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Reformulation of XFEM and its application to fatigue crack simulations in steel structures

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Contents

1	General Introduction	1
1.1	BACKGROUND	1
1.2	OBJECTIVE	6
1.2.1	Effective Modeling of Crack Tip Enrichment and J -integral in Standard XFEM	7
1.2.2	Reformulation of XFEM based on PUFEM	7
1.2.3	Application to Fatigue Crack Simulations in Real Structures	8
1.3	OVERVIEW	9
2	Standard XFEM	13
2.1	INTRODUCTION	13
2.2	FORMULATION OF STANDARD XFEM	14
2.2.1	Partition of Unity	14
2.2.2	Standard XFEM Approximation	15
2.2.3	Blending Elements	17
2.3	APPLICATION TO LINEAR FRACTURE MECHANICS	20
2.3.1	Governing Equations	20
2.3.2	Enrichment Functions	22
a)	Strong discontinuities	22
b)	Fields near crack tips	22
2.3.3	Approximation of Displacement Fields	27
2.3.4	Integration	28
2.3.5	Blending Elements	31

2.3.6	Evaluation of Fracture Mechanics Parameters	32
a)	J -integral method	32
b)	Equivalent domain integral method	33
c)	M -integral method	34
3	Effective Modeling of Crack Tip Enrichment and J -integral in Standard XFEM	39
3.1	INTRODUCTION	39
3.2	INFLUENCE OF BLENDING ELEMENTS IN CRACK ANALYSES	41
3.2.1	Blending Elements in Crack Analyses	41
3.2.2	Influence of Blending Elements	43
a)	Model descriptions	43
b)	Evaluation of stress fields	46
c)	Evaluation of fracture mechanics parameters	49
3.3	IMPROVEMENT OF NUMERICAL ACCURACY OF FRACTURE MECHANICS PARAMETERS	53
3.3.1	Modeling of Crack Tip Enrichment	53
3.3.2	Evaluation of Numerical Results	57
3.3.3	Effective Definition of J -integral Domain	60
3.4	CONCLUDING REMARKS	60
4	Reformulation of XFEM Based on PUFEM	63
4.1	INTRODUCTION	63
4.2	STANDARD XFEM	65
4.2.1	Standard XFEM Approximation	65
4.2.2	Blending elements	66
4.2.3	Approximate accuracy of the standard XFEM	67
4.3	REFORMULATION OF XFEM BASED ON PUFEM	75
4.3.1	PUFEM Approximation	75
4.3.2	Formulation of XFEM based on PUFEM	77
4.3.3	Approximate accuracy of the PU-XFEM	78
4.3.4	Consideration for the weighted XFEM	81

4.4	APPLICATION TO LINEAR FRACTURE MECHANICS	84
4.5	NUMERICAL EXAMPLES	91
4.5.1	Numerical Models in infinite plates	91
a)	Model descriptions	92
b)	Convergence study using L_2 norm	95
c)	Dominant ratio of enrichment in displacement field	97
d)	Evaluation of stress fields	102
e)	Evaluation on linear fracture mechanics parameters	105
4.5.2	Numerical Models of finite plates	109
a)	Model descriptions	109
b)	Evaluation on linear fracture mechanics parameters	112
4.6	CONCLUDING REMARKS	119
5	Application to Fatigue Crack Simulations in Real Structures	121
5.1	INTRODUCTION	121
5.2	MODELING OF CRACK USING PU-XFEM	124
5.3	IMPLEMENTATION INTO GENERAL-PURPOSE FEM CODE	129
5.3.1	Flow of Analysis	129
5.3.2	Implementation of PU-XFEM Approximation into ABAQUS	132
a)	Method to increase nodal degrees of freedom by adopting multiple-nodes	134
b)	Outline of the program PRE-PRO	137
5.3.3	Evaluation on Fracture Mechanics Parameters and Criteria of Fatigue Propagation	138
5.4	NUMERICAL EXAMPLES	141
5.4.1	Application to Fatigue Crack Detected in Intermediate Transverse Beam of I-girder Bridge	141
5.4.2	Application to Fatigue Test for Orthotropic Steel Deck Specimen Using Bulb Rib	148
5.5	CONCLUDING REMARKS	156

Contents

6	Conclusions	159
	References	163
	Associated Publications	173

Chapter 1

General Introduction

1.1 BACKGROUND

Recently, there are several reports on fatigue damages of aged steel bridges. It is therefore urgently required to assess the cause of such damages and to rationalize the maintenance, such as inspection, repair and reinforcement (Japanese Society of Steel Construction 1995, Japan Road Association 2002).

