

順序集合における不動点定理の非整数階微分方程式 境界値問題への適用例

Applied results of a fixed point theorem in partially ordered sets to fractional order boundary
value problems

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1 Introduction

Let X be a partially ordered set with a metric d and let T be a mapping from X into itself. We say that T is monotone nondecreasing if for any $x, y \in X$, $x \leq y$ implies $Tx \leq Ty$. Nieto and López [3] consider the following fixed point theorem in partially ordered sets.

Theorem 1. *Let X be a partially ordered set with a metric d such that (X, d) is a complete metric space. If a nondecreasing sequence $\{x_n\}$ converges to x , then we have $x_n \leq x$ for any n . Let T be a monotone nonincreasing mapping from X into itself such that there exists $k \in [0, 1)$ such that for any $x, y \in X$,*

$$x \geq y \text{ implies } d(Tx, Ty) \leq kd(x, y).$$

Assume that there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Then there exists a fixed point of T . Moreover, if for any $x, y \in X$, there exists $z \in X$ which is comparable to x and y , then the fixed point of T is unique.

In this paper, using Theorem 1, we show the existence and uniqueness of solutions of fractional order boundary value problems.

2 Riemann-Liouville fractional derivative and integral

The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function u of $(0, \infty)$ into $\mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α and $\Gamma(\alpha)$ denotes the gamma function. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function u of $(0, \infty)$ into $\mathbb{R}$ is defined by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds.$$

For the proof of Lemmas 2 and 3, we use the following: For $p, q > 0$ and $a \in \mathbb{R}$,

$$\int_a^t (t-s)^{p-1} (s-a)^{q-1} ds = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} (t-a)^{p+q-1}. \quad (1)$$

In fact, we have (1) since

$$\begin{aligned} \int_a^t (t-s)^{p-1} (s-a)^{q-1} ds &= \int_0^{t-a} (t-a-\tau)^{p-1} \tau^{q-1} d\tau \\ &= \int_0^1 (t-a-(t-a)u)^{p-1} (t-a)^{q-1} u^{q-1} (t-a) du \\ &= (t-a)^{p+q-1} \int_0^1 (1-u)^{p-1} u^{q-1} du \\ &= (t-a)^{p+q-1} B(p, q) \\ &= (t-a)^{p+q-1} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \end{aligned}$$

For $\alpha, \beta > 0$, we have

$$I_{0+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \quad (2)$$

In fact, by (1), we have

$$I_{0+}^{\alpha} t^{\beta} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\beta} ds = \frac{1}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.$$

Moreover for $\beta > \alpha > 0$, we have

$$D_{0+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}. \quad (3)$$

In fact, since $D_{0+}^n t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} t^{\beta-n}$ for $n = 1, 2, 3, \dots, [\beta]$, by (1), we have

$$\begin{aligned} D_{0+}^\alpha t^\beta &= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} s^\beta ds \\ &= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \left(\frac{\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} t^{n-\alpha+\beta} \right) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} \cdot \frac{d^n}{dt^n} t^{n-\alpha+\beta} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} \cdot \frac{\Gamma(n-\alpha+\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}. \end{aligned}$$

Lemma 2. Let $\alpha > 0$. If $u(t) = t^{\alpha-n}$ ($n = 1, 2, \dots, [\alpha] + 1$), then $D_{0+}^\alpha u = 0$. Conversely, if $D_{0+}^\alpha u(t) = 0$, then there exists $C_1, C_2, \dots, C_n \in \mathbb{R}$ such that

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n}$$

where $n = [\alpha] + 1$.

Proof. Let $n = 1, 2, \dots, [\alpha] + 1$ and $u(t) = t^{\alpha-n}$. By (1), we have

$$\begin{aligned} D_{0+}^\alpha u(t) &= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} s^{\alpha-n} ds \\ &= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \left(\frac{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}{\Gamma(1)} \right) \\ &= 0. \end{aligned}$$

Conversely, assume that $D_{0+}^\alpha u(t) = 0$. Then we have

$$\frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = 0.$$

Since $\frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = 0$, we have

$$\frac{d^{n-1}}{dt^{n-1}} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = C_1.$$

Moreover we have

$$\frac{d^{n-2}}{dt^{n-2}} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = C_1 t + C_2.$$

