

Ascending Chain Condition for Minimal Log Discrepancies for Generalized Fano Pairs

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Introduction

Generalized pairs, as a generalization of log pairs, play an important role in Birkar's proof of the Borisov-Alexeev-Borisov conjecture (the BBAB theorem) [Birkar19, Birkar21a]. On the other hand, the ascending chain condition (ACC) for minimal log discrepancies (mld) is conjectured to hold for pairs with the coefficients of their boundaries in a fixed DCC set. The ACC conjecture and the lower semi-continuity conjecture for mlds imply the termination of flips [Shokurov].

Using the lower bound for the log canonical thresholds (lct) and the finiteness of mlds for bounded generalized pairs, we show the ACC conjecture holds for bounded generalized pairs. As a consequence, applying the BBAB theorem, the ACC for mlds for generalized Fano pairs holds.

Notation and Conventions

- We work over $\mathbb{C}$. Every variety is assumed to be normal, projective and $\mathbb{Q}$ -factorial.
- A generalized pair $(X, B + M)$ consists of a variety X , an effective divisor B on X and a b-nef b-divisor M over X such that $K_X + B + M$ is $\mathbb{R}$ -Cartier. Equivalently, M can be viewed as the image of a nef divisor M' on a birational model X' over X . We say the coefficients of M are in some set $I \subseteq \mathbb{R}_{\geq 0}$ if $M' = \sum m_i M'_i$ for some $m_i \in I$ and some Cartier divisors M'_i on X' .

BBAB Theorem (Birkar)

Let d be a natural number and let ϵ be a positive real number. Then the varieties X such that, with some generalized boundaries $B + M$,

- $(X, B + M)$ is generalized ϵ -lc of dimension d , and
- $-(K_X + B + M)$ is nef and big,

form a bounded family.

Lower bound of lct [Birkar21b]

Let d and n be natural numbers and let ϵ be a positive real number. Then there exist a positive real number t such that if $(X, B + M)$ is an ϵ -lc generalized pair of dimension d with

- $\dim X = d$,
- there is a very ample divisor A on X with $A^d \leq n$,
- $B + D + M + N$ is a generalized boundary for some effective divisor D on X and some b-nef b-divisor N over X , and
- $A - (B + D + M + N) \geq 0$,

then $(X, B + tD + M + tN)$ is generalized lc.

Finiteness of mlds [CGN]

Let d and k be natural numbers and let $I \subseteq \mathbb{R}_{\geq 0}$ be a finite set. Then the set of mlds $\{\text{mld}(X, B + M)\}$ with

- $\dim X = d$,
- the coefficients of B and M are in I , and
- kK_X is Cartier,

is finite.

Acc for mlds for generalized Fano pairs

Let d be a natural number and let $0 \in I \subseteq \mathbb{R}_{\geq 0}$ be a DCC set. Then the set of mlds $\{\text{mld}(X, B + M)\}$ with

- $\dim X = d$,
- the coefficients of B and M in I , and
- $-(K_X + B + M)$ is nef and big,

satisfies ACC.

Sketch of Proof.

- It is enough to show this for a sequence $\{(X_i, B_i + M_i)\}_{i=1}^{\infty}$ of such generalized pairs.
- We may assume that $\{a_i := \text{mld}(X_i, B_i + M_i)\}_{i=1}^{\infty}$ is strictly increasing. We may assume that $(X_i, B_i + M_i)$ is generalized ϵ -lc for every i for some fixed positive real number ϵ .
- Applying the BBAB theorem, this then implies that the generalized pairs $(X_i, B_i + M_i)$ form a bounded family.
- There exists a positive real number t such that $(X_i, (1+t)B_i + (1+t)M_i)$ is generalized lc for each i .
- The numbers of terms in both B_i and M_i are bounded. So we may assume that $B_i = \sum_{j=1}^k b_{ij} B_{ij}$ and $M_i = \sum_{j=1}^k m_{ij} M'_{ij}$ for some real numbers $b_{ij}, m_{ij} \in I$, some integral divisors B_{ij} on X_i and some Cartier divisors M'_{ij} on a model X' over X .
- We may assume that $\{b_{ij}\}_{i=1}^{\infty}$ and $\{m_{ij}\}_{i=1}^{\infty}$ are non-decreasing for all j . Let $b_{0j} = \lim_{i \rightarrow \infty} b_{ij}$ and $m_{0j} = \lim_{i \rightarrow \infty} m_{ij}$ for each j .
- Let $\hat{B}_i = \sum_{j=1}^k b_{0j} B_{ij}$ and $\hat{M}'_i = \sum_{j=1}^k m_{0j} M'_{ij}$ for each i . Then $B_i + M_i \leq \hat{B}_i + \hat{M}_i \leq (1 + c_i t)(B_i + M_i)$ for some $c_i \geq 0$ with $\lim_{i \rightarrow \infty} c_i = 0$.
- This implies that $\text{mld}(X_i, \hat{B}_i + \hat{M}_i) \in [(1 - c_i)a_i, a_i]$.
- We may assume that $\{\text{mld}(X_i, \hat{B}_i + \hat{M}_i)\}_{i=1}^{\infty}$ is strictly increasing, which contradicts the finiteness of mlds.

□

References

- [Birkar19] C. Birkar, Anti-pluricanonical systems on Fano varieties, Ann. of Math. (2) 190 (2019), no. 2, 345–463.
[Birkar21a] C. Birkar, Singularities of linear systems and boundedness of Fano varieties, Ann. of Math. (2) 193 (2021), no. 2, 347–405.
[Birkar21b] C. Birkar, Boundedness and volume of generalised pairs, arXiv:2103.14935.
[CGN] Weichung Chen, Yoshinori Gongyo, Yusuke Nakamura, On generalized minimal log discrepancy, to appear in J. Math. Soc. Japan.
[Shokurov] V. V. Shokurov, Letters of a bi-rationalist. V. Minimal log discrepancies and termination of log flips, Tr. Mat. Inst. Steklova 246 (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 328–351; English transl., Proc. Steklov Inst. Math. 3 (246) (2004), 315–336.