa \rightarrow \mathbf{0}} K_X^{\mathbb{R}}(\kappa)$,

$$\int_{\gamma(z, \eta_0; \varrho, \theta)} \psi_1(z, \tilde{w}) \psi_2(\tilde{w}, w) d\tilde{w} \notin \varinjlim_{\kappa \rightarrow \mathbf{0}} K_X^{\mathbb{R}}(\kappa),$$

for any fixed η_0 (see Kamimoto-Kataoka [5], [6]). In order to solve this problem, Kamimoto-Kataoka [5], [6] introduce a new class of kernel functions. On the other hand, our starting point is a new representation of cohomology as in § 1.

Our parameter η controls the convergent radius with respect to z_1 variable. Note that

$$\psi'(z, w, \eta) := \int_{\gamma(z, \eta; \varrho, \theta)} \psi_1(z, \tilde{w}) \psi_2(\tilde{w}, w) d\tilde{w} \in \varinjlim_{\kappa \rightarrow \mathbf{0}} \widehat{K}_X^{\mathbb{R}}(\kappa).$$

§ 4. Symbols and Formal Symbols

For an explicit calculus of operators in $\mathcal{E}_X^{\mathbb{R}}$, it is very useful to introduce the notions of symbols and formal symbols. In this section, we recall definitions of symbols and formal symbols of $\mathcal{E}_X^{\mathbb{R}}$ (see [1], [3], [4]).

Let $X := \mathbb{C}^n$ and consider $T^*X \simeq X \times \mathbb{C}^n = \{(z; \zeta)\}$. Let $\pi: T^*X \rightarrow X$ be a canonical projection. If $V \subset T^*X$ is a conic set and $d > 0$, we set

$$V[d] := \{(z, \zeta) \in V; \|\zeta\| \geq d\}.$$

For any open conic subset $\Omega \subset T^*X$ and $\rho \geq 0$, we set

$$\Omega_\rho := \text{Cl} \left[\bigcup_{(z, \zeta) \in \Omega} \{(z + z'; \zeta + \zeta') \in \mathbb{C}^{2n}; \|z'\| \leq \rho, \|\zeta'\| \leq \rho \|\zeta\|\} \right].$$

Here Cl means the closure. In particular, $\Omega_0 = \text{Cl } \Omega$. For any $d > 0$ and $\rho \in [0, 1[$, we set $d_\rho := d(1 - \rho)$ for short. Let U, V be conic subsets of T^*X . Then we write $V \underset{\text{conic}}{\subseteq} U$ if V is generated by a compact subset of U .

Definition 4.1. Let $\Omega \underset{\text{conic}}{\subseteq} T^*X$ be an open conic subset.

(1) We call $P(z, \zeta)$ a *symbol* on Ω if there exist $d > 0$ and $\rho \in]0, 1[$ such that $P(z, \zeta) \in \Gamma(\Omega_\rho[d_\rho]; \mathcal{O}_{T^*X})$, and for any $h > 0$ there exists $C_h > 0$ such that

$$|P(z, \zeta)| \leq C_h e^{h\|\zeta\|} \quad ((z; \zeta) \in \Omega_\rho[d_\rho]).$$

We denote by $\mathcal{S}(\Omega)$ the set of symbols on Ω .

(2) We call $P(z, \zeta)$ a *null-symbol* on Ω if there exist $d > 0$ and $\rho \in]0, 1[$ such that $P(z, \zeta) \in \Gamma(\Omega_\rho[d_\rho]; \mathcal{O}_{T^*X})$, and there exist $C, \delta > 0$ such that

$$|P(z, \zeta)| \leq C e^{-\delta\|\zeta\|} \quad ((z; \zeta) \in \Omega_\rho[d_\rho]).$$

We denote by $\mathcal{N}(\Omega)$ the set of null-symbols on Ω .

(3) For any $z_0^* \in T^*X$, we set

$$\mathcal{S}_{z_0^*} := \varinjlim_{\Omega \ni z_0^*} \mathcal{S}(\Omega) \supset \mathcal{N}_{z_0^*} := \varinjlim_{\Omega \ni z_0^*} \mathcal{N}(\Omega)$$

where $\Omega \underset{\text{conic}}{\subseteq} T^*X$ ranges through open conic neighborhoods of z_0^* .

Definition 4.2. Let t be an indeterminate.

(1) $P(t; z, \zeta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z, \zeta) \in \widehat{\mathcal{S}}_{\text{cl}}(\Omega)$ if $P(t; z, \zeta) \in \Gamma(\Omega_{\rho}[d_{\rho}]; \mathcal{O}_{T^*X})[[t]]$ for some $d > 0$ and $\rho \in]0, 1[$, and there exists a constant $A > 0$ satisfying the following: for any $h > 0$ there exists a constant $C_h > 0$ such that

$$|P_{\nu}(z, \zeta)| \leq \frac{C_h A^{\nu} \nu! e^{h\|\zeta\|}}{\|\zeta\|^{\nu}} \quad (\nu \in \mathbb{N}_0, (z; \zeta) \in \Omega_{\rho}[d_{\rho}]).$$

(2) $P(t; z, \zeta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z, \zeta, \eta) \in \widehat{\mathcal{S}}_{\text{cl}}(\Omega)$ is an element of $\widehat{\mathcal{N}}_{\text{cl}}(\Omega)$ if there exists a constant $A > 0$ satisfying the following: for any $h > 0$ there exists a constant $C_h > 0$ such that

$$\left| \sum_{\nu=0}^{m-1} P_{\nu}(z, \zeta) \right| \leq \frac{C_h A^m m! e^{h\|\zeta\|}}{\|\zeta\|^m} \quad (m \in \mathbb{N}, (z; \zeta) \in \Omega_{\rho}[d_{\rho}]).$$

(3) We set

$$\widehat{\mathcal{S}}_{\text{cl}, z_0^*} := \varinjlim_{\Omega} \widehat{\mathcal{S}}_{\text{cl}}(\Omega) \supset \widehat{\mathcal{N}}_{\text{cl}, z_0^*} := \varinjlim_{\Omega} \widehat{\mathcal{N}}_{\text{cl}}(\Omega).$$

We call each element of $\widehat{\mathcal{S}}_{\text{cl}}(\Omega)$ (resp. $\widehat{\mathcal{N}}_{\text{cl}}(\Omega)$) a *classical formal symbol* (resp. *classical formal null-symbol*) on Ω .

