

ON KNESER TYPE THEOREM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS
WITH THE PHASE SPACE $\mathcal{C}_\gamma$ IN BANACH SPACES

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§ 1. Introduction.

Let $R = (-\infty, \infty)$ and E be an infinite dimensional Banach space with norm $\|\cdot\|_E$. Let $X = E$ or R . Denote by $\mathcal{C}_\gamma^X$, $\gamma \in R$, the space of continuous functions $\psi : (-\infty, 0] \rightarrow X$ having the limit $\lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \psi(\theta)$ with the norm

$$\|\psi\|_{\mathcal{C}_\gamma^X} = \sup_{-\infty < \theta \leq 0} e^{\gamma\theta} \|\psi(\theta)\|_X.$$

If $x : (-\infty, \sigma+a) \rightarrow X$, $0 < a \leq \infty$, then for any $t \in (-\infty, \sigma+a)$ we define $x_t : (-\infty, 0] \rightarrow X$ by $x_t(\theta) = x(t+\theta)$, $-\infty < \theta \leq 0$.

The purpose of this paper is to give Kneser type theorem on the set of solutions for the Cauchy problem of the functional differential equation(FDE) with infinite delay in a Banach space (for brevity, CP(1.1)),

$$\frac{dx}{dt} = f(t, x_t), \quad x_\sigma = \varphi \in \mathcal{E}_\gamma^E, \quad (1.1)$$

under the condition that $f : [\sigma, \sigma+a] \times \mathcal{E}_\gamma^E(\varphi, r) \rightarrow E$, $\mathcal{E}_\gamma^E(\varphi, r) := \{\psi \in \mathcal{E}_\gamma^E \mid |\varphi - \psi|_{\mathcal{E}_\gamma^E} \leq r\}$, is a uniformly continuous mapping. The

argument in the proof of the main theorem (Theorem 3.4) is based on the idea in [9] and on properties of $\mathcal{E}_\gamma^E$. Our result extends the one obtained in [10] and is closely related to the one due to Kubiacyk [2] for ordinary differential equations (ODE's).

§ 2. Some Lemmas.

In this section, we shall show a differential inequality and a comparison theorem. For a continuous function $w : (a, b) \rightarrow \mathbb{R}$ and for $t \in (a, b)$, $(D_+ w)(t)$, $(D_- w)(t)$ and $(\bar{D}_+ w)(t)$ denote the right hand derivative, the left hand lower derivative and the right hand upper derivative, respectively.

Lemma 2.1. Let $w : [\sigma, \sigma+a) \rightarrow \mathbb{R}$ be a continuous function such that $(D_+ w)(t)$ exists for all $t \in [\sigma, \sigma+a)$. Then the following inequalities hold :

1)

$$\bar{D}_+ \sup_{\sigma \leq s \leq t} w(s) \leq |(D_+ w)(t)|.$$

2) If $w(t) \geq 0$, then

$$\bar{D}_+ \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w(s) \leq \begin{cases} |(D_+ w)(t)| & \text{if } \gamma \geq 0 \\ |(D_+ w)(t)| - \gamma \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w(s) & \text{if } \gamma < 0. \end{cases}$$

Proof. For a proof of the assertion 1) refer to [1,6].

Set $u(t) = \sup\{w(s) | \sigma \leq s \leq t\}$, $z(t) = \sup\{e^{\gamma s} w(s) | \sigma \leq s \leq t\}$ and $I = [\sigma, \sigma+a)$. Clearly, $z(t)$ is nondecreasing in $t \in I$. Let any $\tau \in I$ be a fixed number and $\gamma \in (-\infty, 0)$. Then we have, for $h > 0$,

$$\begin{aligned} z(\tau+h) - z(\tau) &= e^{\gamma t_0} w(t_0) - z(\tau) \quad \text{for some } t_0 \in [\tau, \tau+h] \\ &\leq e^{\gamma \tau} \sup_{\sigma \leq s \leq \tau+h} w(s) - e^{\gamma \tau} \sup_{\sigma \leq s \leq \tau} w(s) \\ &= e^{\gamma \tau} \{u(\tau+h) - u(\tau)\}, \end{aligned}$$

from which it follows that

$$\bar{D}_+ \sup_{\sigma \leq s \leq t} e^{\gamma s} w(s) \leq e^{\gamma t} \bar{D}_+ \sup_{\sigma \leq s \leq t} w(s).$$

