

ON THEORIES HAVING A FINITE NUMBER OF
NON-ISOMORPHIC COUNTABLE MODELS

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§0. Introduction. In this paper we shall state some interesting facts concerning non- ω -categorical theories which have only finitely many countable models. Although many examples of such theories are known, almost all of them are essentially the same in the following sense: they are obtained from ω -categorical theories, called base theories below, by adding axioms for infinitely many constant symbols. Moreover all known base theories have the (strict) order property in the sense of [6], and so they are unstable. For example, well-known Ehrenfeucht's example which have three countable models has the theory of dense linear order as its base theory.

Many papers including [4] and [5] are motivated by the conjecture that every non- ω -categorical theory with a finite number of countable models has the (strict) order property, but this conjecture still remains open. (Of course there are partial positive solutions. For example, in [4], Pillay showed that if such a theory has few links (see [1]), then it has the strict order property.) In this paper we prove the instability of the base theory T_0 of such a theory T rather than T itself.

Our main theorem is a strengthening of the following which is also our result: if a theory T_0 is stable and κ -categorical, then T_0 cannot be extended to a theory T which has κ countable models ($1 < \kappa < \omega$), by adding axioms for new constants.

§1. Preliminaries. Our notations and conventions are fairly standard. T, T_n ($n < \omega$) will denote complete theories formulated in some countable languages. M, M_n ($n < \omega$) will denote countable models of such theories. $\bar{a}, \bar{b}, \dots$ will be used to denote finite sequences of elements in some models. Types are complete types without parameters, and will be denoted by $p, q, \dots$. $I(\omega, T)$ is the number of countable models of T .

Definition 1. A type $q(\bar{x}, \bar{y})$ is said to be an order expression if $q(\bar{x}, \bar{y})$ is principal over the first variables $\bar{x}$ and non-principal over the second variables $\bar{y}$.

Definition 2 (Benda). A type $p(\bar{x})$ is said to be a powerful type of T if every model of T which realizes it realizes every type $q(\bar{y}) \in S(T)$.

We now state some facts which are necessary for proving our results.

Fact (i). If $I(\omega, T) < \omega$, then a powerful type $p(\bar{x})$ of T exists.

Fact (ii). Let $1 < I(\omega, T) < \omega$ and $p(\bar{x}), q(\bar{y}) \in S(T)$. If $p(\bar{x})$ is a powerful type of T , then there is an order expression $r(\bar{x}, \bar{y})$ which extends $p(\bar{x})$ and $q(\bar{y})$.

Fact (iii). A theory T is ω -categorical if and only if it has only finitely many non-equivalent formulas $\varphi(\bar{x})$, for each $\bar{x}$.

Fact (i) and Fact (ii) can be obtained by easy observations (see [4] for reference). Fact (iii) can be seen, e.g. in [2].

§2. Main theorem and its corollary. We prove the following theorem which will show the difficulty in constructing a stable theory T with $1 < I(\omega, T) < \omega$, even if such a theory exists.

Theorem. Let T_i ($i < \omega$) and T be theories with the following properties:

- (i) $T_i \subseteq T_{i+1}$ for all $i < \omega$; $T = \bigcup_{i < \omega} T_i$;
- (ii) T_i is ω -categorical for all $i < \omega$;
- (iii) $1 < I(\omega, T) < \omega$.

Then T has the order property, and so T is unstable. (So some T_i is unstable.)

First we prove the following lemma:

Lemma. Let $p(\bar{x})$ be a powerful type of T with $1 < I(\omega, T) < \omega$ and $q(\bar{x}, \bar{y})$ an order expression which extends $p(\bar{x}) \cup p(\bar{y})$. Then there is a sequence $\check{\text{tp}}(\bar{a}_i, \bar{a}_{i+1}) = q(\bar{x}, \bar{y})$ for all $i < \omega$, and $\text{tp}(\bar{a}_i, \bar{a}_j)$ is an order expression iff $i < j < \omega$.