Definition 4.3. Let t be an indeterminate.

(1) $P(t; z, \zeta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z, \zeta) \in \widehat{\mathcal{S}}(\Omega)$ if $P_{\nu}(z, \zeta) \in \Gamma(\Omega_{\rho}[(\nu+1)d_{\rho}]; \mathcal{O}_{T^*X})$ for some $d > 0$ and $\rho \in]0, 1[$, and there exists a constant $A \in]0, 1[$ satisfying the following: for any $h > 0$ there exists a constant $C_h > 0$ such that

$$|P_{\nu}(z, \zeta)| \leq C_h A^{\nu} e^{h\|\zeta\|} \quad (\nu \in \mathbb{N}_0, (z; \zeta) \in \Omega_{\rho}[(\nu+1)d_{\rho}]).$$

(2) Let $P(t; z, \zeta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z, \zeta) \in \widehat{\mathcal{S}}(\Omega)$. Then $P(t; z, \zeta)$ is an element of $\widehat{\mathcal{N}}(\Omega)$ if there exists a constant $A \in]0, 1[$ satisfying the following: for any $h > 0$ there exists a constant $C_h > 0$ such that

$$\left| \sum_{\nu=0}^{m-1} P_{\nu}(z, \zeta) \right| \leq C_h A^m e^{h\|\zeta\|} \quad (m \in \mathbb{N}, (z; \zeta) \in \Omega_{\rho}[md_{\rho}]).$$

(3) For $z_0^* \in T^*X$, we set

$$\widehat{\mathcal{S}}_{z_0^*} := \varinjlim_{\Omega} \widehat{\mathcal{S}}(\Omega) \supset \widehat{\mathcal{N}}_{z_0^*} := \varinjlim_{\Omega} \widehat{\mathcal{N}}(\Omega).$$

We call each element of $\widehat{\mathcal{S}}(\Omega)$ (resp. $\widehat{\mathcal{N}}(\Omega)$) a *formal symbol* (resp. *formal null-symbol*) on Ω . Then we can prove:

Theorem 4.4 (see [4]). (1) $\mathcal{S}(\Omega) \subset \widehat{\mathcal{S}}_{\text{cl}}(\Omega) \subset \widehat{\mathcal{S}}(\Omega)$ and $\mathcal{N}(\Omega) \subset \widehat{\mathcal{N}}_{\text{cl}}(\Omega) \subset \widehat{\mathcal{N}}(\Omega)$.

(2) For any $z_0^* \in \dot{T}^*X$, the inclusions $\mathcal{S}_{z_0^*} \subset \widehat{\mathcal{S}}_{\text{cl}, z_0^*} \subset \widehat{\mathcal{S}}_{z_0^*}$ and $\mathcal{N}_{z_0^*} \subset \widehat{\mathcal{N}}_{\text{cl}, z_0^*} \subset \widehat{\mathcal{N}}_{z_0^*}$ induce

$$\mathcal{E}_{X, z_0^*}^{\mathbb{R}} = \varinjlim_{\kappa} (K_X^{\mathbb{R}}(\kappa) / N_X^{\mathbb{R}}(\kappa)) \simeq \mathcal{S}_{z_0^*} / \mathcal{N}_{z_0^*} \simeq \widehat{\mathcal{S}}_{\text{cl}, z_0^*} / \widehat{\mathcal{N}}_{\text{cl}, z_0^*} \simeq \widehat{\mathcal{S}}_{z_0^*} / \widehat{\mathcal{N}}_{z_0^*}.$$

Take any $K(z, w) dw = [\psi(z, w) dw] \in \varinjlim_{\kappa \rightarrow 0} (K_X^{\mathbb{R}}(\kappa) / N_X^{\mathbb{R}}(\kappa))$. Then we set

$$\sigma(\psi)(z, \zeta) := \int_{\gamma(0, \eta_0; \varrho, \theta)} \psi(z, z + w) e^{\langle w, \zeta \rangle} dw.$$

Here $0 < \eta_0 \ll 1$ is fixed. Then σ induces an isomorphism (see [1], [4]):

$$\mathcal{E}_{X, z_0^*}^{\mathbb{R}} = \varinjlim_{\kappa \rightarrow 0} (K_X^{\mathbb{R}}(\kappa) / N_X^{\mathbb{R}}(\kappa)) \simeq \mathcal{S}_{z_0^*} / \mathcal{N}_{z_0^*}.$$

§ 5. Symbols with an Apparent Parameter

For our new cohomological definition of $\mathcal{E}_X^{\mathbb{R}}$ in § 3, we shall develop the symbol theory as in § 4. First, we give the definition of symbols with an apparent parameter. We inherit the notation from the preceding section. Set for short

$$(5.1) \quad S := S_{\kappa}$$

for some $r, \theta \in]0, \frac{1}{2}[$ (recall (1.2)). In particular we always assume that $|\eta| < \frac{1}{2}$ for any $\eta \in S$. For $Z \Subset S$, we set $m_Z := \min_{\eta \in Z} |\eta| > 0$.