Based on the statements given below, the numerical simulation of the behavior of fatigue crack in an aged steel bridge is effective in order to clarify the mechanism of fatigue crack propagation and to study the rationalization of maintenance for the bridge:

- (a) The detailed evaluations are possible in order to examine the causes of fatigue damages.
- (b) The critical aspect of fatigue cracks is quantitatively evaluated. Therefore the interval of inspections and the priority of repairs and reinforcement are effectively determined.
- (c) A larger number of numerical simulations of fatigue damage are possible in comparisons with fatigue tests using specimens.
- (d) The numerical simulations are easily applicable to large and complex structures.

The finite element method (FEM) is one of the most popular and powerful tool of the numerical analyses for the investigation of behavior in various engineering problems (Zienkiewicz & Taylor 2005, Reddy 2006). According to this fact, the functions of general-purpose FEM analysis softwares have recently been enhanced. And this enables the numerical investigation of the behavior of large-scale civil engineering structures (Hanganu *et al.* 2002, Miki *et al.* 2005).

Thus, several applications to the simulation of the fatigue crack propagation in large or complex structures using the FEM have been reported in recent studies (Taniguchi 1991, Kiss & Dunai 2002, Ichige & Utsunomiya 2006, Ohkawa & Sumi 2006, Tateishi *et al.* 2008). However, it is not a simple task to evaluate the behavior of the fatigue crack propagation in the local parts of the complex and large-scaled civil engineering structures, such as steel deck systems of bridges, without advanced knowledge and experience in numerical modeling, due to the following reasons.

The approximation in the FEM is based on the piecewise continuous polynomial functions and is highly suited for smooth problems. However, a special care is required to be taken for well approximating non-smooth, high gradients or large deformation problems. For example, the element edges have to be modeled along a discontinuity, and considerable mesh refinement is required where the solution of the problem is expected to have singularities or large gradients. In addition, in modeling of a moving discontinuity or a boundary, the adequate remeshing procedures (which are generally difficult or even impossible) are required. Therefore, the behavior of the fatigue cracks observed in the local regions of large-scale steel bridges are one of the typical examples to be hard for evaluation using the FEM.

In order to overcome the above inconveniences of meshing procedures in the FEM, it was proposed the node-based analysis method called ‘the meshfree methods’, e.g., the element free Galerkin method (EFGM) (Belytschko *et al.* 1994), and the free mesh method (FMM) (Yagawa & Yamada 1996). These methods have successfully modified the above-mentioned problems in the FEM. However, due to the lack of meshes, these node-based methods have included certain difficulties in the complex data structures, the application to composite material and the extension to the three dimensional problem. As a result, such advantages in the FEM as a mesh-based method have been lost.

On the other hand, the partition of unity finite element method (PUFEM) (Melenk & Babuška 1996, Babuška & Melenk 1997) have been proposed as ‘the generalization of the mesh-based FEM’ based on a concept of partition of unity (PU). In the method, the enhanced approximations using the ‘*priori* knowledge’ of the solution are directly introduced into the mesh-based formulation. In this method, the enhanced approximation using the solution has a potential to calculate with high accuracy independent of refinement of the meshes like the meshfree method, so that the problem of the complicated meshing procedures can be avoided.

The extended finite element method (XFEM) (Belytschko & Black 1999, Moës *et al.* 1999) is one of the typical representations of successful developments of the PUFEM. The XFEM is a numerical method which employs the approximation called as ‘enrichment’, which includes the *priori* knowledge of the solution in a local area of the problem. It means that the discontinuities, high gradients and singularities can be directly introduced in the approximation through the enrichment independently from mesh generation. The enrichment is defined in a local area through the introduction of enrichment functions and corresponding additional degrees of freedom. These degrees of freedom have been usually defined at the respective nodes. It means that the enrichment is defined on a per node basis. The advantages of the XFEM approximation can be found in its clear concept and its simple formulation. In addition, the XFEM can be easily implemented by simple revision of the classical FEM programming codes in the same framework.