It is easy to prove the assertion 2) in case where γ is a negative number. Let $\gamma \geq 0$. Then by the assertion 1) we have

$$\begin{aligned} \bar{D}_+ \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w(s) &\leq -\gamma e^{-\gamma t} \sup_{\sigma \leq s \leq t} e^{\gamma s} w(s) + e^{-\gamma t} \bar{D}_+ \sup_{\sigma \leq s \leq t} e^{\gamma s} w(s) \\ &\leq -\gamma e^{-\gamma t} \sup_{\sigma \leq s \leq t} e^{\gamma s} w(s) + e^{-\gamma t} |D_+(e^{\gamma t} w(t))| \end{aligned}$$

$$\leq |(D_+ w)(t)|$$

as required.

Lemma 2.2. Let $\gamma \geq 0$ and $U : [\sigma, \sigma+a] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function, where $\mathbb{R}^+ = [0, \infty)$. Assume that

(1) $u^* : [\sigma, \sigma+a] \rightarrow \mathbb{R}^+$ is the maximal solution of the scalar differential equation

$$\frac{du}{dt} = U(t, u(t)), \quad u(\sigma) = u_0 \geq 0; \text{ and}$$

(2) $m : (-\infty, \sigma+a] \rightarrow \mathbb{R}$ is a continuous function such that $m_\sigma \in \mathcal{E}_\gamma^{\mathbb{R}}$ and $m(t) \geq 0$ on $[\sigma, \sigma+a]$, and that, for every $t_1 \in [\sigma, \sigma+a]$ such that $|m_{t_1}|_{\mathcal{E}_\gamma^{\mathbb{R}}} = m(t_1)$, the differential inequality

$$(D_- m)(t_1) \leq U(t_1, m(t_1))$$

is satisfied.

If $|m_\sigma|_{\mathcal{E}_\gamma^{\mathbb{R}}} \leq u_0$, then

$$m(t) \leq u^*(t) \quad \text{for } t \in [\sigma, \sigma+a].$$

Proof. For any $\varepsilon > 0$ we denote by $u(t, \varepsilon)$ any solution of the differential equation

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$$\frac{d}{dt} u(t) = U(t, u(t)) + \varepsilon, \quad u(\sigma) = u_0 + \varepsilon. \quad (2.1)$$

Then, by Lemma 1.3.1 in [4], we have

$$\lim_{\varepsilon \rightarrow 0+} u(t, \varepsilon) = u^*(t)$$

uniformly on $[\sigma, \sigma+a]$. Thus it is sufficient to show that for every $\varepsilon > 0$, sufficiently small,

$$m(t) \leq u(t, \varepsilon) \quad \text{on} \quad [\sigma, \sigma+a].$$

Suppose, on the contrary, that the set

$$Z = \{t \in [\sigma, \sigma+a] \mid m(t) > u(t, \varepsilon)\}$$

is nonempty and define $t_1 = \inf Z$. Then we have $t_1 > \sigma$, because $\lim_{\varepsilon \rightarrow 0+} m_\sigma \leq u_0 < u_0 + \varepsilon$. Moreover, since $m(t_1) = u(t_1, \varepsilon)$ and $m(t) < u(t, \varepsilon)$ for $t \in [\sigma, t_1)$, it is easy to see that

$$\begin{aligned} \underline{D}_m(t_1) &\geq \liminf_{h \rightarrow 0-} \frac{1}{h} (u(t_1+h, \varepsilon) - u(t_1, \varepsilon)) \\ &= U(t_1, m(t_1)) + \varepsilon \quad \text{by (2.1)}. \end{aligned}$$