Definition 5.1. We define a set $\mathfrak{N}(\Omega; S)$ as follows: $P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$ if

- (i) $P(z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \mathcal{O}_{T^*X \times \mathbb{C}})$ for some $d > 0$ and $\rho \in]0, 1[$,
- (ii) there exists $\delta > 0$ so that for any $Z \Subset S$, there exists a constant $C_Z > 0$ satisfying

$$|P(z, \zeta, \eta)| \leq C_Z e^{-\delta \|\eta \zeta\|} \quad ((z, \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z).$$

Lemma 5.2. If $P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$, it follows that $\partial_{\eta} P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$.

Proposition 5.3. Assume that $P(z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \mathcal{O}_{T^*X \times \mathbb{C}})$ satisfies that $\partial_{\eta} P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$.

- (1) The following conditions are equivalent:

- (i) *there exists a constant $v > 0$ satisfying the following: for any $Z \Subset S$ there exists a constant $C_Z > 0$ such that*

$$|P(z, \zeta, \eta)| \leq C_Z e^{v\|\eta\zeta\|} \quad ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z).$$

- (ii) *for any $h > 0$ and $Z \Subset S$ there exists constant $C_{h,Z} > 0$ such that*

$$|P(z, \zeta, \eta)| \leq C_{h,Z} e^{h\|\zeta\|} \quad ((z; \zeta, \eta) \in \Omega_\rho[d_\rho] \times Z).$$

(2) *Further assume that $P(z, \zeta, \eta)$ satisfies the equivalent conditions of (1) (resp. $P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$). Then for any $\eta_0 \in S$, it follows that $P(z, \zeta, \eta_0) \in \mathcal{S}(\Omega)$ (resp. $P(z, \zeta, \eta_0) \in \mathcal{N}(\Omega)$) and further $P(z, \zeta, \eta) - P(z, \zeta, \eta_0) \in \mathfrak{N}(\Omega; S)$.*

Definition 5.4. (1) We define a set $\mathfrak{S}(\Omega; S)$ as follows: $P(z, \zeta, \eta) \in \mathfrak{S}(\Omega; S)$ if

- (i) $P(z, \zeta, \eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathcal{O}_{T^*X \times \mathbb{C}})$ for some $d > 0$ and $\rho \in]0, 1[$,
- (ii) $\partial_\eta P(z, \zeta, \eta) \in \mathfrak{N}(\Omega; S)$,
- (iii) $P(z, \zeta, \eta)$ satisfies the equivalent conditions of Proposition 5.3 (1).

Note that $\mathfrak{N}(\Omega; S) \subset \mathfrak{S}(\Omega; S)$ holds by Lemma 5.2.

(2) For $z_0^* \in \dot{T}^*X$, we set

$$\mathfrak{S}_{z_0^*} := \varinjlim_{\Omega, S} \mathfrak{S}(\Omega; S) \supset \mathfrak{N}_{z_0^*} := \varinjlim_{\Omega, S} \mathfrak{N}(\Omega; S).$$

Here $\Omega \Subset T^*_{\text{conic}} X$ ranges through open conic neighborhoods of z_0^* , and the inductive limits with respect to S are taken by $r, \theta \rightarrow 0$ in (5.1).

We call each element of $\mathfrak{S}(\Omega; S)$ (resp. $\mathfrak{N}(\Omega; S)$) a *symbol* (resp. *null-symbol*) on Ω with an apparent parameter in S . It is easy to see that $\mathfrak{S}(\Omega; S)$ is a $\mathbb{C}$ -algebra under the ordinary operations of functions, and $\mathfrak{N}(\Omega; S)$ is a subalgebra. By definition, we can regard that

$$\begin{aligned} \mathcal{S}(\Omega) &= \{P(z, \zeta, \eta) \in \mathfrak{S}(\Omega; S); \partial_\eta P(z, \zeta, \eta) = 0\} \subset \mathfrak{S}(\Omega; S), \\ \mathcal{N}(\Omega) &= \mathcal{S}(\Omega) \cap \mathfrak{N}(\Omega; S) \subset \mathfrak{N}(\Omega; S). \end{aligned}$$

Hence we have an injective mapping $\mathcal{S}(\Omega)/\mathcal{N}(\Omega) \hookrightarrow \mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S)$. Moreover

Proposition 5.5. *There exists the following isomorphism:*

$$\mathcal{S}(\Omega)/\mathcal{N}(\Omega) \simeq \mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S).$$

Take any $K(z, w, \eta) dw = [\psi(z, w, \eta) dw] \in \varinjlim_{\kappa \rightarrow 0} \widehat{E}_X^{\mathbb{R}}(\kappa)$. We set

$$\sigma(\psi)(z, \zeta, \eta) := \int_{\gamma(0, \eta; \varrho, \theta)} \psi(z, z + w, \eta) e^{\langle w, \zeta \rangle} dw.$$

Then we can prove the following theorem:

Theorem 5.6. *The mapping σ induces an isomorphism*

$$\mathcal{E}_{X, z_0^*}^{\mathbb{R}} = \varinjlim_{\kappa \rightarrow 0} E_X^{\mathbb{R}}(\kappa) \simeq \mathfrak{S}_{z_0^*} / \mathfrak{N}_{z_0^*}.$$

The inverse ϖ of σ is defined as follows: Let $z_0^* = (0; 1, 0, \dots, 0) \in \dot{T}^*X$, and $P(z, \zeta, \eta) \in \mathfrak{S}_{z_0^*}$. By Proposition 5.3, for any sufficiently small $\eta_0 > 0$ we have $P(z, \zeta, \eta) - P(z, \zeta, \eta_0) \in \mathfrak{N}_{z_0^*}$. We develop $P(z, \zeta, \eta_0)$ into the Taylor series with respect to $\zeta'/\zeta_1 = (\zeta_2/\zeta_1, \dots, \zeta_n/\zeta_1)$:

$$P(z, \zeta, \eta_0) = \sum_{\alpha \in \mathbb{N}_0^{n-1}} P_\alpha(z, \zeta_1, \eta_0) \left(\frac{\zeta'}{\zeta_1} \right)^\alpha.$$

