

Striped structures of stable and unstable sets
of expansive homeomorphisms and a theorem
of K. Kuratowski on independent sets

広島大学 加藤久男 (Hisao Kato)

1. Introduction.

All spaces under consideration are assumed to be metric. By a *compactum*, we mean a compact metric space, and by a *continuum*, a connected nondegenerate compactum. A homeomorphism $f: X \rightarrow X$ of a compactum X is called *expansive* if there is a constant $c > 0$ (called an *expansive constant for f*) such that if $x, y \in X$ and $x \neq y$, then there is an integer $n = n(x, y) \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

This property has frequent applications in topological dynamics, ergodic theory and continuum theory [1,3,7,8].

A homeomorphism $f: X \rightarrow X$ of a compactum X is *continuum-wise expansive* if there is a constant $c > 0$ such that if A is a nondegenerate subcontinuum of X , then there is an integer $n = n(A) \in \mathbb{Z}$ such that $\text{diam } f^n(A) > c$. By definitions, we can easily see that every expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many important examples of homeomorphisms which are continuum-wise expansive homeomorphisms, but not expansive homeomorphisms.

In this note, we show that if $f: X \rightarrow X$ is an expansive

homeomorphism of a compactum X with $\dim X > 0$, then the decompositions $\{W^S(x) | x \in X\}$ and $\{W^u(x) | x \in X\}$ of X to stable and unstable sets are uncountable respectively, and moreover there is σ ($\sigma = s$ or u) and a positive number $\rho > 0$ such that the σ -striped set $Z(\sigma, \rho)$ of f is not empty. Hence, by using a theorem of K. Kuratowski on independent sets [6], it is proved that almost every Cantor set C of $Z(\sigma, \rho)$ satisfies the property that for each $x \in C$, $W^\sigma(x)$ contains a nondegenerate subcontinuum containing x and if $x, y \in C$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$. Also, we show that if $f: G \rightarrow G$ is a map of a graph G and the shift map $\tilde{f}: (G, f) \rightarrow (G, f)$ of f is expansive, then for each $\tilde{x} \in (G, f)$, $W^u(\tilde{x})$ is equal to the arc-component of (G, f) containing $\tilde{x}$, and $W^s(\tilde{x})$ is 0-dimensional.

2. Definitions and preliminaries.

Let $f: X \rightarrow X$ be a homeomorphism of a compactum X and let $x \in X$. Then the *stable set* $W^S(x)$ and the *unstable set* $W^u(x)$ are defined as follows:

$$W^S(x) = \{y \in X | \lim_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0\},$$

$$W^u(x) = \{y \in X | \lim_{n \rightarrow \infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

Also, the *continuum-wise stable* and *unstable sets* $V^S(x)$, $V^u(x)$ are defined as follows:

$V^s(x) = \{y \in X \mid \text{there is } A \in C(X) \text{ such that } x, y \in A \text{ and } \lim_{n \rightarrow \infty} \text{diam } f^n(A) = 0\},$

$V^u(x) = \{y \in X \mid \text{there is } A \in C(X) \text{ such that } x, y \in A \text{ and } \lim_{n \rightarrow \infty} \text{diam } f^{-n}(A) = 0\}.$

Clearly, $W^\sigma(x) \supset V^\sigma(x)$, $\{W^\sigma(x) \mid x \in X\}$ and $\{V^\sigma(x) \mid x \in X\}$ are decompositions of X for each $\sigma = s$ and u , i.e.,

$X = \cup \{W^\sigma(x) \mid x \in X\}$ (resp. $X = \cup \{V^\sigma(x) \mid x \in X\}$), and if $W^\sigma(x) \neq W^\sigma(y)$ (resp. $V^\sigma(x) \neq V^\sigma(y)$), then $W^\sigma(x) \cap W^\sigma(y) = \emptyset$ (resp. $V^\sigma(x) \cap V^\sigma(y) = \emptyset$).

