

Indices of vectorfields and Nash blowup

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Introduction

The aim of this lecture is to explicit the behaviour of the index of vectorfields through the Nash procedure. We will consider vectorfields tangent to the strata of a Whitney stratification of an analytic variety X , imbedded in a smooth one M . They will be suitable continuous sections of the tangent bundle of M restricted to X and will be called radial vectorfields. Their construction was given by M.H. Schwartz [S2]. The main property is a proportionality theorem which is an important step in the proof of equality of M.H. Schwartz and R. MacPherson classes of singular algebraic complex varieties. This result was given in [BS] as an intermediary result. Another application, given in [S2], is the Poincaré-Hopf theorem for (real) singular analytic varieties.

1. Stratified vectorfields.

Let X denote a real analytic variety of dimension n , stratified with respect to Whitney conditions. This means that :

- (1) X is union of a locally finite collection of disjoint locally closed subsets V_i , such that $V_i \cap \overline{V_j} \neq \emptyset \Leftrightarrow V_i \subset \overline{V_j}$,
- (2) the subsets $V_i = V_i^s$ are smooth manifolds of dimension s , called the strata,
- (3) whenever $V_i \subset \overline{V_j}$ then the pair satisfies the Whitney conditions :
 - (a) suppose (x_n) is a sequence of points in V_j converging to $y \in V_i$ and suppose that the sequence of tangent spaces $T_{x_n}(V_j)$ converges to some limit L . Then $T_y(V_i) \subset L$.
 - (b) suppose (x_n) is a sequence of points in V_j converging to $y \in V_i$ and (y_n) is a sequence of points in V_i also converging to y , suppose that the sequence of tangent spaces $T_{x_n}(V_j)$ converges to some limit L and that the sequence of secant lines $\overline{x_n y_n}$ converges to some limit λ . Then $\lambda \subset L$.

We will assume that X is imbedded in an analytic manifold M of dimension m . Then, given a Whitney stratification of X , there is a Whitney stratification of M adding $M - X$ as supplementary stratum. Let us denote by $T(M)$ the tangent bundle to M and $\pi : T(M) \rightarrow M$ the natural projection. Let A be a subset of M , the subspace of $T(M)|_A$, union of all restrictions $T(V_i \cap A)|_A$ will be denoted by

$$E(A) = \bigcup_{x \in V_i \cap A} T_x(V_i)$$

Definition. A stratified vectorfield v on a subset $A \subset M$ is a section of $T(M)$ defined on A and such that if $x \in V_i \cap A$, then $v(x) \in E(x) = T_x(V_i)$.

Suppose a is an isolated singular point of the stratified vectorfield v defined on a neighborhood of a in M . There are two well defined indices : the index of v as a section of $T(V_i)$, denoted by $I(v|_{V_i}, a)$ and the index of v as a section of $T(M)$, denoted by $I(v, a)$. In general, these indices do not agree.

2. Radial vectorfields.

In [S2], M.H. Schwartz proved existence of so-called radial vectorfields. They are special cases of stratified vectorfields with isolated singular points, satisfying the following main property : If $a \in V_i$ is an isolated singular point of a radial vectorfield v , then the index of v as a section of $T(V_i)$ and the index of v as a section of $T(M)$ agree.

In this section, we recall the construction of radial vectorfields. They will be obtained as a sum of two extensions :

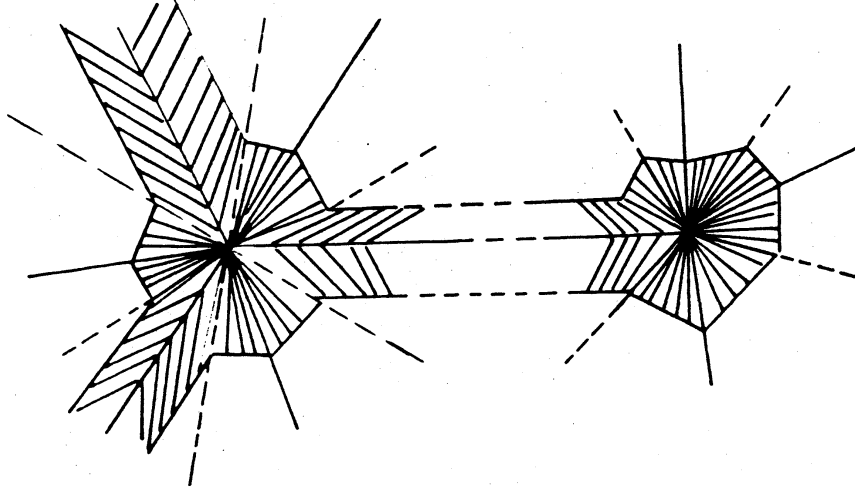
Using the Whitney condition (a) we can extend a stratified vectorfield v , given on a stratum V_i , in suitable neighborhoods $\mathcal{T}_\varepsilon(V_i)$ as a "parallel extension" of v .