The XFEM was originally proposed to be focused on crack analysis as ‘the finite element method for crack growth without remeshing’, by Belytschko and his coworkers (Belytschko & Black 1999, Moës *et al.* 1999). In this application to the elastic linear fracture mechanics, the XFEM has the following features:

- (a) The displacement field of the local body including the strong discontinuity along the crack surface is approximated by the introduction of the enrichment. It enables the simplification of the modeling of the crack as a line in the two dimensional problem or a surface in the three dimensional problem independently from the finite element meshes.

- (b) The approximation near the crack tip employs the enrichment to reproduce the asymptotic solution of the singular field near crack tip. This enables to the accurate evaluation of the fracture mechanics parameters without refinement of the meshes or use of the singular elements around the crack tip (Barsoum 1976).
- (c) The complicated remeshing procedures according to the crack propagation can be avoided because of the independent modeling of the crack from the finite element meshes.
- (d) The procedure of the modeling of the entire structure is the same as that in the FEM, except the crack modeling independent from meshes. It means that we can take advantages of the features of the FEM as one of the most popular and powerful tool of the numerical analyses.

According to the above-mentioned features, the XFEM has a potential ability to solve the complex simulation procedure of the crack problem by numerical analysis.

Therefore, several additional studies in the application of the XFEM to the elastic linear fracture mechanics were performed as follows. Development for the three dimensional crack problem and arbitrary branched and intersecting cracks were studied by Sukumar *et al.* (2000), Daux *et al.* (2000), and Dolbow *et al.* (2000). Efficient discontinuity modeling along crack surface is developed using the level set method, which is a numerical technique for the implicit description of boundary, by Stolarska *et al.* (2001), and Ventura *et al.* (2003). Improvement of the accuracy in the problem with a crack of arbitrary shape was achieved by mapping near the crack tip fields considering the crack shape (Duflo 2007, Shibamura & Utsunomiya 2008). Direct evaluation of the fracture mechanics parameters, i.e., the stress intensity factors, by introducing new definition of the degrees of freedom for the simplification of post processing tasks was proposed by Xiao & Karahaloo (2003), and Liu *et al.* (2004).

In addition, the applications of the XFEM to the following various problems have been reported besides the elastic linear fracture mechanics. Applications to the elastic-plastic problems were given by Li & Yatom (2005), and Elguedj *et al.* (2006). Modeling of growth of multiple cracks is presented by Budyn *et al.* (2004). Large deformation analyses were presented by Dolbow & Devan (2004), and Areias & Belytschko (2005). Simulation of the growth of cohesive cracks was reported by Moës

& Belytschko (2002), and Zi & Belytschko (2003). Application to the elastodynamic crack propagation was given by Réthoré *et al.* (2005) and Zi *et al.* (2005). Analysis of the crack in the orthotropic material was given by Asadpoure *et al.* (2006), and Asadpoure & Mohammadi (2007). Study of the biomaterial interface crack was reported by Nagashima *et al.* (2003), and Sukumar *et al.* (2004). Sloshing analysis was also given by Nagashima (2009).

As described, there are many studies related to applications of the XFEM. However, the essential problem of the XFEM was recently pointed out by Chessa *et al.* (2003) in the following: In the existing XFEM, the enrichment is usually localized to sub-domains through the introduction of the enrichment functions at the respective nodes. It leads that there are elements with some of their nodes being enriched and the remaining nodes being standard (unenriched) elements in the numerical model. These partially enriched elements are called ‘blending elements’. According to the study of Chessa *et al.* (2003), the blending elements have the unwanted parasitic terms in the approximation and cause the problems in the numerical accuracy.

Therefore, the following studies were performed in the past for improvement of the problem of the blending elements in the XFEM approximation.

Chessa *et al.* (2003) and Gracie *et al.* (2008) adopted an assumed strain formulation into the blending elements, in order to eliminate the unwanted parasitic terms in the strain approximation. The primary problem of the assumed strain method is the difficulty in constructing the basis functions for the assumed strain approximation. In the method, a linearly independent basis spanning the unwanted parasitic terms is required. The determination of the basis is difficult for complicated enrichment functions including the asymptotic crack tip enrichment. It means a lack of generality.