Similarly we obtain that

$$\frac{d^{n-3}}{dt^{n-3}} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = C_1 t^2 + C_2 t + C_3$$

where we change $\frac{C_1}{2}$ by C_1 . Hence we obtain that

$$\int_0^t (t-s)^{n-\alpha-1} u(s) ds = C_1 t^{n-1} + C_2 t^{n-2} + \dots + C_n$$

for some $C_1, C_2, \dots, C_n$. By (1), we have

$$\begin{aligned} I_{0+}^\alpha \int_0^t (t-s)^{n-\alpha-1} u(s) ds &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\int_0^s (s-\tau)^{n-\alpha-1} u(\tau) d\tau \right) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t u(\tau) \left(\int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{n-\alpha-1} ds \right) d\tau \\ &= \frac{1}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)\Gamma(n-\alpha)}{\Gamma(n)} (t-\tau)^{n-1} \\ &= \frac{\Gamma(n-\alpha)}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} u(\tau) d\tau \\ &= \Gamma(n-\alpha) I_{0+}^n u(t). \end{aligned}$$

Since

$$I_{0+}^\alpha (C_1 t^{n-1} + C_2 t^{n-2} + \dots + C_n) = C_1 t^{\alpha+n-1} + C_2 t^{\alpha+n-2} + \dots + C_n t^\alpha,$$

we have

$$\Gamma(n-\alpha) I_{0+}^n u(t) = C_1 t^{\alpha+n-1} + C_2 t^{\alpha+n-2} + \dots + C_n t^\alpha.$$

Hence, for some $C_1, C_2, \dots, C_n$, we obtain that

$$I_{0+}^n u(t) = C_1 t^{\alpha+n-1} + C_2 t^{\alpha+n-2} + \dots + C_n t^\alpha.$$

Since $D_{0+}^n I_{0+}^n u(t) = u(t)$ and $D_{0+}^n (C_1 t^{\alpha+n-1} + C_2 t^{\alpha+n-2} + \dots + C_n t^\alpha) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n}$, we have

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n}.$$

□

The following lemma can be found in [4].

Lemma 3. Let $\alpha > 0$ and let $u \in L(a, b)$. Then we have

$$D_{0+}^\alpha I_{0+}^\alpha u = u.$$

Proof. Let $n = [\alpha] + 1$. By (1), we have

$$\begin{aligned} D_{0+}^\alpha I_{0+}^\alpha u &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_0^x \frac{1}{(t-s)^{\alpha-n+1}} \left(\int_\tau^t (s-\tau)^{\alpha-1} u(\tau) d\tau \right) ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_0^t \left(\int_\tau^t (t-s)^{n-\alpha-1} (s-\tau)^{\alpha-1} u(\tau) ds \right) d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_0^t u(\tau) \left(\int_\tau^t (t-s)^{n-\alpha-1} (s-\tau)^{\alpha-1} ds \right) d\tau \\ &= \frac{1}{\Gamma(n)} \cdot \frac{d^n}{dx^n} \int_0^t (t-\tau)^{n-1} u(\tau) d\tau. \end{aligned}$$

Since $\int_0^t \left(\int_0^t \cdots \left(\int_0^t u(s) ds \right) \cdots ds \right) ds = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} u(s) ds$, we have $D_{0+}^\alpha I_{0+}^\alpha u = u$. $\square$

The following lemma can be found in [1]. See also [2].

Lemma 4. *Let $\alpha > 0$. Let $u \in C(0, 1) \cap L(0, 1)$ satisfying $D_{0+}^\alpha u \in C(0, 1) \cap L(0, 1)$. Then*

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for some $C_1, C_2, \dots, C_n \in \mathbb{R}$ and $n = [\alpha] + 1$.

Proof. By Lemma 2, we have $D_{0+}^\alpha (I_{0+}^\alpha D_{0+}^\alpha u - u) = D_{0+}^\alpha I_{0+}^\alpha D_{0+}^\alpha u - D_{0+}^\alpha u = D_{0+}^\alpha u - D_{0+}^\alpha u = 0$. By Lemma 3, there exists $C_1, C_2, \dots, C_n \in \mathbb{R}$ such that $I_{0+}^\alpha D_{0+}^\alpha u(t) - u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$. $\square$

3 Applied results to fractional order boundary value problems

Using Lemma 4, we obtain the following [1]. For the sake of completeness, we show the proof.