Using the Whitney condition (b), we can define a "transverse" vectorfield to every stratum V_i . This vectorfield is tangent to the geodesic arcs issued from V_i (relatively to a given Riemannian metric).

a) The parallel extension.

Let us denote by (K) a triangulation of M compatible with the stratification and denote by σ^q an (open) q -simplex in (K) . Fix $\varepsilon > 0$, for every $y \in \sigma^q$, we will denote by $\mathcal{T}_\varepsilon(y)$ the set of points of the (open) star of σ^q whose barycentric coordinates relatively to vertices of σ^q are proportional to those of y with ratio $\geq 1 - \varepsilon$. The set $\mathcal{T}_\varepsilon(y)$ is $(m - q)$ -dimensional and it admits a natural partition in radii.

If A is a closed subset of V_i^s , we will write $T_\varepsilon(A) = \bigcup_{y \in A} T_\varepsilon(y)$.



The tube $T_\varepsilon(A)$ with radii.

Consider a stratified vectorfield v defined on A , we will construct an extension v' of v on $T_\varepsilon(A)$ as following : Let us denote by U_k the open star of the vertex $a_k \in V_i^s$ relatively to (K) and by $\{\phi_k\}$ a partition of unity associated to the covering of V_i^s by the subsets $U_k \cap V_i^s$. The Whitney condition (a) shows that, for every k , there is a continuous map $\Psi_k: U_k \times \mathbf{R}^s \rightarrow E(U_k)$ whose restriction to $y \in U_k \cap V_i^s$ is a complex linear isomorphism $\Psi_k: \{y\} \times \mathbf{R}^s \rightarrow E(y)$.

Every point x of $T_\varepsilon(A) - A$ belongs to an unique radius issued from a point $y \in A$. For every k , such that $y \in U_k \cap A$, there is an unique vector $v^{(k)}(x)$ in $E(x)$ such that $p_2 \circ \Psi_k^{-1}(v(y)) = p_2 \circ \Psi_k^{-1}(v^{(k)}(x))$ where $p_2: U_k \times \mathbf{R}^s \rightarrow \mathbf{R}^s$ is the second projection.

The parallel extension v' of v is defined by :

$$v'(x) = \sum_k \phi_k(y) v^{(k)}(x)$$

It satisfies :

Lemma. [S2] Let v be a section of E over a closed subset $A \subset V_i$, then :

a) for every $\varepsilon > 0$, the parallel extension v' of v is a section of E on $T_\varepsilon(A)$. If v is non zero, then v' is also non zero,

b) if two sections of E are homotopic on $A \subset V_i$, then their parallel extensions are also homotopic on $T_\varepsilon(A)$.

b) *The transversal vectorfield.*

Fix a closed subset $A \subset V_i^s$, we choose a Riemannian metric and a function μ of class $\mathcal{C}^2$ on an open neighborhood of A in M . Fix $\eta > 0$, for every point $y \in A$, we call $\Theta_\eta(y)$ the $(m-s)$ -disc whose radii are the geodesic arcs issued from y , orthogonal to V_i and with common length $\mu(y) = \eta$. Write $\Theta_\eta(A) = \bigcup_{y \in A} \Theta_\eta(y)$ where η is so small that all the discs $\Theta_\eta(y)$ are disjoint.

The transversal vectorfield is constructed using the canonical vectorfield $g(x) = \overrightarrow{\text{grad } x}$ tangent to the radii of $\Theta_\eta(A)$. This vectorfield is not tangent to the strata and its projection $g'(x)$ on E does not define a continuous vectorfield.