Definition 5.7. Under the preceding notation, we set

$$\varpi_\alpha(P)(z, w_1, \eta) := \int_d^\infty P_\alpha(z, \zeta_1, \eta_0) \Gamma_{|\alpha|}(\zeta_1, \eta) e^{-w_1 \zeta_1} d\zeta_1,$$

where we set (see [10]):

$$\Gamma_\nu(\tau, \eta) := \begin{cases} 1 & (\nu = 0), \\ \frac{1}{(\nu - 1)!} \int_0^\eta e^{-s\tau} s^{\nu-1} ds & (\nu \in \mathbb{N}). \end{cases}$$

Further we define

$$\varpi(P)(z, w, \eta) := \sum_{\alpha \in \mathbb{N}_0^{n-1}} \frac{\alpha! \varpi_\alpha(P)(z, w_1 - z_1, \eta)}{(2\pi \sqrt{-1})^n (w' - z')^{\alpha + \mathbf{1}_{n-1}}}.$$

Here we set $w' := (w_2, \dots, w_n)$ and $\mathbf{1}_{n-1} := (1, \dots, 1)$.

We can show that ϖ induces a mapping $\mathfrak{S}_{z_0^*} / \mathfrak{N}_{z_0^*} \rightarrow \mathcal{E}_{X, z_0^*}^{\mathbb{R}}$ and

$$\varpi \circ \sigma = \mathbb{1}: \mathfrak{S}_{z_0^*} / \mathfrak{N}_{z_0^*} \rightarrow \mathfrak{S}_{z_0^*} / \mathfrak{N}_{z_0^*}, \quad \sigma \circ \varpi = \mathbb{1}: \mathcal{E}_{X, z_0^*}^{\mathbb{R}} \rightarrow \mathcal{E}_{X, z_0^*}^{\mathbb{R}}.$$

§ 6. Classical Formal Symbols with an Apparent Parameter

In order to describe several operations of $\mathcal{E}_X^{\mathbb{R}}$ (coordinate transformations, products and formal adjoints) in terms of symbols, we need a notion of classical formal symbols as in non-parameter case: In this section, we give the definition of classical formal symbols with an apparent parameter.

Definition 6.1. Let t be an indeterminate. Then we define the set $\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$ as follows: $P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$ if

- (i) $P(t; z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \mathcal{O}_{T^*X \times \mathbb{C}})[[t]]$ for some $d > 0$ and $\rho \in]0, 1[$,
- (ii) there exists a constant $A > 0$, and for any $Z \Subset S$, $h > 0$, there exists $C_{h,Z} > 0$ such that

$$\left| \sum_{\nu=0}^{m-1} P_{\nu}(z, \zeta, \eta) \right| \leq \frac{C_{h,Z} A^m m! e^{h\|\zeta\|}}{\|\eta\zeta\|^m} \quad (m \in \mathbb{N}, (z; \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z).$$

Definition 6.2. We say that $P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$ if

- (i) $P(t; z, \zeta, \eta) \in \Gamma(\Omega_{\rho}[d_{\rho}] \times S; \mathcal{O}_{T^*X \times \mathbb{C}})[[t]]$ for some $d > 0$ and $\rho \in]0, 1[$,
- (ii) there exists a constant $A > 0$, and for any $Z \Subset S$, $h > 0$ there exists $C_{h,Z} > 0$ such that

$$|P_{\nu}(z, \zeta, \eta)| \leq \frac{C_{h,Z} A^{\nu} \nu! e^{h\|\zeta\|}}{\|\eta\zeta\|^{\nu}} \quad (\nu \in \mathbb{N}_0, (z; \zeta, \eta) \in \Omega_{\rho}[d_{\rho}] \times Z).$$

- (iii) $\partial_{\eta} P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$.

We call each element of $\widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$ (resp. $\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$) a *classical formal symbol* (resp. *classical formal null-symbol*) on Ω with an apparent parameter in S . We remark that $\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S) \subset \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$. For any $z_0^* \in \dot{T}^*X$, we set

$$\widehat{\mathfrak{S}}_{\text{cl}, z_0^*} := \varinjlim_{\Omega, S} \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S) \supset \widehat{\mathfrak{N}}_{\text{cl}, z_0^*} := \varinjlim_{\Omega, S} \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S).$$

Proposition 6.3. Let $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$. Then for any $\eta_0 \in S$, it follows that $P(t; z, \zeta, \eta_0) \in \widehat{\mathcal{S}}_{\text{cl}}(\Omega)$ and $P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0) \in \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$.

We can regard that

$$\begin{aligned} \widehat{\mathcal{S}}_{\text{cl}}(\Omega) &= \{P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S); \partial_{\eta} P(t; z, \zeta, \eta) = 0\} \subset \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S), \\ \widehat{\mathcal{N}}_{\text{cl}}(\Omega) &= \widehat{\mathcal{S}}_{\text{cl}}(\Omega) \cap \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S) \subset \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S). \end{aligned}$$

Hence we have an injective mapping $\widehat{\mathcal{S}}_{\text{cl}}(\Omega)/\widehat{\mathcal{N}}_{\text{cl}}(\Omega) \hookrightarrow \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$. Moreover

Proposition 6.4. $\widehat{\mathcal{S}}_{\text{cl}}(\Omega)/\widehat{\mathcal{N}}_{\text{cl}}(\Omega) \simeq \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$.