Hence, we have

$$\underline{D}_m(t_1) > U(t_1, m(t_1)). \quad (2.2)$$

On the other hand, since $U(t,s) \geq 0$ and $u(t,\varepsilon)$ is nondecreasing in t , we have

$$\begin{aligned}
 |m_{t_1}|_{\mathcal{G}_\gamma^R} &= \sup_{\theta \leq 0} e^{\gamma\theta} |m(t_1+\theta)| \\
 &= \max\left\{ \sup_{\theta \leq \sigma-t_1} e^{\gamma\theta} |m(t_1+\theta)|, \sup_{\sigma-t_1 \leq \theta \leq 0} e^{\gamma\theta} |m(t_1+\theta)| \right\} \\
 &= \max\left\{ \sup_{s \leq 0} e^{\gamma(s-t_1+\sigma)} |m(\sigma+s)|, m(t_1) \right\} \\
 &= \max\left\{ e^{\gamma(\sigma-t_1)} |m_\sigma|_{\mathcal{G}_\gamma^R}, m(t_1) \right\} \\
 &= m(t_1).
 \end{aligned}$$

Thus, from the assumption 2) we are led to the inequality

$$\underline{D}_m(t_1) \leq U(t_1, m(t_1)),$$

which is incompatible with (2.2). This implies that the set Z is empty. Therefore the proof is completed.

A function $\eta : (\sigma, \sigma+a] \times [0, 2r] \rightarrow \mathbb{R}$ is said to be a Kamke-type function if the following conditions hold :

(η_1) $\eta = \eta(t,s)$ is a real-valued function, defined on

$(\sigma, \sigma+a] \times [0, 2r]$, which is Lebesgue measurable in t for each fixed $s \in [0, 2r]$ and is continuous in s for a.a. $t \in (\sigma, \sigma+a]$:

(η_2) There exists a function α , defined on $(\sigma, \sigma+a]$ and locally integrable there, such that $|\eta(t, s)| \leq \alpha(t)$ for a.a. $t \in (\sigma, \sigma+a]$ and all $s \in [0, 2r]$.

The following result is a modification of the one given by [8, Lemma 3.1]. The proof is obvious.

Lemma 2.3. Let $\eta(t, s) : (\sigma, \sigma+a] \times [0, 2r] \rightarrow \mathbb{R}$ be a Kamke-type function and let $\{w^n\}$ and $\{z^n\}$ converge pointwise to functions w^0 and z^0 on $[\sigma, \sigma+a]$ as $n \rightarrow \infty$, respectively. Assume that

1) there are a constant $H > 0$ such that

$|w^n(t) - w^n(s)| \leq H|t-s|$ for all $t, s \in (\sigma, \sigma+a]$ and all $n \in \mathbb{N}$; and

2) w^n and z^n are related to each other as

$$\frac{d}{dt} w^n(t) \leq \eta(t, z^n(t)) + \sigma_n \quad \text{for a.a. } t \in (\sigma, \sigma+a),$$

where $\sigma_n \geq 0$ and $\sigma_n \rightarrow 0$ as $n \rightarrow \infty$.

Then

$$\frac{d}{dt} w^0(t) \leq \eta(t, z^0(t)) \quad \text{for a.a. } t \in (\sigma, \sigma+a).$$

§ 3. Main results.

For a bounded set Ω of E , the α -measure of Ω is defined as follows :

$$\alpha(\Omega) = \inf\{d > 0 \mid \Omega \text{ has a finite cover of diameter } < d\}.$$

Let $\mathcal{I}$ be the set of functions x on $(-\infty, \sigma+a]$, $0 < a < \infty$, into E such that x is continuous on $[\sigma, \sigma+a]$ and $x_\sigma \in \mathcal{E}_\gamma$. For a subset $\mathcal{I} \subset \mathcal{I}$, we will use the following notations :

$$\mathcal{I}(t) = \{x(t) \in E \mid x \in \mathcal{I}\}, \quad \mathcal{I}_t = \{x_t \mid x \in \mathcal{I}\} \quad \text{for } t \in [\sigma, \sigma+a]$$

and

$$\mathcal{I}[[c, d]] = \{x|_{[c, d]} \mid x \in \mathcal{I}\},$$

where $c, d \in [\sigma, \sigma+a)$ and $x|_{[c, d]}$ is the restriction of x to $[c, d]$. We denote by $C([a, b], E)$ the set of all the continuous functions $x : [a, b] \rightarrow E$ with supremum norm. For brevity, we denote $\mathcal{E}_\gamma$ the phase space $\mathcal{E}_\gamma^E$ when $X = E$. The following lemma is concerned with the phase space $\mathcal{E}_\gamma$.