To obtain a continuous stratified vectorfield $w(x)$, we proceed by induction on the dimension of the strata, as following : Let V_j be a stratum such that $V_i \subset \overline{V_j}$. Suppose that $w = w^{(j)}$ is already known on V_j and suppose V_k is a stratum such that $V_i \subset \overline{V_j} \subset \overline{V_k}$. We may consider, in $\Theta_\eta(A)$, a neighborhood P of V_j in $\overline{V_k}$ and whose radii are those of $\mathcal{T}_\varepsilon(V_j)$ in a neighborhood of V_i . Fix a radius $]y, z]$ of P such that $y \in V_j$ and $z \in V_k$; for every point $x \in]y, z]$, we define :

$$w(x) = (1 - \lambda(x))w^{(j)}(x) + \lambda(x)g'(x)$$

where $\lambda(x) = \frac{\overline{yx}}{\overline{yz}}$ and $w^{(j)}$ is the field w already built on V_j . The vectorfield w is called transversal vectorfield. Obviously, $w(x)$ is $w^{(j)}(x)$ on V_j and $g'(x)$ on $V_k - P$.

The Whitney condition (b) shows that :

Lemma. [S2] If A is a compact subset of V_i^s , for every fixed $\varepsilon > 0$, there are a riemannian metric μ on M , a tubular neighborhood $\Theta_\eta(A)$ of A such that the transversal vectorfield w in $\Theta_\eta(A)$ satisfies :

(a) for every point $x \in \Theta_\eta(A)$, we have : $\text{angle } \langle w(x), g(x) \rangle < \varepsilon$,

(b) the transversal vectorfield w is a section of E defined as an extension of the zero section on A . It does not have zeroes in $\Theta_\eta(A) - A$ and it points outward of $\Theta_\eta(A)$ on $\partial\Theta_\eta(A) - \Theta_\eta(\partial A)$.

c) *The local radial extension of a vectorfield.*

Let A denote a closed subset of V_i and A' a closed neighborhood of A in V_i such that, for $\varepsilon > 0$ and $\eta' > 0$, we have $\mathcal{T}_\varepsilon(A) \subset \Theta_{\eta'}(A')$. Consider a section v of $T(V_i)$ over A , such that $\|v\| < 1$, relatively to the previous Riemannian metric. We can define a parallel extension v' of v in $\mathcal{T}_\varepsilon(A)$ and a transversal vectorfield w in $\Theta_{\eta'}(A')$.

Following MH Schwartz [S2], we denote by v^{rad} and we call radial vectorfield the extension $v^{rad} = v' + w$. The radial vectorfield satisfies the following properties :

Proposition. [S2] If $\eta > 0$ and $\varepsilon > 0$ are sufficiently small, the radial vectorfield v^{rad} defined in $\mathcal{T}_\varepsilon(A)$ satisfies :

i) if $B \subset A$, the vectorfield v^{rad} points outward of every tube $\Theta_\eta(B) \subset \mathcal{T}_\varepsilon(A)$ on $\partial\Theta_\eta(B) - \Theta_\eta(\partial B)$,

ii) if a point $a \in B \subset V_i$ is an isolated singularity of v , it is also an isolated singularity of v^{rad} and v^{rad} satisfies the “conservation of indices” property :

$$I(v^{rad}, a) = I(v^{rad}|_{V_i}, a) \quad \text{with } v^{rad}|_{V_i} = v$$

iii) if two sections v_0 and v_1 of $T(V_i)|_B$ are homotopic, then their radial extensions v_0^{rad} and v_1^{rad} are also homotopic on $\Theta_\eta(B)$.

3. Poincaré-Hopf theorems.

The radial vectorfields are the good ones to recover a Poincaré-Hopf theorem for singular varieties. Let D be a compact subset in M such that ∂D is smooth and transverse to all strata V_i . Let w be a stratified vectorfield pointing outward of $D \cap X$ on $\partial D \cap X$, without singularity on $\partial D \cap X$ and with isolated singularities $a_k \in V_{i(a_k)} \cap D$, where $V_{i(a_k)}$ denotes the stratum containing the point a_k . For such a vectorfield, and using the usual definition of index, the Poincaré-Hopf theorem is false in general, i.e.

$$\sum_{a_k \in X \cap D} I(w|_{V_{i(a_k)}}, a_k) \neq \chi(X \cap D)$$

It is easy to construct such an example on the pinched torus, see also the example in [S2], 6.2.1.

To recover the Poincaré-Hopf theorem, one has either to use the radial vectorfields with the usual definition of index, or to modify the definition of index.