Proof. We construct two sequences $\{\bar{a}_i\}_{i < \omega}$ of realizations of p and $\{M_i\}_{i < \omega}$ of models of T such that for each $i < \omega$,

- (1) $M_i \succ M_{i+1}$;
- (2) M_i is prime over $\bar{a}_i$ (hence $\bar{a}_i \in M_i$);
- (3) $\text{tp}(\bar{a}_i, \bar{a}_{i+1}) = q(\bar{x}, \bar{y})$.

This is done inductively. Let $\bar{a}_0$ be a realization of p and M_0 a prime model over $\bar{a}_0$. (M_0 exists since $I(\omega, T) < \omega$.)

Assume that we have already defined $\{\bar{a}_i\}_{i < n}$ and $\{M_i\}_{i < n}$.

Since M_{n-1} is prime over $\bar{a}_{n-1}$ and $q(\bar{x}, \bar{y})$ is an order expression, we can choose $\bar{a}_{n-1} \in M_{n-1}$ such that $\text{tp}(\bar{a}_{n-1}, \bar{a}_n) = q(\bar{x}, \bar{y})$. Let $M_n \prec M_{n-1}$ be a prime model over $\bar{a}_n$. It is then clear that (1) - (3) are satisfied by $\{\bar{a}_i\}_{i \leq n}$ and $\{M_i\}_{i \leq n}$. Thus the construction can be carried out. We prove

that $\bar{a}_i$ $i < \omega$ has the desired properties. It is sufficient to prove that $\text{tp}(\bar{a}_i, \bar{a}_j)$ is an order expression if $i < j$.

Let $i < j$. Then clearly $q_{i,j}(\bar{x}, \bar{y}) = \text{tp}(\bar{a}_i, \bar{a}_j)$ is principal over $p(\bar{x})$. So we only have to show that $q_{i,j}(\bar{x}, \bar{y})$ is non-principal over $p(\bar{y})$. But this is clear, since $\bar{a}_{j-1}$ is prime over $\bar{a}_i \wedge \bar{a}_j$ and $\text{tp}(\bar{a}_{j-1}, \bar{a}_j)$ is an order expression.

Proof of Theorem. By Fact (i) and Fact (ii), we can choose a powerful type $p(\bar{x})$ and an order expression $q(\bar{x}, \bar{y})$ which extends $p(\bar{x}) \cup p(\bar{y})$. So by Lemma, there is a sequence $\{\bar{a}_i\}_{i < \omega}$ of realizations of $p(\bar{x})$ such that all $tp(\bar{a}_i, \bar{a}_j)$ ($i < j < \omega$) are order expressions and all $\bar{a}_i \wedge \bar{a}_{i+1}$ ($i < \omega$) realizes the same type $q(\bar{x}, \bar{y})$. Choose a number $m < \omega$ and a formula $\varphi(\bar{x}, \bar{y}) \in L(T_m)$ such that $p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{y})\}$ proves $q(\bar{x}, \bar{y})$. For each $i < \omega$, let $\varphi_i(\bar{x}, \bar{y})$ be the formula $\exists \bar{x}_0, \dots, \bar{x}_{i-1} [\varphi(\bar{x}, \bar{x}_0) \wedge \varphi(\bar{x}_0, \bar{x}_1) \wedge \dots \wedge \varphi(\bar{x}_{i-1}, \bar{y})]$. Since each φ_i is an $L(T_m)$ -formula and T_m is ω -categorical, by Fact (iii), there are only finitely many non-equivalent formulas in $\{\varphi_i\}_{i < \omega}$. So $\Psi(\bar{x}, \bar{y}) = \bigvee_{i < \omega} \varphi_i(\bar{x}, \bar{y})$ is a first order formula. Now it is a routine to check that, for all $i, j < \omega$, $\Psi(\bar{a}_i, \bar{a}_j)$ holds in M_0 iff $i < j$. Thus T has the order property.

Corollary. Let T_0 be stable and ω -categorical. Let T be an extension of T_0 obtained by the addition of axioms for new constant symbols. Then $I(\omega, T) = 1$ or $I(\omega, T) \geq \omega$.

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