

Further extension of an order preserving operator inequality and its application

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An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

The celebrated Löwner-Heinz inequality asserts that if $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ holds for any $\alpha \in [0, 1]$, but $A^p \geq B^p$ does not always hold for $p > 1$. From this point of view, we shall show an order preserving operator inequality in Theorem 2.3 and we shall state an implication among Theorem 2.3 and the previous known results.

Theorem 2.3. Let $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, \dots, p_{2n} \geq 1$ for natural number n . Then the following inequality holds for $r \geq t$:

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left[\underbrace{A^{-\frac{t}{2}} \{ A^{\frac{1}{2}} \dots \dots \dots [A^{-\frac{t}{2}} \{ A^{\frac{1}{2}} (A^{-\frac{t}{2}} B^{p_1} A^{-\frac{t}{2}}) A^{\frac{1}{2}} \}^{p_2} A^{\frac{1}{2}} \}^{p_3} A^{-\frac{t}{2}}]^{p_4} A^{\frac{1}{2}} \dots \dots \dots A^{-\frac{t}{2}}}_{\leftarrow A^{-\frac{t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n-1 \text{ times by turns}} \rightarrow A^{-\frac{t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n-1 \text{ times by turns}} \right]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{q[2n]+r-t}}$$

where $q[2n] = \underbrace{\{ \dots \{ [(p_1 - t)p_2 + t]p_3 - t \} p_4 + t \} p_5 - \dots - t \}_{p_{2n} + t}}_{-t \text{ and } t \text{ alternately } n \text{ times appear}}$

Remark 1. Consider the following operator function $f(\square)$ defined by

$$f(\square) = \left\{ A^{\frac{r}{2}} \left[\underbrace{A^{-\frac{t}{2}} \{ A^{\frac{1}{2}} \dots \dots \dots [A^{-\frac{t}{2}} \{ A^{\frac{1}{2}} (A^{-\frac{t}{2}} \square A^{-\frac{t}{2}}) A^{\frac{1}{2}} \}^{p_2} A^{\frac{1}{2}} \}^{p_3} A^{-\frac{t}{2}}]^{p_4} A^{\frac{1}{2}} \dots \dots \dots A^{-\frac{t}{2}}}_{\leftarrow A^{-\frac{t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n-1 \text{ times by turns}} \rightarrow A^{-\frac{t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n-1 \text{ times by turns}} \right]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{q[2n]+r-t}}.$$

Theorem 2.3 asserts that $f(A^{p_1}) \geq f(B^{p_1})$ holds for any $p_1 \geq 1$ whenever $A \geq B \geq 0$ with $A > 0$ under some suitable conditions, that is, $f(\square)$ can be considered as order preserving operator function. Theorem 2.3 is further extension of the following previous one: if $A \geq B \geq 0$ with $A > 0$, then for $t \in [0, 1]$ and $p \geq 1$, $A^{1-t+r} \geq \{ A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}} \}^{\frac{1-t+r}{(p-t)s+r}}$ holds for $r \geq t$ and $s \geq 1$, in particular, $A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0$. Finally an application of Theorem 2.3 is shown.

§1 Introduction

An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

Theorem LH (Löwner-Heinz inequality, denoted by (LH) briefly).

If $A \geq B \geq 0$ holds, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$. (LH)

Although (LH) asserts that $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$, unfortunately $A^p \geq B^p$ does not always hold for $p > 1$. The following result has been obtained from this point of view.

Theorem F.

If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

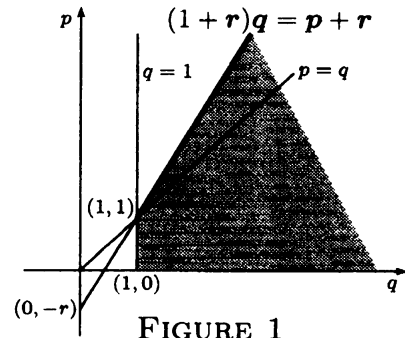


FIGURE 1

The original proof of Theorem F is shown in [F1], an elementary one-page proof is in [F2] and alternative ones are in [MF],[K1]. It is shown in [T1] that the conditions p , q and r in FIGURE 1 are best possible.