Theorem. [S2] (Generalization of the Poincaré-Hopf theorem). Let X be a Whitney stratified analytic variety imbedded in an analytic manifold M . Let D be a compact subset of M with smooth boundary, transverse to the strata. Let v^{rad} be a radial vectorfield pointing outward of $D \cap X$ on $\partial D \cap X$ and with isolated singularities $a_k \in V_{i(a_k)} \cap D$, then :

$$\sum_{a_k \in X \cap D} I(v^{rad}|_{V_{i(a_k)}}, a_k) = \chi(X \cap D)$$

where, if $\dim V_{i(a_k)} = 0$ then $I(v^{rad}|_{V_{i(a_k)}}, a_k) = +1$.

On the same way, we can construct a radial vectorfield v^{-rad} pointing inward of $D \cap X$ on $\partial D \cap X$ and with isolated singularities $a_h \in V_{i(a_h)} \cap D$.

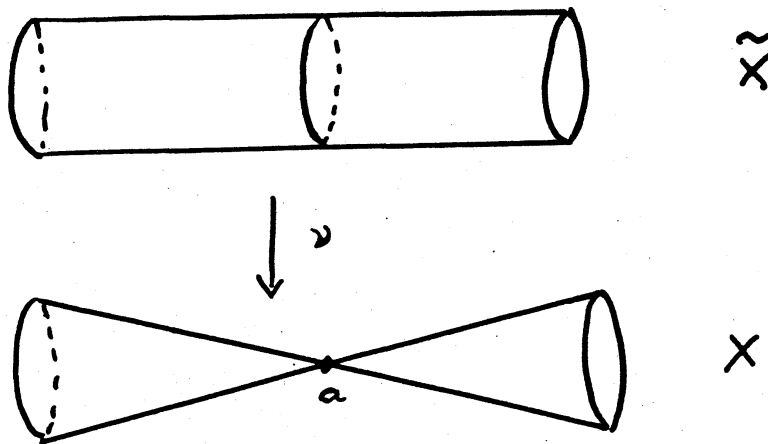
Theorem. [S2] Under the same hypotheses but if v^{-rad} points inward of $D \cap X$ on $\partial D \cap X$ then :

$$\begin{aligned} \sum_{a_h \in X \cap D} I(v^{-rad}|_{V_{i(a_h)}}, a_h) &= \chi(X \cap \text{int} D) \\ &= \chi(X \cap D) - \chi(X \cap \partial D) \end{aligned}$$

We now consider the complex case. The previous constructions of radial vectorfields are also valid in the complex case, relatively to Whitney complex analytic stratifications (see [BS] and [S1]). From now on, let us denote by n, m and s the complex dimensions of X, M and V_i respectively. The tangent spaces will be complex tangent spaces.

4. Nash construction.

Let $G_n(TM)$ denote the Grassmannian bundle associated to the complex tangent bundle $\pi : TM \rightarrow M$. The fiber of $G_n(TM)$ over $x \in M$ is the set of n -complex planes in $T_x M$. The projection $\nu : G_n(TM) \rightarrow M$ has a canonical section σ over the regular part X_{reg} of X , defined by $\sigma(x) = T_x(X_{\text{reg}})$. The Nash blowup of X , denoted by $\tilde{X}$, is, by definition, the closure of $\text{Im}(\sigma)$ in $G_n(TM)$. We denote also by ν the analytic map $\tilde{X} \rightarrow X$, restriction of ν (see [McP]).



The Nash blowup of a cone

The Nash blowup of a cone is a cylinder, but in general the Nash blowup is not a smooth manifold (see [McP]).

Consider the tautological bundle ξ over $G_n(TM)$. The fiber ξ_P over an n -plane $P \in G_n(TM)$ is the set of vectors $v \in P$. We will denote by $\tilde{\xi}$ the restriction $\xi|_{\tilde{X}}$. It is a subspace of

$$\Lambda = \{(v, P) : v \in TM|_X, P \in \tilde{X} \subset G_n(TM), \pi(v) = \nu(P)\},$$

and we will use the symbol $\nu_* : \tilde{\xi} \rightarrow T(M)|_X$ to denote the restriction to $\tilde{\xi}$ of the canonical projection $\Lambda \rightarrow T(M)$.