Lemma 5. *Let $h \in C[0, 1]$ and $1 < \alpha \leq 2$. Then the unique solution of the problem*

$$\begin{cases} D_{0+}^\alpha u(t) + h(t) = 0, \\ u(0) = u(1) = 0 \end{cases}$$

is

$$u(t) = \int_0^1 G_\alpha(t, s) h(s) ds$$

where

$$G_\alpha(t, s) = \begin{cases} \frac{1}{\Gamma(\alpha)} (t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}) & (0 \leq s \leq t \leq 1), \\ \frac{1}{\Gamma(\alpha)} (t^{\alpha-1}(1-s)^{\alpha-1}) & (0 \leq t \leq s \leq 1). \end{cases}$$

Proof. By Lemma 4, we have

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}$$

for some $C_1, C_2 \in \mathbb{R}$. By $u(0) = 0$, we have $C_2 = 0$. Moreover, by $u(1) = 0$, we have $C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds$. Thus we obtain that $u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} t^{\alpha-1} h(s) ds = \int_0^1 G_\alpha(t, s) h(s) ds$. $\square$

Using Lemma 5, we obtain the following.

Theorem 6. Let f be a mapping of $[0, 1] \times [0, \infty)$ into $[0, \infty)$ such that f is continuous and nondecreasing with respect to second argument. Let $1 < \alpha, \beta \leq 2$. Assume that there exists $\lambda \in [0, \Gamma(\alpha)\Gamma(\beta))$, for any $u, v \in [0, \infty)$ with $u \geq v$ and $t \in [0, 1]$,

$$0 \leq f(t, u) - f(t, v) \leq \lambda(u - v).$$

Then the problem

$$\begin{cases} D_{0+}^{\beta}(D_{0+}^{\alpha}u(t)) = f(t, u(t)), \\ u(0) = u(1) = (D_{0+}^{\alpha}u)(0) = (D_{0+}^{\alpha}u)(1) = 0 \end{cases} \quad (4)$$

has a unique nonnegative solution.

Proof. We first show that the unique solution of the problem (4) is

$$u(t) = \int_0^1 G_{\alpha}(t, s) \left(\int_0^1 G_{\beta}(s, r) f(r, u(r)) dr \right) ds$$

where G_{α} is the function in Lemma 5. In fact, let $y(t) = -D_{0+}^{\alpha}u(t)$. Then the problem

$$\begin{cases} D_{0+}^{\beta}(D_{0+}^{\alpha}u(t)) = f(t, u(t)), \\ (D_{0+}^{\alpha}u)(0) = (D_{0+}^{\alpha}u)(1) = 0 \end{cases}$$

is equal to the problem

$$\begin{cases} D_{0+}^{\beta}y(t) + f(t, u(t)) = 0, \\ y(0) = y(1) = 0. \end{cases}$$

By Lemma 5, we have the unique solution

$$y(t) = \int_0^1 G_{\beta}(t, s) f(s, u(s)) ds,$$

that is,

$$D_{0+}^{\alpha}u(t) + \int_0^1 G_{\beta}(t, s) f(s, u(s)) ds = 0.$$

Furthermore, by Lemma 5, the problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + \int_0^1 G_{\beta}(t, s) f(s, u(s)) ds = 0, \\ u(0) = u(1) = 0 \end{cases}$$

has the unique solution

$$u(t) = \int_0^1 G_{\alpha}(t, s) \left(\int_0^1 G_{\beta}(s, r) f(r, u(r)) dr \right) ds.$$

Let $X = \{u \in C[0, 1] \mid u(t) \geq 0\}$. Then (X, d) is a complete metric space where d is defined by $d(u, v) = \sup_{0 \leq t \leq 1} |u(t) - v(t)|$ for $u, v \in X$. We define a mapping T of X by

$$(Tu)(t) = \int_0^1 G_\alpha(t, s) \left(\int_0^1 G_\beta(s, r) f(r, u(r)) dr \right) ds$$

for $u \in X$. Using Theorem 1, we obtain the unique fixed point of T . This is the unique solution of (4). For more details, see [5]. $\square$

In the case that $\alpha = \beta = 2$ in Theorem 6, we have the following.

Corollary 7. *Let f be a mapping of $[0, 1] \times [0, \infty)$ into $[0, \infty)$ such that f is continuous and nondecreasing with respect to second argument. Assume that there exists $\lambda \in [0, 1)$, for any $u, v \in [0, \infty)$ with $u \geq v$ and $t \in [0, 1]$,*

$$0 \leq f(t, u) - f(t, v) \leq \lambda(u - v).$$

Then the problem

$$\begin{cases} u''' = f(t, u(t)), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

has a unique nonnegative solution.

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