Theorem G. *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0, 1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \quad (1.1)$$

holds for $r \geq t$ and $s \geq 1$.

The original proof of Theorem G is in [F3], and an alternative one is in [MF-K]. An elementary one-page proof of (1.1) is in [F4]. We mention that further extensions of Theorem G and related results to Theorem F are in [MF-K-N],[F5],[F-H-I],[F-Y-Y], [K2] and etc. It is shown in [T2] that the exponent value $\frac{1-t+r}{(p-t)s+r}$ of the right hand of (1.1) is best possible and alternative ones are in [MF-M-N],[Y]. It is known that the operator inequality (1.1) interpolates Theorem F and an inequality (see Theorem AH under below at

4 page) equivalent to the main result of Ando-Hiai log majorization [A-H] by the parameter $t \in [0, 1]$.

§2 Further extension of Theorem G

Theorem 2.1 [F7]. Let $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, \dots, p_{2n} \geq 1$. Then the following inequality holds,

$$A \geq \left[\underbrace{A^{\frac{1}{2}} \{ A^{-\frac{t}{2}} [A^{\frac{1}{2}} \dots [A^{-\frac{t}{2}} \{ A^{\frac{1}{2}} (A^{-\frac{t}{2}} B^{p_1} A^{-\frac{t}{2}})^{p_2} A^{\frac{1}{2}} \}^{p_3} A^{-\frac{t}{2}}]^{p_4} \dots A^{\frac{1}{2}}]^{p_{2n-1}} A^{-\frac{t}{2}} \}^{p_{2n}} A^{\frac{1}{2}}}_{\leftarrow A^{-\frac{t}{2}} \text{ and } A^{\frac{1}{2}} \text{ alternately } n \text{ times}} \right]^{\frac{1}{q[2n]}} \quad (2.1)$$

$\rightarrow A^{-\frac{t}{2}} \text{ and } A^{\frac{1}{2}} \text{ alternately } n \text{ times}$

where $q[2n] = q[2n; p_1, p_2, \dots, p_n, \dots, p_{2(n-1)}, p_{2n-1}, p_{2n}]$

$$= \underbrace{\{ \dots [\{ [(p_1 - t)p_2 + t] p_3 - t \} p_4 + t \} p_5 - \dots - t \} p_{2n} + t}_{-t \text{ and } t \text{ alternately } n \text{ times appear}} \quad (2.2)$$

Corollary 2.2 [F6]. If $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, p_3, p_4 \geq 1$, then the following inequality holds,

$$A \geq \left\{ A^{\frac{1}{2}} [A^{-\frac{t}{2}} \{ A^{\frac{1}{2}} (A^{-\frac{t}{2}} B^{p_1} A^{-\frac{t}{2}})^{p_2} A^{\frac{1}{2}} \}^{p_3} A^{-\frac{t}{2}}]^{p_4} A^{\frac{1}{2}} \right\}^{\frac{1}{[(p_1 - t)p_2 + t] p_3 - t p_4 + t}}.$$

Theorem 2.3 [F7]. Let $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, \dots, p_{2n} \geq 1$ for natural number n . Then the following inequality holds for $r \geq t$,

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left[\underbrace{A^{-\frac{t}{2}} \{ A^{\frac{1}{2}} \dots [A^{-\frac{t}{2}} \{ A^{\frac{1}{2}} (A^{-\frac{t}{2}} B^{p_1} A^{-\frac{t}{2}})^{p_2} A^{\frac{1}{2}} \}^{p_3} A^{-\frac{t}{2}}]^{p_4} A^{\frac{1}{2}} \dots A^{-\frac{t}{2}} }_{\leftarrow A^{-\frac{t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n - 1 \text{ times by turns}} \right]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{q[2n]+r-t}} \quad (2.3)$$

$\rightarrow A^{-\frac{t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n - 1 \text{ times by turns}$

where $q[2n]$ is defined in (2.2).