Gracie *et al.* (2008) tried to avoid the problem of the accuracy reduction caused by the blending elements by applying the approach of the discontinuous Galerkin method. In this method, the domain is decomposed into fully enriched and standard (unenriched) patches, which are independently discretized. Continuity between the patches is enforced with an internal penalty method. This proposal provided a certain improvement. However, in the application to fracture mechanics, a sufficient numerical accuracy cannot be achieved when coarse meshes are used near the crack tip. This is due to the fact that the application of the coarse meshes to the crack analyses is one of

the advantages of the XFEM. In addition, complicated processing is required for application of the discontinuous Galerkin method to the XFEM. This method may therefore not be a realistic solution for the problem of the blending elements.

Fries (2008) proposed the weighted XFEM as ‘the corrected XFEM’ employing a new definition of enrichment. The weighted XFEM showed the effective improvement of the accuracy reduction caused by the blending elements with a lot of numerical results. In addition, this method can be implemented by simple revision of the standard XFEM programming codes. Therefore, this proposal has been an effective method to improve the problem caused by the blending elements. However, this proposal is based on a simple redefinition of enrichment function satisfying to be zero on the boundary between standard elements and blending elements by using a ramp function in the framework of the standard XFEM approximation. Therefore, the usefulness of this proposal has been unfortunately based only on the results of the inductive evaluations because the theoretical approach was not sufficiently described.

According to the above-mentioned facts, it is found that most of the applications of the XFEM reported in the past potentially involve the problem of the blending elements without any effective improvement. This problem of the blending elements is the potentially serious problem of the existing XFEM to ensure the accuracy in actual applications. However, the problem of the blending elements has not been sufficiently solved in the past studies from the view points of the actual application and the theoretical improvement. It is therefore concluded that the present standard XFEM approximation has not been completed and still contains the essential problem due to the blending elements.

1.2 OBJECTIVE

According to the background of this study presented in Section 1.1, a number of studies related to the XFEM have been reported particularly in the applications to crack analysis. However, the essential problem of the blending elements in the existing XFEM

has not been sufficiently solved according to past studies.

The objectives of this thesis are to propose a solution of the problem of the blending elements in the XFEM approximation and to develop an effective fatigue crack simulation code in order to apply to the actual use for real structures.

1.2.1 *Effective Modeling of Crack Tip Enrichment and J -integral in Standard XFEM*

The elastic linear fracture mechanics is the main application of the XFEM. However, the influence of the problem of the blending elements in the application of the standard XFEM to the linear fracture mechanics is not sufficiently investigated in past studies. In addition, the method to effectively avoid the influence of the problem of blending elements in the framework of the standard XFEM will be useful, because there are a lot of studies related to crack analysis employing the XFEM reported using as a basement the standard XFEM approximation.

According to the above-mentioned statements, the first objective is thorough evaluations of the influence of the problem of the blending elements in the application of the standard XFEM to linear fracture mechanics and a new proposal of effective modeling of the crack tip enrichment and J -integral in the framework of the standard XFEM.

1.2.2 *Reformulation of XFEM based on PUFEM*

The proposed modeling in the framework of the standard XFEM in the previous objective is not a foundational solution of the problem of the blending elements in the XFEM. In order to solve the accuracy decrement caused by the blending elements, a reformulation of the XFEM is required based on the useful theory.

The PUFEM was proposed as a generalization of the FEM by Melenk & Babuška (1996). It is important that the foundational concept of the PUFEM is ‘assurance of approximate accuracy’. The XFEM was proposed as one of the developments of the

PUFEM. However, this fact conflicts the decrement of numerical accuracy caused by the problem of the blending elements in the standard XFEM. Therefore, as a theoretical approach to solve the problem caused by the blending elements, a reconsideration of the application of the PUFEM to the XFEM will be useful.

According to the above statements, the second objective of the thesis is the reformulation of the XFEM based on the PUFEM as ‘PU-XFEM’ in order to solve the problem caused by the blending elements. The reformulation is led by establishing the logical relationship between the XFEM and the PUFEM. In addition, we try to find the reason why the weighted XFEM proposed by Fries (2008) can improve the accuracy reduction caused by the blending elements.