Lemma 3.1 (Shin [7]). If $\mathcal{I}_\sigma$ is relatively compact in $\mathcal{E}_\gamma$ and if $\mathcal{I}[[\sigma, t]]$ is a bounded and equicontinuous set in $C([\sigma, t], E)$, then

$$\alpha(x_t) = e^{-\gamma t} \sup_{\sigma \leq s \leq t} e^{\gamma s} \alpha(x(s)).$$

Lemma 3.2 (Shin [9]). Let $\{S_n\}$ be a family of nonempty bounded subsets of a Banach space Y such that $S_{n+1} \subset S_n$ for $n \in \mathbb{N}$. If S_n is connected for every $n \in \mathbb{N}$ and if $\alpha(S_n) \rightarrow 0$ as $n \rightarrow \infty$, then the set $\bigcap_{n=1}^{\infty} \text{cl } S_n$ is nonempty, compact and connected, where $\text{cl } A$ stands for the closure of A .

Assume that $f : [\sigma, \sigma+a] \times \mathcal{E}_\gamma(\varphi, r) \rightarrow E$ is a uniformly continuous function such that $\|f\|_E \leq M$. Then a function $u : (-\infty, \sigma+\xi] \rightarrow E$, $0 < \xi \leq a$, said to be an $\frac{1}{n}$ -approximate solution for CP(1.1) if the following conditions hold :

- (1) u is continuous on J , $J = [\sigma, \sigma+\xi]$, and $u_\sigma = \varphi \in \mathcal{E}$;
- (2) u has the right hand derivative $(D_+ u)(t)$ such that $\|(D_+ u)(t)\|_E \leq M$ on $[\sigma, \sigma+\xi)$, and satisfies

$$u(t) = \varphi(0) + \int_{\sigma}^t (D_+ u)(s) ds \quad \text{for } t \in J ; \text{ and}$$

- (3) $\|(D_+ u)(t) - f(t, u_t)\|_E \leq \frac{1}{n}$ for $t \in [\sigma, \sigma+\xi)$.

We denote by $Q^n[d]$ the set of all the $\frac{1}{n}$ -approximate solutions, defined on $(-\infty, \sigma+d]$, for CP(1.1). Then there is a $\xi > 0$ and the set $Q^n := Q^n[\xi]$ is nonempty (see [8, Lemma 2.1]).

Lemma 3.3 (Shin [9]). Let $f : [\sigma, \sigma+a] \times \mathcal{E}_\gamma(\varphi, r) \rightarrow E$ be

uniformly continuous and $\|f\|_E \leq M$ on $[\sigma, \sigma+a] \times \mathcal{G}_\gamma(\varphi, r)$. Then Q^n is nonempty and $Q^n|J$ is connected in $C(J, E)$ for every $n \in \mathbb{N}$.

Now, we state the main result in this paper, which is related to the result due to Kubiacyk[2] for ODE's.

Theorem 3.4. Assume that $f : [\sigma, \sigma+a] \times \mathcal{G}_\gamma(\varphi, r) \rightarrow E$ is a uniformly continuous function such that $\|f\|_E \leq M$ on $[\sigma, \sigma+a] \times \mathcal{G}_\gamma(\varphi, r)$, and that there exists a Kamke-type function $\omega(t, s) : (\sigma, \sigma+a] \times [0, 2r] \rightarrow \mathbb{R}^+$ such that

1) $\omega(t, s)$ is nondecreasing in s ;

2) $\omega(t, z(t)) \rightarrow 0$ as $t \rightarrow \sigma+0$ and $\int_{\sigma}^t \omega(s, z(s)) ds < \infty$

whenever $z : [\sigma, \sigma+a] \rightarrow [0, 2r]$ is an absolutely continuous function satisfying the condition $(D^+z)(\sigma) = z(\sigma) = 0$, where

$$(D^+z)(\sigma) := \lim_{t \rightarrow \sigma+} \frac{z(t)}{t-\sigma} ;$$

3) $z \equiv 0$ is the unique absolutely continuous function, mapping $[\sigma, \sigma+a]$ into $\mathbb{R}^+$, which satisfies the initial condition $(D^+z)(\sigma) = z(\sigma) = 0$ and the scalar differential equation

$$\frac{dz}{dt} = \begin{cases} \omega(t, z(t)) & \gamma \geq 0 \\ \omega(t, z(t)) - \gamma z(t) & \gamma < 0 \end{cases} \quad \text{for a.a. } t \in (\sigma, \sigma+a) ; \text{ and}$$