Theorem 6.5. Let $\Omega \Subset T^*X$ be any sufficiently small neighborhood of $z_0^* \in \dot{T}^*X$. Then $\mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S) \simeq \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$.

Sketch of Proof. We may assume $z_0^* = (z_0; 1, 0, \dots, 0)$. For any $P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$, we set

$$P(z, \zeta) := \sum_{\nu=0}^{\infty} P_\nu(z, \zeta, \eta_0) \zeta_1^\nu \Gamma_\nu(\zeta_1, a).$$

Here we fix any $\eta_0 \in S$ and choose a sufficiently small $0 < a < 1$. Then we can show that $P(z, \zeta) \in \mathcal{S}(\Omega) \subset \mathfrak{S}(\Omega; S)$ and $P(t; z, \zeta, \eta) \equiv P(z, \zeta) \bmod \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$. $\square$

Definition 6.6. As in the case of $\mathfrak{S}(\Omega; S)$, for any $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$ we set

$$:P(t; z, \zeta, \eta): := P(t; z, \zeta, \eta) \bmod \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S) / \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S).$$

Take $\Omega \subseteq T_{\text{conic}}^* \mathbb{C}^n$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be local coordinates on a neighborhood of $\text{Cl } \pi(\Omega) \subset X$, and $(z; \zeta)$, $(w; \lambda)$ corresponding local coordinates on a neighborhood of $\text{Cl } \Omega$. Let $z = \Phi(w)$ be the coordinate transformation. We define $J_\Phi^*(z', z)$ by the relation $\Phi^{-1}(z') - \Phi^{-1}(z) = J_\Phi^*(z', z)(z' - z)$. Then ${}^t J_\Phi^*(z, z)\lambda = {}^t \left[\frac{\partial w}{\partial z}(z) \right] \lambda = \zeta$. Let $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$ with respect to $(z; \zeta)$. Then we set $\Phi^* P(t; w, \lambda, \eta) = \sum_{\nu=0}^{\infty} t^\nu \Phi^* P_\nu(w, \lambda, \eta)$ by

$$\Phi^* P(t; w, \lambda, \eta) := e^{t \langle \partial_{\zeta'}, \partial_{z'} \rangle} P(t; \Phi(w), \zeta' + {}^t J_\Phi^*(\Phi(w) + z', \Phi(w))\lambda, \eta) \Big|_{\substack{z'=0 \\ \zeta'=0}},$$

i.e.

$$\Phi^* P_\nu(w, \lambda, \eta) = \sum_{k+|\alpha|=\nu} \frac{1}{\alpha!} \partial_{\zeta'}^\alpha \partial_{z'}^\alpha P_k(\Phi(w), \zeta' + {}^t J_\Phi^*(\Phi(w) + z', \Phi(w))\lambda, \eta) \Big|_{\substack{z'=0 \\ \zeta'=0}}.$$

Theorem 6.7. (1) $\Phi^* P(t; w, \lambda, \eta)$ defines an element of $\widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$ with respect to $(w; \lambda)$. Moreover if $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$, then $\Phi^* P(t; w, \lambda, \eta) \in \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$.

(2) $\mathbb{1}^*$ is the identity, and if $z = \Phi(w)$ and $w = \Psi(v)$ are complex coordinate transformations, $\Psi^* \Phi^* = (\Phi \Psi)^*$ holds.

Definition 6.8. Under the notation above, we define a coordinate transformation Φ^* associated with Φ by

$$\Phi^* (:P:) (t; w, \lambda, \eta) := :\Phi^* P(t; w, \lambda, \eta):.$$

Theorem 6.9. For any $P(t; z, \zeta, \eta)$, $Q(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$, set

$$\begin{aligned} Q \circ P(t; z, \zeta, \eta) &:= e^{t \langle \partial_{\zeta'}, \partial_{z'} \rangle} Q(t; z, \zeta', \eta) P(t; z', \zeta, \eta) \Big|_{\substack{z'=z \\ \zeta'=\zeta}} \\ &= e^{t \langle \partial_{\zeta'}, \partial_{z'} \rangle} Q(t; z, \zeta + \zeta', \eta) P(t; z + z', \lambda, \eta) \Big|_{\substack{z'=0 \\ \zeta'=0}}. \end{aligned}$$

(1) $Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$. Moreover if either $P(t; z, \zeta, \eta)$ or $Q(t; z, \zeta, \eta)$ is an element of $\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$, it follows that $Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$.

(2) $R \circ (Q \circ P) = (R \circ Q) \circ P$ holds.

(3) Let $\Phi(w) = z$ be a holomorphic coordinate transformation. Then

$$\Phi^* Q \circ \Phi^* P(t; w, \lambda, \eta) = \Phi^*(Q \circ P)(t; w, \lambda, \eta).$$

Definition 6.10. For any $:P(t; z, \zeta, \eta):, :Q(t; z, \zeta, \eta): \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$, we define the product by:

$$:Q(t; z, \zeta, \eta): :P(t; z, \zeta, \eta): := :Q \circ P(t; z, \zeta, \eta):.$$

Theorem 6.11. For any $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$, set

$${}^tP(t; z, \zeta, \eta) := e^{t\langle \partial_\zeta, \partial_z \rangle} P(t; z, -\zeta, \eta).$$

(1) ${}^tP(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega^a; S)$, where $\Omega^a := \{(z; \zeta); (z; -\zeta) \in \Omega\}$, and ${}^t{}^tP = P$. Moreover if $P(t; z, \zeta) \in \widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$, it follows that ${}^tP(t; z, \zeta) \in \widehat{\mathfrak{N}}_{\text{cl}}(\Omega^a; S)$.