Corollary 2.4 [F6]. If $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, p_3, p_4 \geq 1$,

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left[A^{-\frac{t}{2}} \{ A^{\frac{1}{2}} (A^{-\frac{t}{2}} B^{p_1} A^{-\frac{t}{2}})^{p_2} A^{\frac{1}{2}} \}^{p_3} A^{-\frac{t}{2}} \right]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[(p_1 - t)p_2 + t] p_3 - t p_4 + r}}$$

holds for $r \geq t$.

Corollary 2.4 obtained by Theorem 2.3 and previous known results

Corollary 2.4. If $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, p_3, p_4 \geq 1$,

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left[A^{\frac{-t}{2}} \left\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \right\}^{p_3} A^{\frac{-t}{2}} \right]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[(p_1-t)p_2+t]p_3-t)p_4+r}}$$

holds for $r \geq t$.

$$p_2 = p_3 = 1 \quad \downarrow$$

Theorem G. If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0, 1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}} \}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ and $s \geq 1$.

$$t = 0 \text{ and } s = 1$$

$$t = 1 \text{ and } r = s$$

Theorem F. $A \geq B \geq 0 \implies$

$$A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

for $p \geq 1$ and $r \geq 0$.

Theorem AH. If $A \geq B \geq 0$ with $A > 0 \implies$

$$A^r \geq \{ A^{\frac{r}{2}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r A^{\frac{r}{2}} \}^{\frac{1}{p}}$$

for $r, p \geq 1$.



Theorem AH-L. For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$,

$$(A \#_{\alpha} B)^r \underset{(\log)}{>} A^r \#_{\alpha} B^r \quad \text{for } r \geq 1.$$

Theorem F.

If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

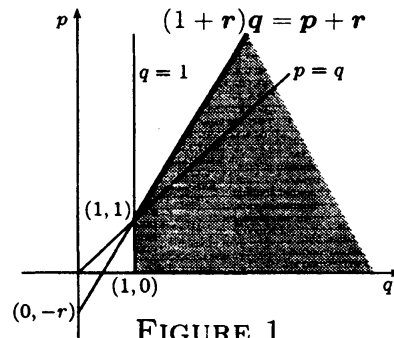


FIGURE 1

§3 Further extension of an operator function implying Theorem G

Theorem H. If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0, 1]$ and $p \geq 1$,

$$F_{A,B}(r, s) = A^{\frac{-r}{2}} \{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}} \}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is a decreasing function for $r \geq t$ and $s \geq 1$, and $F_{A,A}(r, s) \geq F_{A,B}(r, s)$ holds, that is,

$$A^{1-t+r} \geq \{ A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}} \}^{\frac{1-t+r}{(p-t)s+r}} \quad (1.1)$$

holds for $t \in [0, 1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$.

$F_{A,B}(r, s)$ in Theorem H is an operator function implying (1.1) in Theorem G. The original proof of Theorem H is in [F3], and an alternative one is in [MF-K]. Further extensions of Theorem H and related results are in [MF-K-N], [F5], [F-H-I], [F-Y-Y], [K2] and etc.

We shall state further extension of Theorem H as an application of Theorem 2.3.

Theorem 3.1 [F8]. Let $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, \dots, p_{2n} \geq 1$ for natural number n . Then

$$G_{A,B}[r, p_{2n}] \quad (3.1)$$

$$= A^{\frac{-r}{2}} \left\{ A^{\frac{r}{2}} \left[\underbrace{A^{\frac{-t}{2}} \{ A^{\frac{1}{2}} \dots \dots \dots [A^{\frac{-t}{2}} \{ A^{\frac{1}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{1}{2}} \}^{p_3} A^{\frac{-t}{2}}]^{p_4} A^{\frac{1}{2}} \dots \dots \dots A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n-1 \text{ times by turns}} \rightarrow A^{\frac{-t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n-1 \text{ times by turns}} \right]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{q[2n]+r-t}} A^{\frac{-r}{2}}$$

is a decreasing function of $p_{2n} \geq 1$ and $r \geq t$, and the following inequality holds