4)

$$\underline{D}_- \alpha(A(t)) := \liminf_{h \rightarrow 0^-} \frac{1}{h} [\alpha(A(t)) - \alpha(\{x(t) - hf(t, x_t) : x \in A\})]$$

$$\leq \omega(t, \alpha(A_t))$$

for a.a. $t \in (\sigma, \sigma+a]$ and for any subset $A \subset \mathcal{I}$ such that $A|[\sigma, \sigma+a]$ is equicontinuous and that $A_t \subset \mathcal{G}_\gamma(\varphi, r)$ for all $t \in [\sigma, \sigma+a]$.

Then the set of all the solutions for CP(1.1) defined on J ($=[\sigma, \sigma+\xi]$) is nonempty, compact and connected in $C(J, E)$.

Proof. From Lemma 3.2 and Lemma 3.3 it is sufficient to see that $\alpha(Q^n|J) \rightarrow 0$ as $n \rightarrow \infty$. Since $Q^n|J$ is an equicontinuous subset of $C(J, E)$, we have $\alpha(Q^n|J) \leq \sup\{\alpha(Q_t^n) | t \in J\}$ by Theorem 2.1 in [5]. Thus we must prove that $\alpha(Q_t^n) \rightarrow 0$ uniformly on J as $n \rightarrow \infty$. From the properties of the α -measure of noncompactness, we have, for $t \in (\sigma, \sigma+\gamma]$ and $h > 0$,

$$\begin{aligned} & \frac{1}{h} \{\alpha(Q^n(t)) - \alpha(Q^n(t-h))\} \\ & \leq \frac{1}{h} \{\alpha(Q^n(t)) - \alpha(\{x(t) - hf(t, x_t) | x \in Q^n\})\} \\ & \quad + \frac{1}{h} \alpha(\{x(t) - x(t-h) - hf(t, x_t) | x \in Q^n\}). \end{aligned} \quad (3.1)$$

By the uniform continuity of f , for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|f(s, \varphi_1) - f(t, \varphi_2)|_E \leq \varepsilon/2$ if $|t-s| < \delta$ and

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$|\varphi_1 - \varphi_2| < \delta$. Since $\{x_t^n \mid x^n \in Q^n\}$ is uniformly equicontinuous on $[\sigma, \sigma + \xi]$, we have, for any $x \in Q^n$ and $h \in (0, \delta)$,

$$\begin{aligned} & |x(t) - x(t-h) - hf(t, x_t)| \\ & \leq \left| \int_{t-h}^t [D_+ x(s) - f(s, x_s)] ds \right| + \left| \int_{t-h}^t [f(s, x_s) - f(t, x_t)] ds \right| \\ & \leq \frac{h}{n} + \frac{\varepsilon}{2} h. \end{aligned} \quad (3.2)$$

Set $w^n(t) = \alpha(Q^n(t))$ and $z^n(t) = \alpha(Q_t^n)$ for all $t \in J$.

Clearly, we have, for $t, s \in J$ and any $n \in \mathbb{N}$,

$$w^{n+1}(t) \leq w^n(t), \quad |w^n(t) - w^n(s)| \leq 2M|t-s|$$

and, Lemma 3.1,

$$z^n(t) = \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w^n(s). \quad (3.3)$$

Hence we get

$$z^{n+1}(t) \leq z^n(t) \quad \text{and} \quad |z^n(t) - z^n(s)| \leq 2M \sup_{-a \leq \theta \leq 0} e^{\gamma\theta} |t-s|.$$

These imply that the sequences $\{w^n(t)\}$ and $\{z^n(t)\}$ converges to functions $w^0(t)$ and $z^0(t)$ uniformly on J , respectively.