(2)

$${}^t(Q \circ P)(t; z, \zeta, \eta) = {}^tP(t; z, \zeta) \circ {}^tQ(t; z, \zeta, \eta).$$

(3) For any holomorphic coordinate transformation $\Phi(w) = z$, on $\widehat{\mathfrak{S}}_{\text{cl}}(\Omega^a; S) \otimes_{\mathcal{O}_X} \Omega_X$ it follows that

$$\det \frac{\partial z}{\partial w} \Phi^*({}^tP) \otimes dw = {}^t(\Phi^* P) \circ \det \frac{\partial z}{\partial w} \otimes dw.$$

Here $\det \frac{\partial z}{\partial w}$ stands for the Jacobian determinant.

Definition 6.12. For any $:P(t; z, \zeta, \eta): \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$, we define the formal adjoint by

$${}^t(:P(t; z, \zeta, \eta):) := :e^{t\langle \partial_\zeta, \partial_z \rangle} P(t; z, -\zeta, \eta): \in \widehat{\mathfrak{S}}_{\text{cl}}(\Omega^a; S)/\widehat{\mathfrak{N}}_{\text{cl}}(\Omega^a; S).$$

Theorem 6.13. Let $[\psi(z, w, \eta) dw], [\varphi(z, w, \eta) dw] \in \mathcal{E}_{X, z_0^*}^{\mathbb{R}}$. Then the following hold:

$$(1) \sum_{\alpha} \frac{1}{\alpha!} \partial_\zeta^\alpha \sigma(\psi)(z, \zeta, \eta) \partial_z^\alpha \sigma(\varphi)(z, \zeta, \eta) \in \mathfrak{S}_{z_0^*}.$$

$$(2) \sum_{\alpha} \frac{1}{\alpha!} \partial_\zeta^\alpha \sigma(\psi)(z, \zeta, \eta) \partial_z^\alpha \sigma(\varphi)(z, \zeta, \eta) - \sigma(\psi) \circ \sigma(\varphi)(z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\text{cl}, z_0}.$$

$$(3) \sigma(\mu(\psi \otimes \varphi))(z, \zeta, \eta) - \sum_{\alpha} \frac{1}{\alpha!} \partial_\zeta^\alpha \sigma(\psi)(z, \zeta, \eta) \partial_z^\alpha \sigma(\varphi)(z, \zeta, \eta) \in \mathfrak{N}_{z_0^*}.$$

This theorem justifies the following formal calculations:

$$\sigma(\mu(\psi \otimes \varphi))(z, \zeta, \eta) = \int \mu(\psi \otimes \varphi)(z, z + \tilde{z}, \eta) e^{\langle \tilde{z}, \zeta \rangle} d\tilde{z}$$

$$\begin{aligned}
&= \int d\tilde{z} e^{\langle \tilde{z}-z, \zeta \rangle} \int \psi(z, w, \eta) \varphi(w, \tilde{z}, \eta) dw \\
&= \int d\tilde{z} e^{\langle \tilde{z}-z, \zeta \rangle} \int \psi(z, z+w, \eta) \varphi(z+w, \tilde{z}, \eta) dw \\
&= \iint \psi(z, z+w, \eta) e^{\langle w, \zeta \rangle} \varphi(z+w, \tilde{z}, \eta) e^{\langle \tilde{z}-w, \zeta \rangle} d\tilde{z} dw \\
&= \int dw \psi(z, z+w, \eta) e^{\langle w, \zeta \rangle} \int \varphi(z+w, \tilde{z}, \eta) e^{\langle \tilde{z}-w, \zeta \rangle} d\tilde{z} \\
&= \int dw \psi(z, z+w, \eta) e^{\langle w, \zeta \rangle} \int \varphi(z+w, \tilde{z}+w, \eta) e^{\langle \tilde{z}, \zeta \rangle} d\tilde{z} \\
&= \int \psi(z, z+w, \eta) e^{\langle w, \zeta \rangle} \sigma(\varphi)(z+w, \zeta, \eta) dw \\
&= \int \psi(z, z+w, \eta) e^{\langle w, \zeta \rangle} \sum_{\alpha} \frac{w^{\alpha}}{\alpha!} \partial_z^{\alpha} \sigma(\varphi)(z, \zeta, \eta) dw \\
&= \sum_{\alpha} \left(\int \psi(z, z+w, \eta) e^{\langle w, \zeta \rangle} \frac{w^{\alpha}}{\alpha!} dw \right) \partial_z^{\alpha} \sigma(\varphi)(z, \zeta, \eta) \\
&= \sum_{\alpha} \frac{1}{\alpha!} \partial_{\zeta}^{\alpha} \left(\int \psi(z, z+w, \eta) e^{\langle w, \zeta \rangle} dw \right) \partial_z^{\alpha} \sigma(\varphi)(z, \zeta, \eta) \\
&= \sum_{\alpha} \frac{1}{\alpha!} \partial_{\zeta}^{\alpha} \sigma(\psi)(z, \zeta, \eta) \partial_z^{\alpha} \sigma(\varphi)(z, \zeta, \eta).
\end{aligned}$$

Remark 6.14. Let $[\psi(z, w, \eta) dw] \in \mathcal{E}_{X, z_0}^{\mathbb{R}}$. Then we can also prove the following:

(1) We have

$$:P^*(t; z, \zeta, \eta): = : \int_{\gamma(0, \eta; \varrho, \theta)} \psi(z-w, z, \eta) e^{-\langle w, \zeta \rangle} dw :.$$

(2) Let $z = \Phi(w)$ be a complex coordinate transformation. Then

$$:\Phi^* P(t; w, \lambda, \eta): = : \int_{\gamma(z, \eta; \varrho, \theta)} \psi(z, z', \eta) e^{\langle \Phi^{-1}(z') - \Phi^{-1}(z), \lambda \rangle} dz' :.$$

§ 7. Formal Symbols with an Apparent Parameter

For the analysis of $\mathcal{E}_X^{\mathbb{R}}$ by using symbols, it is useful to introduce a wider class $\widehat{\mathfrak{S}}(\Omega; S)$ than $\widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)$. In this section, we introduce this class $\widehat{\mathfrak{S}}(\Omega; S)$; that is, formal symbols with an apparent parameter.