We are interested in the structures of the decompositions $\{W^\sigma(x) \mid x \in X\}$ and $\{V^\sigma(x) \mid x \in X\}$ ($\sigma = s$ and u) of X . Let $f: X \rightarrow X$ be a homeomorphism of a compactum X with $\dim X > 0$. Let $\rho > 0$ be a positive number. Consider the family $\Phi(\sigma) = \{Z \mid Z \text{ is a closed subset of } X \text{ satisfying that (i) for each } x \in Z \text{ there is a subcontinuum } A_x \text{ of } X \text{ such that } \text{diam } A_x \geq \rho, x \in A_x \subset W^\sigma(x), \text{ and (ii) for any neighborhood } U \text{ of } x \text{ in } X, \text{ there is } y \in Z \cap U \text{ such that } W^\sigma(x) \neq W^\sigma(y)\}$. Clearly, $\Phi(\sigma)$ has the maximal element $Z(\sigma, \rho)$ ($= \text{Cl}(\cup \{Z \mid Z \in \Phi(\sigma)\})$). The set $Z(\sigma, \rho)$ is said to be a σ -striped set of f . Note that if $0 < \rho_1 < \rho_2$, then $Z(\sigma, \rho_1) \supset Z(\sigma, \rho_2)$. Also, note that if $Z(\sigma, \rho) \neq \emptyset$ for some $\rho > 0$, then X contains an uncountable collection of mutually disjoint, nondegenerate subcontinua of X each of which is contained in a different element of $\{W^\sigma(x) \mid x \in X\}$.

Let $f: X \rightarrow X$ be a map of a compactum X with metric d . Consider the following inverse limit space:

$$(X, f) = \{(x_i)_{i=0}^{\infty} \mid x_i \in X, f(x_{i+1}) = x_i \text{ for each } i \geq 0\}.$$

Define a metric $\tilde{d}$ for (X, f) by

$$\begin{aligned} \tilde{d}(\tilde{x}, \tilde{y}) &= \sum_{i=0}^{\infty} d(x_i, y_i) / 2^i \text{ for } \tilde{x} = (x_i)_{i=0}^{\infty}, \\ \tilde{y} &= (y_i)_{i=0}^{\infty} \in (X, f). \end{aligned}$$

The space (X, f) is called the *inverse limit of the map f* . Define a map $\tilde{f}: (X, f) \rightarrow (X, f)$ by

$$\tilde{f}(x_0, x_1, \dots) = (f(x_0), x_0, x_1, \dots), \text{ for } (x_i)_{i=0}^{\infty} \in (X, f).$$

Then the map $\tilde{f}$ is a homeomorphism and it is called the *shift map of f* .

(2.1) Example. Let S^1 be the unit circle and let $f: S^1 \rightarrow S^1$ be the natural covering map with degree 2. Consider the inverse limit (S^1, f) of f and the shift map $\tilde{f}: (S^1, f) \rightarrow (S^1, f)$. The continuum (S^1, f) is well-known as the 2-adic solenoid and $\tilde{f}$ is an expansive homeomorphism. In this case, for each $\tilde{x} \in (S^1, f)$, $W^u(\tilde{x}) = V^u(\tilde{x})$ is the arc-component of (S^1, f) containing $\tilde{x}$. Also, $V^s(\tilde{x}) = \{\tilde{x}\} \subsetneq W^s(\tilde{x})$ for each $\tilde{x} \in (S^1, f)$. Then the decomposition $\{W^{\sigma}(\tilde{x}) \mid \tilde{x} \in (S^1, f)\}$ ($\sigma = s$ and u) is uncountable.

Note that $\dim W^S(\tilde{x}) = 0$, because $W^S(\tilde{x})$ is an F_σ -set and $W^S(\tilde{x})$ does not contain a nondegenerate subcontinuum.

Note that the continuum (S^1, f) itself is a u-stripped set $Z(\sigma, \rho)$ of $\tilde{f}$ for some $\rho > 0$, but $Z(s, \rho) = \emptyset$ for each $\rho > 0$.

(2.2) Example. There is an expansive homeomorphism $f: X \rightarrow X$ such that $\text{Int}_X W^\sigma(x) \neq \emptyset$ for some $x \in X$. Let G be the one point union of the unit interval I and a circle S^1 , i.e., $(G, *) = (I, 1) \vee (S^1, *)$. Define a map $g: G \rightarrow G$ such that $g|_{S^1}: S^1 \rightarrow S^1$ is the natural covering map with degree 2 and $g(0)=0$, $g(1) = *$ and $g(I) = G$. We can choose $g: G \rightarrow G$ so that $\tilde{g}: X=(G, g) \rightarrow X=(G, g)$ is expansive. Then $W^u(\tilde{\theta})$ is a dense open set of X , where $\tilde{\theta} = (0, 0, \dots)$. Hence X itself is not a u-stripped set of $\tilde{g}$.

A subset E of a space X is called to be an F_σ -set in X if E is a union of countable closed subsets F_n of X , i.e., $E = \bigcup_{n=1}^{\infty} F_n$. A subset E of X is called to be an $F_{\sigma\delta}$ -set in X if E is an intersection of countable F_σ -sets E_n , i.e., $E = \bigcap_{n=1}^{\infty} E_n$.