1.2.3 *Application to Fatigue Crack Simulations in Real Structures*

In order to quantitatively evaluate a behavior of fatigue crack in the local part of large-scale civil engineering structure, a development of useful numerical analysis code is required.

The functions of general-purpose FEM analysis softwares have recently been enhanced. These assist the finite element modeling of the complex and large-scale structure. In addition, the reformulated PU-XFEM enables the crack analysis with significant accuracy and ease in the finite element model. Therefore, the implementation of the PU-XFEM approximation into the general-purpose FEM analysis softwares will be effective for evaluation of the behavior of actual fatigue cracks in the complex and large-scale structures.

According to the above statements, the third objective of this thesis is development of a fatigue crack simulation code based on ABAQUS (Dassault Systems Simulia Corp. 2007), which is one of the most well known general-purpose FEM analysis software, through implementation of the PU-XFEM approximation and validations of the developed PU-XFEM analysis code through its applications to the numerical simulations of behaviors of fatigue crack propagation in real structures. In particular, it is important that the method of implementation consists with the convention rules defined in the software in order to enable additional extensions in the future.

1.3 OVERVIEW

In this doctor thesis, we propose a solution of the essential problem of the blending elements in the XFEM approximation and develop the fatigue crack simulation code for an actual application to real structures. The overviews of the following respective chapters are described below:

In Chapter 2, the definition of the standard XFEM approximation is reviewed according to past studies, as the basis of the study in this thesis. The standard XFEM approximation is formulated by addition of the local enrichment to the classical finite element approximation. The blending elements are defined as partially enriched elements. In the blending elements, the unwanted parasitic terms, which cannot be generally compensated by the classical finite element approximation, are introduced in the approximation. The linear fracture mechanics is the most principal application of the XFEM. In this application, two types of the enrichment functions are employed, i.e., the strong discontinuities along the crack surface and the fields near the crack tip. The J -integral method is generally used for the evaluation of the fracture mechanics parameters.

In Chapter 3, we first evaluate the influence of the blending elements in the application of the standard XFEM to two dimensional linear fracture mechanics by the verification using basic numerical models. The numerical results show large errors of the stress fields in the domain of the blending elements. And it is found that the fracture mechanics parameters can be evaluated with good accuracies under the condition that the J -integral domain is defined inside the fully enriched domain. Based on the condition for good accuracies, we then propose the effective modeling of the crack tip enrichment and J -integral in the frameworks of the standard XFEM. Using this proposal, the numerical results show stable and sufficient accuracies in the evaluation of the fracture mechanics parameters, compared with those using the existing method. According to the above-mentioned results, it is found that the proposed modeling is effective in the application of the standard XFEM application to the linear fracture mechanics.

In Chapter 4, in order to essentially solve the problem of the blending elements in

the standard XFEM, we propose the reformulation of the XFEM based on the concept of the PUFEM, which assures the numerical accuracy in the entire domain, as the ‘PU-XFEM’. In the proposed reformulation, we define the two sets of the PU and the approximate functions, relating to the classical finite element approximation and the approximation of enrichment, respectively. The blending elements are redefined as the elements in the common part of the respective two supports of the PU. In the evaluation of numerical accuracy using L_2 norm of the interpolation error for arbitral one dimensional problem, it is shown that the insufficient convergence rate in the standard XFEM is completely overcome in the proposed PU-XFEM. It means that the PU-XFEM provides a ability to solve the problem of blending elements in the standard XFEM. The PU-XFEM is applied to the two dimensional linear fracture mechanics and the validation is investigated by the basic evaluations of the numerical accuracy using the models in the infinite plates and the models of the finite plates. The verification of the numerical results shows the effectiveness of the proposed PU-XFEM in the actual application to crack analysis. It can be concluded that the problem of the blending elements has been solved by the PU-XFEM. According to the aforementioned facts, the proposed formulation of the PU-XFEM can be a new basis of the XFEM instead of the standard XFEM.