For $U \subset S$ and $m \in \mathbb{N}$, we set

$$(\Omega_{\rho} * U)[md_{\rho}] := \{(z; \zeta, \eta) \in \Omega_{\rho} \times U; \|\eta\zeta\| \geq md_{\rho}\} \subset \Omega_{\rho}[md_{\rho}] \times S.$$

Definition 7.1. Let t be an indeterminate. Then, we say that $P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z, \zeta, \eta)$ is an element of $\widehat{\mathfrak{N}}(\Omega; S)$ if

- (i) $P_{\nu}(z, \zeta, \eta) \in \Gamma((\Omega_{\rho} * S)[(\nu + 1)d_{\rho}]; \mathcal{O}_{T^*X \times \mathbb{C}})$ for some $d > 0$ and $\rho \in]0, 1[$,
- (ii) there exists a constant $A \in]0, 1[$, and for any $Z \Subset S$, $h > 0$ there exists $C_{h,Z} > 0$ such that

$$\left| \sum_{\nu=0}^{m-1} P_{\nu}(z, \zeta, \eta) \right| \leq C_{h,Z} A^m e^{h\|\zeta\|} \quad (m \in \mathbb{N}, (z; \zeta, \eta) \in (\Omega_{\rho} * Z)[md_{\rho}]).$$

Definition 7.2. (1) We say that $P(t; z, \zeta, \eta) = \sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(z, \zeta, \eta)$ is an element of $\widehat{\mathfrak{S}}(\Omega; S)$ if

- (i) $P_{\nu}(z, \zeta, \eta) \in \Gamma((\Omega_{\rho} * S)[(\nu + 1)d_{\rho}]; \mathcal{O}_{T^*X \times \mathbb{C}})$ for some $d > 0$ and $\rho \in]0, 1[$,
- (ii) there exists a constant $A \in]0, 1[$, and for any $Z \Subset S$, $h > 0$, there exists $C_{h,Z} > 0$ such that

$$|P_{\nu}(z, \zeta, \eta)| \leq C_{h,Z} A^{\nu} e^{h\|\zeta\|} \quad (\nu \in \mathbb{N}_0, (z; \zeta, \eta) \in (\Omega_{\rho} * Z)[(\nu + 1)d_{\rho}]).$$

- (iii) $\partial_{\eta} P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}(\Omega; S)$.

We call each element of $\widehat{\mathfrak{S}}(\Omega; S)$ (resp. $\widehat{\mathfrak{N}}(\Omega; S)$) a *formal symbol* (resp. *formal null-symbol*) on Ω with an apparent parameter in S .

We set

$$\widehat{\mathfrak{S}}_{z_0^*} := \lim_{\substack{\longrightarrow \\ \Omega, S}} \widehat{\mathfrak{S}}(\Omega; S) \supset \widehat{\mathfrak{N}}_{z_0^*} := \lim_{\substack{\longrightarrow \\ \Omega, S}} \widehat{\mathfrak{N}}(\Omega; S).$$

Proposition 7.3. (1) Let $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega; S)$. Then for any $\eta_0 \in S$, it follows that $P(t; z, \zeta, \eta_0) \in \widehat{\mathcal{S}}(\Omega)$ and $P(t; z, \zeta, \eta) - P(t; z, \zeta, \eta_0) \in \widehat{\mathfrak{N}}(\Omega; S)$.

(2) There exists the following isomorphism:

$$\widehat{\mathcal{S}}(\Omega)/\widehat{\mathcal{N}}(\Omega) \simeq \widehat{\mathfrak{S}}(\Omega; S)/\widehat{\mathfrak{N}}(\Omega; S).$$

Theorem 7.4. (1) $\widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S) \subset \widehat{\mathfrak{S}}(\Omega; S)$ and $\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S) \subset \widehat{\mathfrak{N}}(\Omega; S)$.

(2) For any $z_0^* \in \dot{T}^*X$, the inclusions $\mathfrak{S}_{z_0^*} \subset \widehat{\mathfrak{S}}_{\text{cl}, z_0^*} \subset \widehat{\mathfrak{S}}_{z_0^*}$ and $\mathfrak{N}_{z_0^*} \subset \widehat{\mathfrak{N}}_{\text{cl}, z_0^*} \subset \widehat{\mathfrak{N}}_{z_0^*}$ induce

$$\begin{array}{ccccccc} \mathcal{E}_{X, z_0^*}^{\mathbb{R}} & \xrightarrow{\sim} & \mathcal{S}_{z_0^*} / \mathcal{N}_{z_0^*} & \simeq & \widehat{\mathcal{S}}_{\text{cl}, z_0^*} / \widehat{\mathcal{N}}_{\text{cl}, z_0^*} & \simeq & \widehat{\mathcal{S}}_{z_0^*} / \widehat{\mathcal{N}}_{z_0^*} \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \varinjlim_{\kappa \rightarrow 0} E_X^{\mathbb{R}}(\kappa) & \simeq & \mathfrak{S}_{z_0^*} / \mathfrak{N}_{z_0^*} & \simeq & \widehat{\mathfrak{S}}_{\text{cl}, z_0^*} / \widehat{\mathfrak{N}}_{\text{cl}, z_0^*} & \simeq & \widehat{\mathfrak{S}}_{z_0^*} / \widehat{\mathfrak{N}}_{z_0^*}. \end{array}$$

Remark 7.5. (1) It is sometimes convenient to deal with formal symbols than classical formal symbols. However, in Theorem 6.5, any sufficiently small neighborhood of $z_0^* \in \dot{T}^*X$ we can obtain an isomorphism $\mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S) \simeq \widehat{\mathfrak{S}}_{\text{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\text{cl}}(\Omega; S)$, and in Theorem 7.4, we only obtain an isomorphism between stalks at this stage.