We use a theorem of K. Kuratowski on independent sets [6]. A subset F of X is said to be *independent in* $R \subset X^n$, if for every system $x_1, x_2, \dots, x_n$ of different points of F the point $(x_1, x_2, \dots, x_n) \in F^n$ never belongs to R . In [6], K. Kuratowski proved the following theorem.

(2.3) Theorem ([6, Main theorem and Corollary 3]).

If X is a complete space and $R \subset X^n$ is an F_σ -set of the first category, then the set $J(R)$ of all compact subsets F of X independent in R is a dense G_δ -set in 2^X of all compact subsets of X . Moreover, if X has no isolated points, then almost every Cantor set of X is independent in R .

For the proof of the main theorem of this note, we need the following.

(2.4) Proposition. *Let $f: X \rightarrow X$ be a homeomorphism of a compactum X . Then $W^\sigma(x)$ is an $F_{\sigma\delta}$ -set in X ($\sigma = s, u$).*

(2.5) Proposition. *Let $f: X \rightarrow X$ be an expansive homeomorphism of a compactum X . Then $W^\sigma(x)$ is an F_σ -set in X ($\sigma = s, u$).*

(2.6) Proposition. *Let $f: X \rightarrow X$ be a continuum-wise expansive homeomorphism of a compactum X . Then $V^\sigma(x)$ is an F_σ -set in X ($\sigma = s, u$).*

3. Striped structures of stable and unstable sets.

In this section, we study striped structures of stable and unstable sets of expansive homeomorphisms and continuum-wise expansive homeomorphisms. The main result of this section is the following theorem.

(3.1) Theorem. *Let $f: X \rightarrow X$ be an expansive homeomorphism of a compactum X with $\dim X > 0$. Then the decomposition $\{W^\sigma(x) | x \in X\}$ ($\sigma = s$ and u) of X is uncountable. Moreover, there exists σ ($\sigma = s$ or u) and a positive number $\rho > 0$ such that the σ -striped set $Z(\sigma, \rho)$ is not empty. In particular, almost every Cantor set C of $Z(\sigma, \rho)$ satisfies the property that for any $x \in C$, there exists a nondegenerate subcontinuum A_x of X such that $x \in A_x \subset W^\sigma(x)$, and if $x, y \in C$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$.*

To prove (3.1), we need the following facts. The next lemma is obvious.

(3.2) Lemma. *Let $f: X \rightarrow X$ be a map of a compactum X and let $N \geq 1$ be a natural number. Suppose that there is $\gamma > 0$ such that $d(f^{iN}(x), f^{iN}(y)) \geq \gamma$ for each $i = 0, 1, 2, \dots$. Then there is a positive number $\eta > 0$ such that $d(f^i(x), f^i(y)) \geq \eta$ for each $i = 0, 1, 2, \dots$.*

(3.3) Lemma ([4, (2.3)]). *Let $f: X \rightarrow X$ be a continuum-wise expansive homeomorphism of a compactum X with an expansive constant $c > 0$ and let $0 < \varepsilon < c/2$. Then there is $\delta > 0$ such that if A is any nondegenerate subcontinuum of X such that $\text{diam } A \leq \delta$ and $\text{diam } f^m(A) \geq \varepsilon$ for some integer $m \in \mathbb{Z}$, then one of the following conditions holds:*

(a) If $m \geq 0$, then $\text{diam } f^n(A) \geq \delta$ for each $n \geq m$.

More precisely, there is a subcontinuum B of A such that $\text{diam } f^j(B) \leq \varepsilon$ for $0 \leq j \leq n$ and $\text{diam } f^n(B) = \delta$.

(b) If $m < 0$, then $\text{diam } f^{-n}(A) \geq \delta$ for each $n \geq -m$.

More precisely, there is a subcontinuum B of A such that $\text{diam } f^{-j}(B) \leq \varepsilon$ for $0 \leq j \leq n$ and $\text{diam } f^{-n}(B) = \delta$.

(3.4) Lemma ([4, (2.4)]). Let f , c , ε , δ be as in (3.3). Then for any $\gamma > 0$, there is $N > 0$ such that if $A \in C(X)$ and $\text{diam } A \geq \gamma$, then $\text{diam } f^n(A) \geq \delta$ for each $n \geq N$ or $\text{diam } f^{-n}(A) \geq \delta$ for each $n \geq N$.