Now, we shall show that $z^0(t) \equiv 0$ on J . From (3.1), (3.2) and the assumption 4) it follows that

$$\frac{dw^n(t)}{dt} \leq \omega(t, z^n(t)) + \frac{2}{n} + \varepsilon \quad \text{for a.a. } t \in (\sigma, \sigma + \xi).$$

Using Lemma 2.3 and the relation (3.3), we can obtain

$$\frac{dw^0(t)}{dt} \leq \omega(t, z^0(t)) + \varepsilon \quad \text{for a.a. } t \in (\sigma, \sigma + \xi) \quad (3.4)$$

and

$$z^0(t) = \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} w^0(s). \quad (3.5)$$

Moreover, it is easy to see that $(D_+ z^0)(\sigma) = z^0(\sigma) = 0$. Using the assumption 2), we can put

$$u(t) = \int_{\sigma}^t \omega(s, z^0(s)) ds + \varepsilon(t - \sigma),$$

from which it follows that $w^0(t) \leq u(t)$ for $t \in J$. Therefore we can obtain

$$0 \leq \frac{du}{dt} = \omega(t, z^0(t)) + \varepsilon \quad \text{for a.a. } t \in (\sigma, \sigma + \xi).$$

Put $v(t) = \sup_{\sigma \leq s \leq t} e^{\gamma(s-t)} u(s)$. Then, by Lemma 2.1 we can see that

$$\frac{dv}{dt} \leq \begin{cases} \omega(t, z^0(t)) + \varepsilon \\ \omega(t, z^0(t)) - \gamma z^0(t) + \varepsilon \end{cases} \quad \text{for a.a. } t \in (\sigma, \sigma + \xi), \quad (3.6)$$

On the other hand, since $w^0(t) \leq u(t)$, we have $z^0(t) \leq v(t)$ by (3.5). Letting $\varepsilon \rightarrow 0$ and using the assumption 1), we see that the relation (3.6) becomes

$$\frac{dv}{dt} \leq \begin{cases} \omega(t, v(t)) \\ \omega(t, v(t)) - \gamma v(t) \end{cases} \quad \text{for a.a. } t \in (\sigma, \sigma + \xi).$$

It is easy to see that $(D_+ v)(\sigma) = v(\sigma) = 0$. Thus, by Lemma 4.1 in [6] and the assumption 3), we have $v(t) \equiv 0$ and so, $z^0(t) \equiv 0$. This implies $\alpha(Q^n | J) \rightarrow 0$ as $n \rightarrow \infty$. Hence the proof is complete.

Corollary 3.5. The conditions 1) - 4) in Theorem 3.4 can be replaced as follows :

- 1) the condition 3) in Theorem 3.4 is satisfied ;and
- 2)

$$\alpha(f(t, B)) \leq \omega(t, \alpha(B)) \quad \text{for a.a. } t \in (\sigma, \sigma + a) \quad \text{and all } B \in \mathcal{B}_\gamma(\varphi, r).$$

Combining the argument in the proof of Theorem 3.4 and Lemma 2.2, we have the following result.

Proposition 3.6. Let $\gamma \geq 0$. Assume that f :

$[\sigma, \sigma+a] \times \mathcal{G}_\gamma(\varphi, r) \rightarrow E$ is a uniformly continuous function such that $\|f\|_E \leq M$ on $[\sigma, \sigma+a] \times \mathcal{G}_\gamma(\varphi, r)$, and that there exists a continuous function $\omega(t, s) : [\sigma, \sigma+a] \times [0, 2r] \rightarrow \mathbb{R}^+$ such that

1) for every $t_1 \in [\sigma, \sigma+a]$ such that $\alpha(A_{t_1}) = \alpha(A(t_1))$, where

A is as in Theorem 3.4, the differential inequality

$$D_- \alpha(A(t_1)) \leq \omega(t, \alpha(A(t_1))),$$

is satisfied ; and

2) $u(t) \equiv 0$ is the unique continuous function, mapping $[\sigma, \sigma+a]$ into $[0, 2r]$, which satisfies the scalar differential equation

$$\frac{du(t)}{dt} = \omega(t, u(t)), \quad u(\sigma) = 0.$$

Then the conclusion of Theorem 3.4 remains valid.

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