(2) We obtain an explicit description of the isomorphism $\mathcal{E}_{X, z_0^*}^{\mathbb{R}} \simeq \varinjlim_{\kappa \rightarrow 0} E_X^{\mathbb{R}}(\kappa)$ as follows: We may assume that $z_0^* = (z_0; 1, 0, \dots, 0) \in \dot{T}^*X$. Let $P \in \mathfrak{S}_{z_0^*}$, and consider $[\varpi(P)(z, w, \eta) dw] \in \varinjlim_{\kappa} E_X^{\mathbb{R}}(\kappa)$. Then we define

$$\begin{aligned} \varpi_{0, \alpha}(P)(z, w_1) &:= \int_d^\infty P_\alpha(z, \zeta_1, \eta_0) \Gamma_{|\alpha|}(\zeta_1, c_0 \varepsilon_{|\alpha|} - w_1) e^{-w_1 \zeta_1} d\zeta_1, \\ \varpi_0(P)(z, z+w) &:= \sum_{\alpha \in \mathbb{N}_0^{n-1}} \frac{\alpha! \varpi_{0, \alpha}(P)(z, w_1)}{(2\pi \sqrt{-1})^n (w')^{\alpha + \mathbf{1}_{n-1}}}. \end{aligned}$$

Here $c_0 > 1 \gg \varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_\nu \xrightarrow{\nu} 0$ satisfying some conditions (cf. [3], [4]). Then we can show that $[\varpi_0(P)(z, w) dw] \in \mathcal{E}_{X, z_0^*}^{\mathbb{R}}$ and

$$[\varpi(P)(z, w, \eta) dw] = [\varpi_0(P)(z, w) dw] \in \varinjlim_{\kappa \rightarrow 0} E_X^{\mathbb{R}}(\kappa).$$

We use the notation of Theorem 6.7. For any $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega; S)$, we also set

$$\Phi^* P(t; w, \lambda, \eta) := e^{t\langle \partial_{\zeta'}, \partial_{z'} \rangle} P(t; \Phi(w), \zeta' + {}^t J_\Phi^*(\Phi(w) + z', \Phi(w))\lambda, \eta) \Big|_{\substack{z'=0 \\ \zeta'=0}}.$$

Theorem 7.6. *Let $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega; S)$.*

(1) *$\Phi^* P(t; w, \lambda, \eta) \in \widehat{\mathfrak{S}}(\Omega; S)$ with respect to coordinate system $(w; \lambda)$. Further if $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}(\Omega; S)$, it follows that $\Phi^* P(t; w, \lambda, \eta) \in \widehat{\mathfrak{N}}(\Omega; S)$.*

(2) *$\mathbb{1}^*$ is the identity, and for complex coordinate transformations $z = \Phi(w)$ and $w = \Psi(v)$, it follow that $\Psi^* \Phi^* P(t; v, \xi, \eta) - (\Phi\Psi)^* P(t; v, \xi, \eta) \in \widehat{\mathfrak{N}}_{(v; \xi)}$.*

Theorem 7.7. *For any $P(t; z, \zeta, \eta), Q(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega; S)$, set*

$$\begin{aligned} Q \circ P(t; z, \zeta, \eta) &:= e^{t\langle \partial_{\zeta'}, \partial_{z'} \rangle} Q(t; z, \zeta', \eta) P(t; z', \zeta, \eta) \Big|_{\substack{z'=z \\ \zeta'=\zeta}} \\ &= e^{t\langle \partial_{\zeta'}, \partial_{z'} \rangle} Q(t; z, \zeta + \zeta', \eta) P(t; z + z', \zeta, \eta) \Big|_{\substack{z'=0 \\ \zeta'=0}}. \end{aligned}$$

(1) *$Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega; S)$. Moreover if either $P(t; z, \zeta, \eta)$ or $Q(t; z, \zeta, \eta)$ is an element of $\widehat{\mathfrak{N}}(\Omega; S)$, it follows that $Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}(\Omega; S)$.*

(2) *$R \circ (Q \circ P) = (R \circ Q) \circ P$ holds.*

(3) *Let $\Phi(w) = z$ be a holomorphic coordinate transformation. Then*

$$\Phi^* Q \circ \Phi^* P(t; w, \lambda, \eta) - \Phi^* (Q \circ P)(t; w, \lambda, \eta) \in \widehat{\mathfrak{N}}_{(w; \lambda)}.$$

Theorem 7.8. For any $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega; S)$ set

$${}^tP(t; z, \zeta, \eta) := e^{t\langle \partial_\zeta, \partial_z \rangle} P(t; z, -\zeta, \eta).$$

(1) ${}^tP(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}(\Omega^a; S)$ and ${}^tP = P$. Moreover if $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}(\Omega; S)$, it follows that ${}^tP(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}(\Omega^a; S)$.

(2) ${}^t(Q \circ P)(t; z, \zeta, \eta) = {}^tP(t; z, \zeta, \eta) \circ {}^tQ(t; z, \zeta, \eta)$.

(3) For any holomorphic coordinate transformation $\Phi(w) = z$, on $\widehat{\mathfrak{S}}(\Omega^a; S) \otimes_{\mathcal{O}_X} \Omega_X$ it follows that

$$\det \frac{\partial z}{\partial w} \Phi^*({}^tP) \otimes dw = {}^t(\Phi^*P) \circ \det \frac{\partial z}{\partial w} \otimes dw.$$

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