For the case of continuum-wise expansive homeomorphism, we have

(3.5) Theorem. Let $f: X \rightarrow X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. Then the decompositions $\{V^\sigma(x) | x \in X\}$ ($\sigma = s$ and u) are uncountable. Moreover, there is σ ($\sigma = s$ or u) and a positive number $\rho > 0$ such that there is a nonempty closed set Z' of X satisfying that (i) for each $x \in Z'$ there is a subcontinuum A_x of X satisfying that $\text{diam } A_x \geq \rho$, $x \in A_x \subset V^\sigma(x)$, (ii) for any neighborhood U of x in X , there is $y \in Z' \cap U$ such that $V^\sigma(x) \neq V^\sigma(y)$. In particular, almost every Cantor set C of $Z(\sigma)$ satisfies the property that for any $x \in C$, there is a nondegenerate subcontinuum A_x of X with $x \in A_x \subset V^\sigma(x)$, and if $x, y \in C$ and $x \neq y$, then

$$V^\sigma(x) \neq V^\sigma(y).$$

(3.6) Theorem. *Let X be a locally connected continuum (= Peano continuum). If $f: X \rightarrow X$ is an expansive homeomorphism (resp. a continuum-wise expansive homeomorphism) of X , then there is an uncountable subset Z of X such that $\text{Cl}(Z) = X$, and (1) for each $x \in Z$ and $\sigma = s$ and u , there is a nondegenerate subcontinuum $A_x \in V^\sigma$ with $x \in A_x$ and $\text{diam } A_x \geq \delta$ for some $\delta > 0$, (2) if $x, y \in Z$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$ (resp. $V^\sigma(x) \neq V^\sigma(y)$) for each $\sigma = s$ and u .*

To prove (3.6), we need the following.

(3.7) Lemma ([5, (1.6)]). *Let $f: X \rightarrow X$ be a continuum-wise expansive homeomorphism of a Peano continuum X . Then there is $\delta > 0$ such that for each $x \in X$, there are two subcontinua A_x and B_x such that $x \in A_x \cap B_x$, $A_x \in V^s$, $B_x \in V^u$, $\text{diam } A_x = \delta$ and $\text{diam } B_x = \delta$. In particular, $\text{Int}_X(W^\sigma(x)) = \emptyset$ for each $x \in X$ and $\sigma = s, u$.*

For the case of inverse limits of graphs, we have the following theorem.

(3.8) Theorem. *Let $f: G \rightarrow G$ be a map of a graph G (= finite connected 1-dimensional polyhedron). Suppose that the shift map $\tilde{f}: (G, f) \rightarrow (G, f)$ is expansive. Then for each $\tilde{x} \in (G, f)$, (a) $W^u(\tilde{x})$ is equal to the arc-component $A(\tilde{x})$ of $X = (G, f)$*

containing $\tilde{x}$, and (b) $W^S(\tilde{x})$ is 0-dimensional.

To prove (3.8), we need the following notations.

Let A be a closed subset of a compactum X . A map $f: X \rightarrow X$ is called *positively expansive on A* if there is a positive number $c > 0$ such that if $x, y \in A$ and $x \neq y$, then there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) > c$. If a map $f: X \rightarrow X$ is positively expansive on the total space X , we say f is *positively expansive*. Let $\mathcal{A}$ be a finite closed covering of X . A map $f: X \rightarrow X$ is *positively pseudo-expansive with respect to $\mathcal{A}$* if the following conditions hold:

(P₁) f is positively expansive on A for each $A \in \mathcal{A}$.

(P₂) For each $A, B \in \mathcal{A}$ with $A \cap B \neq \emptyset$, one of the following two conditions holds: (*) f is positively expansive on $A \cup B$. (**) If f is not positively expansive on $A \cup B$, then there is a natural number $k \geq 1$ such that for any $A', A'' \in \mathcal{A}$ with $A' \cap A'' \neq \emptyset$, $f^k(A' \cup A'') \cap (A - B) = \emptyset$ or $f^k(A' \cup A'') \cap (B - A) = \emptyset$.

(3.9) Theorem. Let G be a graph and let $f: G \rightarrow G$ be an onto map. Then the shift map $\tilde{f}: (G, f) \rightarrow (G, f)$ is expansive if and only if f is positively pseudo-expansive map with respect to $\mathcal{A}$, where $\mathcal{A} = \{e \mid e \text{ is an edge of some simplicial complex } K \text{ with } |K| = G\}$.

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