In Chapter 5, the PU-XFEM approximation is implemented into ABAQUS in order to model fatigue crack propagation in real structures. In the development of the fatigue crack simulation code, we propose a new method to increase the number of the nodal degrees of freedom, which is consistent with the convention rules defined in the ABAQUS framework and applicable to the extension to three dimensional analysis. Using this developed PU-XFEM analysis code, the behavior of a fatigue crack through thickness of a plate in a three dimensional structure can be quantitatively evaluated. We validate the developed code through its applications to the numerical simulations of the fatigue crack propagation in the intermediate transverse beam of the actual I-girder bridge and the orthotropic steel deck specimen using bulb rib simulating the real bridge structure. The numerical results show that the developed PU-XFEM analysis code is effective for the quantitative evaluation on the path and rate of the fatigue propagation including the estimation of termination of crack propagation. The achievement of the study therefore can be used as a basis for developing numerical simulation tools for

fatigue cracks propagation in large-scale civil engineering structures.

In Chapter 6, we summarize the concluding remarks of the study presented in this thesis.

Chapter 2

Standard XFEM

2.1 INTRODUCTION

The XFEM is a numerical method which employs the local enrichment approximation, which includes the *priori* knowledge of the solution. The local enrichment is defined on a per node basis and can directly introduce the discontinuities, high gradients and singularities in the approximation independently from mesh generation.

In particular, the linear fracture mechanics is the most principal application of the XFEM. In this application, two types of the local enrichment are introduced in order to reproduce the discontinuity along crack surface and the singular field near crack tip in the approximation. In this application, the crack can be modeled independently from the finite element mesh and the complex remeshing procedure according to crack propagation can be avoided. In addition, the fracture mechanics parameters can be accurately evaluated without mesh refinement. Thus, the XFEM has a potential ability to solve the problem of the numerical simulation of the crack problem.

In the XFEM, the enrichment is usually localized to sub-domains thorough the introduction of enrichment functions defined at the respective nodes. It leads the inevitable presence of the partially enriched elements called blending elements. According to the study by Chessa *et al.* (2003), the blending elements have the unwanted parasitic terms in the approximation and cause the problems in the numerical

accuracy.

In this chapter, the standard XFEM approximation and the definition of the blending elements are overviewed based on past studies. In addition, the application of the XFEM to linear fracture mechanics is described. The description in this chapter is the basis of the study in this thesis.

An outline of this chapter is as follows: In Section 2.2, the general form of the standard XFEM approximation and the definition of the blending elements are described. In Section 2.3, we describe the application of the standard XFEM to the elastic linear fracture mechanics including the J -integral method, which is generally used to calculate the fracture mechanics parameters.

2.2 FORMULATION OF STANDARD XFEM

In this section, we describe the definition of the general form of the standard XFEM approximation based on past studies. The contents of this section are as follows: In the section 2.2.1, the concept of partition of unity is described. In the section 2.2.2, the general form of the standard XFEM is described. In the section 2.2.3, the definition of the blending elements is shown.

2.2.1 *Partition of Unity*

A partition of unity is defined as a set of functions $f_k(\mathbf{x})$ ($k=1, \dots, m$) such that

$$\sum_{k=1}^m f_k(\mathbf{x}) = 1 \quad (2.1)$$

By using an arbitrary function $\psi(\mathbf{x})$, the partition of unity is automatically satisfied

$$\sum_{k=1}^m f_k(\mathbf{x}) \psi(\mathbf{x}) = \psi(\mathbf{x}) \quad (2.2)$$

This means that the partition of unity has a property of reproducing condition or completeness. This property is significant in definitions of approximations in various numerical methods (Melenk & Babuška 1996).

The classical finite element approximation $\mathbf{u}_{\text{ap}}(\mathbf{x})$ is defined as

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \sum_{I \in N} \phi_I(\mathbf{x}) \mathbf{u}_I \quad (2.3)$$

where N is the set of all nodes, $\phi_I(\mathbf{x})$ are the standard finite element interpolation functions and $\mathbf{u}_I$ are the nodal degrees of freedom. The set of the interpolation functions $\phi_I(\mathbf{x})$ are also satisfied above condition of partition of unity,

$$\sum_{I \in N} \phi_I(\mathbf{x}) = 1 \quad (2.4)$$

2.2.2 Standard XFEM Approximation

The XFEM is a numerical method with approximation reflecting the *priori* knowledge of the solution of the problem called as ‘enrichment’ in a local area, based on the framework of the finite element method.