$$G_{A,A}[r, p_{2n}] \geq G_{A,B}[r, p_{2n}],$$

that is,

$$A^{1-t+r} \geq$$

$$\left\{ A^{\frac{r}{2}} \left[\underbrace{A^{\frac{-t}{2}} \{ A^{\frac{1}{2}} \dots \dots \dots [A^{\frac{-t}{2}} \{ A^{\frac{1}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{1}{2}} \}^{p_3} A^{\frac{-t}{2}}]^{p_4} A^{\frac{1}{2}} \dots \dots \dots A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n-1 \text{ times by turns}} \rightarrow A^{\frac{-t}{2}} \text{ } n \text{ times and } A^{\frac{1}{2}} \text{ } n-1 \text{ times by turns}} \right]^{p_{2n}} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{q[2n]+r-t}} \quad (3.2)$$

where $q[2n]$ is defined by (2.2).

Corollary 3.2 [F8]. If $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, p_3, p_4 \geq 1$,

$$G_{A,B}[r, p_4] = A^{\frac{-r}{2}} \left\{ A^{\frac{r}{2}} \left[A^{\frac{-t}{2}} \left\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \right\}^{p_3} A^{\frac{-t}{2}} \right]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[(p_1-t)p_2+t]p_3-t)p_4+r}} A^{\frac{-r}{2}}$$

is a decreasing function of $p_4 \geq 1$ and $r \geq t$, and the following inequality holds

$G_{A,A}[r, p_4] \geq G_{A,B}[r, p_4]$, that is,

$$A^{1-t+r} \geq \left\{ A^{\frac{r}{2}} \left[A^{\frac{-t}{2}} \left\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \right\}^{p_3} A^{\frac{-t}{2}} \right]^{p_4} A^{\frac{r}{2}} \right\}^{\frac{1-t+r}{[(p_1-t)p_2+t]p_3-t)p_4+r}}$$

holds for $t \in [0, 1]$, $r \geq t$ and $p_1, p_2, p_3, p_4 \geq 1$.

Remark 3.1. Theorem 3.1 yields Corollary 3.2 by putting $n = 2$ and also Corollary 3.2 yields Theorem H by putting $p_2 = p_3 = 1$.

§4 Satellite inequalities as an application of Theorem 3.1

Theorem 4.1 [F8]. If $A \geq B \geq 0$ with $A > 0$, $t \in [0, 1]$ and $p_1, p_2, \dots, p_{2n} \geq 1$, then the following inequality holds:

$$\begin{aligned} A &\geq B \\ &\geq \left\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \right\}^{\frac{1}{(p_1-t)p_2+t}} \\ &\geq \left\{ A^{\frac{t}{2}} \left[A^{\frac{-t}{2}} \left\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \right\}^{p_3} A^{\frac{-t}{2}} \right]^{p_4} A^{\frac{t}{2}} \right\}^{\frac{1}{[(p_1-t)p_2+t]p_3-t)p_4+t}} \\ &\dots \\ &\geq \\ &\dots \\ &\geq \left[\underbrace{A^{\frac{t}{2}} \left\{ A^{\frac{-t}{2}} \left[A^{\frac{t}{2}} \dots \dots \dots \left[A^{\frac{-t}{2}} \left\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \right\}^{p_3} A^{\frac{-t}{2}} \right]^{p_4} \dots A^{\frac{t}{2}} \right]^{p_{2n-1}} A^{\frac{-t}{2}} \right\}^{p_{2n}} A^{\frac{t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} \right]^{\frac{1}{q[2n]}} \end{aligned} \quad (4.1)$$

$\rightarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}$

where $q[2n]$ is defined in (2.2).

Remark 4.1. Corollary 2 in § 3.2.5 of [F5] states that if $A \geq B > 0$, then

$$A \geq B \geq \left\{ A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}} \right\}^{\frac{1}{(p-t)s+t}} \text{ holds for each } t \in [0, 1] \text{ and } p, s \geq 1 \quad (4.2)$$

and Theorem 4.1 is further extension of (4.2).

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