

On liftability of Du Val del Pezzo surfaces in positive characteristic

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Pathologies on Du Val del Pezzo surfaces

All varieties are defined over $k = \bar{k}$ in characteristic $p \geq 0$.

Definition

X is a Du Val del Pezzo surface $\stackrel{\text{def}}{\iff} X$ is a normal projective surface with only Du Val singularities s.t. $-K_X$ is ample.

Notation

If X has three A_1 -singularities and one D_4 -singularity, then we write $\text{Dyn}(X) = 3A_1 + D_4$ or $X = X(3A_1 + D_4)$.

Pathological Phenomena in $p > 0$

Keel–McKernan constructed a Du Val del Pezzo surface $X := X(7A_1)$ in $p = 2$.

- ▶ $7A_1 \notin \text{Dynkin type in } p = 0$.
- ▶ Violating Bertini's Theorem: all members of $| -K_X |$ are singular.
- ▶ Violating Kodaira type Theorem: $\exists A$: ample $\mathbb{Z}$ -divisor s.t. $H^1(\mathcal{O}_X(K_X + A)) \neq 0$ (Cascini-Tanaka).

Log liftability

It is believed that “**Pathologies in $p > 0 \Leftrightarrow$ Non liftability to the Witt ring**”

Unfortunately, all Du Val del Pezzo surfaces lift to the Witt ring.
Therefore, we consider stronger liftability.

Definition (log lift)

X : Du Val del Pezzo surface in $p > 0$.

X is log liftable $\stackrel{\text{def}}{\iff} (Y, E)$ lifts to the Witt ring $W(k)$, where $\pi: Y \rightarrow X$ is the minimal resolution with reduced exceptional divisor E .

As we will see later, $X(7A_1)$ is not log liftable.

Pathologies and log liftability

We introduce the following properties.

For a Du Val del Pezzo surface X in $p > 0$, we say that X satisfies:

- ▶ (ND) if $\nexists X_{\mathbb{C}}$: Du Val del Pezzo surface over $\mathbb{C}$ s.t.
 $\text{Dyn}(X) = \text{Dyn}(X_{\mathbb{C}})$ and $\rho(X) = \rho(X_{\mathbb{C}})$.
- ▶ (NB) if all members of $| -K_X |$ are singular.
- ▶ (NK) if $\exists A$: ample $\mathbb{Z}$ -divisor s.t. $H^1(\mathcal{O}_X(K_X + A)) \neq 0$.
- ▶ (NL) if X is not log liftable.

Lemma

We have the following implications.

$$\begin{array}{ccccc} ND & \implies & NL & \implies & NB \\ & & \uparrow & & \\ & & NK & & \end{array}$$

By the above lemma, we should focus on Du Val del Pezzo surfaces satisfying NB.

Main Result

Theorem

Du Val del Pezzo surfaces satisfying NB are as follows.

Table 1:

K_X^2				$K_X^2 = 1$				
Dyn(X)				E_8	D_8	$A_1 + E_7$	$2D_4$	$2A_1 + D_6$
Characteristic				$p = 2, 3$	$p = 2$			
No. of isomorphism classes				1	1	1	∞	1
NL?				$\times$	$\times$	$\times$	$\times$	$\times$
ND?				$\times$	$\times$	$\times$	$\times$	$\times$
NK?				$\times$	$\times$	$\times$	$\times$	$\times$
$K_X^2 = 1$				$K_X^2 = 2$				
$4A_1 + D_4$	$8A_1$	$A_2 + E_6$	$4A_2$	E_7	$A_1 + D_6$	$3A_1 + D_4$	$7A_1$	
$p = 2$		$p = 3$		$p = 2$				
∞	∞	1	1	1	1	1	1	
$\bigcirc$	$\bigcirc$	$\times$	$\bigcirc$	$\times$	$\times$	$\times$	$\bigcirc$	
$\bigcirc$	$\bigcirc$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	
$\bigcirc$	$\bigcirc$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	

Remark

(NL) $\iff$ Dyn(X) = $4A_1 + D_4, 8A_1, 7A_1$, or $(p, \text{Dyn}(X)) = (3, 4A_2)$.

Non log liftability of $X(4A_2)$

Sketch of proof

Suppose that $X := X(4A_2)$ is log liftable. Let $\pi: Y \rightarrow X$ be the minimal resolution with reduced exceptional divisor E and (Y_K, E_K) be the generic fiber of $W(k)$ -lifting of (Y, E) . Then the blow-up $Z_K \rightarrow Y_K$ at the base point of $|-K_{Y_K}|$ gives the anti-canonical morphism $f_K: Z_K \rightarrow \mathbb{P}_K^1$. Let $G_{K,i} \subset Z_K$ be the strict transform of $E_{K,i}$, where $E_{K,i}$ ($i = 1, 2, 3, 4$) are connected component of E_K . Then $f_K(G_{K,i})$ is a K -rational point for each i . By using Beauville's result, we have $f_K(G_i) = \omega$ and hence $\omega \in K$, where ω is a primitive cube root of unity. This is a contradiction.

$$\begin{array}{ccc}
 (Z_K, G_K = \sum_{i=1}^4 G_{K,i}) & \longrightarrow & (Y_K, E_K := \sum_{i=1}^4 E_{K,i}) \\
 & \searrow f_K & \downarrow \phi_{|-K_{Y_K}|} \\
 & & \mathbb{P}_K^1
 \end{array}$$