Proposition. [BS] a) Suppose $x \in V_i \subset X$ and $v(x) \in T_x(V_i)$. For every point $\tilde{x}$ in $\nu^{-1}(x)$, there is a unique vector $\tilde{v}(\tilde{x})$ in $\tilde{\xi}(\tilde{x})$ such that $\nu_*(\tilde{v}(\tilde{x})) = v(x)$.

b) If v is a stratified vectorfield on $A \subset X$, then $\tilde{v}$ defines a section of $\tilde{\pi} : \tilde{\xi} \rightarrow \tilde{X}$ over $\tilde{A} = \nu^{-1}(A)$, called the lifting of v .

Proof. We first consider the case $x \in X_{\text{reg}}$, then $\tilde{x} = T_x(X_{\text{reg}})$ is the unique point of $\nu^{-1}(x)$ and the pair $\tilde{v}(\tilde{x}) = (v(x), \tilde{x})$ is a well defined element of $\tilde{\xi}(\tilde{x})$. Suppose now $x \in V_i$ and $v(x) \in T_x(V_i)$. For every point $\tilde{x} \in \nu^{-1}(x)$, there exists a sequence $(\tilde{x}_n) \in \tilde{X}$ converging to $\tilde{x}$ and such that $\nu(\tilde{x}_n) = x_n$ is a regular point of X . The sequence (x_n) converges to x and the limit $L = \lim T_{x_n}(X_{\text{reg}})$ exists and is identified with $\tilde{x}$. The Whitney condition (a) implies that $T_x(V_i) \subset L$ and then $v(x) \in T_x(V_i) \subset \tilde{x}$. The pair $\tilde{v}(\tilde{x}) = (v(x), \tilde{x})$ is a well defined element of $\tilde{\xi}(\tilde{x})$. This proves the part a). The part b) is obvious.

5. Euler obstruction.

Fix a point a in a stratum V_i , we can suppose that the restriction of M to a neighborhood U_a of a is an open subset in $\mathbb{C}^m$ and $V_i \cap U_a$ is an open subset in $\mathbb{C}^s$. Let us denote by b the (Euclidean) ball with center a and (sufficiently small) radius r . The geodesic tube $\Theta = \Theta_\eta(b)$ defined above is transverse to all strata V_j such that $V_i \subset \overline{V_j}$.

Definition. ([McP],[BS]) Let v_0 be a vectorfield tangent to V_i , pointing outward on ∂b , such that a is a unique isolated singular point of v inside b . Let v_0^{rad} be the restriction to $\partial\Theta(b)$ of the radial extension of v_0 and $\widetilde{v_0^{\text{rad}}}$ the lifting of v_0^{rad} as a section of $\tilde{\xi}$ over $\nu^{-1}(\partial\Theta)$. The obstruction to the extension of $\widetilde{v_0^{\text{rad}}}$ inside $\nu^{-1}(\Theta)$ as a non zero section of $\tilde{\xi}$ is called local Euler obstruction of X in a and is denoted $\text{Eu}_a(X)$. It is independent of the choices of v_0, b, η .

If a is a regular point of X then $\text{Eu}_a(X) = 1$.

Proposition. [BS] The local Euler obstruction is constant along each stratum of a Whitney stratification of X . We say that the local Euler obstruction is a constructible function.

Proposition. [BS] (Multiplicativity property) Let v be a radial vectorfield with index $I(v, a)$ in an isolated singularity $a \in V_i$. Let b be a ball, with center a and radius so small that ∂b is transverse to all strata V_j such that $V_i \subset \overline{V_j}$. If v is non zero on ∂b , then the lifting $\tilde{v}$ is a well defined section of $\tilde{\xi}$ on $\nu^{-1}(\partial b \cap X)$. The obstruction to the extension of $\tilde{v}$ as a non zero section of $\tilde{\xi}$ on $\nu^{-1}(b \cap X)$ is given by :

$$\text{Obs}(\tilde{v}, \tilde{\xi}, b) = \text{Eu}_a(X) \times I(v, a).$$

The local Euler obstruction is one of the main tool in the construction of Chern classes of algebraic varieties (with singularities), due to R. MacPherson [McP]. These classes are the same, via an Alexander isomorphism, than the classes previously defined by M.H. Schwartz in 1965, using the obstruction theory [S1]. The multiplicativity property is one of the main steps for the demonstration of the equality of these classes (see [BS]).

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