The standard XFEM approximation $\mathbf{u}_{\text{ap}}(\mathbf{x})$ has the form

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \mathbf{u}^{\text{std}}(\mathbf{x}) + \mathbf{u}^{\text{enr}}(\mathbf{x}) \quad (2.5)$$

where, $\mathbf{u}^{\text{std}}(\mathbf{x})$ is the standard finite element approximation and $\mathbf{u}^{\text{enr}}(\mathbf{x})$ is the approximation using the enrichment.

The approximation of the standard finite element part is defined as the same as that in Eq. (2.3) as

$$\mathbf{u}^{\text{std}}(\mathbf{x}) = \sum_{I \in N} \phi_I^{\text{std}}(\mathbf{x}) \mathbf{u}_I \quad (2.6)$$

where N is the set of all nodes, $\phi_I^{\text{std}}(\mathbf{x})$ are the standard finite element interpolation functions and $\mathbf{u}_I$ are the nodal degrees of freedom associated for standard finite element approximation.

The approximation of the enrichment part is

$$\mathbf{u}^{\text{enr}}(\mathbf{x}) = \sum_{I \in N^{\text{enr}}} \phi_I^{\text{enr}}(\mathbf{x}) \Psi(\mathbf{x}) \mathbf{a}_I \quad (2.7)$$

where $\phi_I^{\text{enr}}(\mathbf{x})$ are the standard finite element interpolation functions. $\Psi(\mathbf{x})$ is called enrichment function, which is generally a basis function including a *priori* knowledge of the solution. We employ a scalar function as the enrich function $\Psi(\mathbf{x})$ for simplification, though it is possible to introduce multiple or vector enrich functions. N^{enr} is the set of nodes enriched by $\Psi(\mathbf{x})$ and $\mathbf{a}_I$ are the nodal degrees of freedom associated for enrichment. That is, the enrichment is defined on a per node basis in the standard XFEM approximation.

The advantages of the XFEM approximation can be found in its easily understandable concept and its simple formulation. In addition, the XFEM can be easily implemented by simple revision of the classical FEM programming codes in the same framework.

Although $\phi_I^{\text{enr}}(\mathbf{x})$ are not necessarily the same as $\phi_I^{\text{std}}(\mathbf{x})$ of the standard part of the approximation in (2.6) (Stazi *et al.* 2003, Laborde *et al.* 2005), we employ the first order interpolation functions as the functions $\phi_I^{\text{std}}(\mathbf{x})$ and $\phi_I^{\text{enr}}(\mathbf{x})$ for simplification. In the following descriptions, the functions $\phi_I^{\text{std}}(\mathbf{x})$ and $\phi_I^{\text{enr}}(\mathbf{x})$ are therefore unified as $\phi_I(\mathbf{x})$ except the cases that their distinction is required.

2.2.3 Blending Elements

In the approximation of the XFEM, three sets of elements are defined. The first type is a set of elements E^{std} , in which none of the nodes is enriched. The second type is a set of elements E^{enr} , in which all of the nodes are enriched. The third type is a set of elements E^{blnd} , in which only some of the nodes are enriched. The sub-domain composed of each set of elements is defined as

$$\begin{aligned}\Omega^{\text{std}} &= \bigcup_{e \in E^{\text{std}}} \Omega_e \\ \Omega^{\text{enr}} &= \bigcup_{e \in E^{\text{enr}}} \Omega_e \\ \Omega^{\text{blnd}} &= \bigcup_{e \in E^{\text{blnd}}} \Omega_e\end{aligned}\tag{2.8}$$

where Ω_e is the domain composed of the element e . The elements composing the domain Ω^{blnd} are called ‘blending elements’. They blend the approximation in the enriched elements in the domain Ω^{enr} with that in the standard elements in the domain Ω^{std} . Respective examples of the blending elements and sub-domain Ω^{std} , Ω^{enr} and Ω^{blnd} in one and two dimensional problems are shown in Fig. 2.1. The presence of the blending elements in the standard XFEM is inevitable as long as the enrichment is defined in the local part of the entire domain.

In the component i of the approximation $\mathbf{u}^{\text{enr}}(\mathbf{x})$ in Eq. (2.7), if $a_{Ii} = 1$ then

$$u_i^{\text{enr}}(\mathbf{x}) = \sum_{I \in N^{\text{enr}}} \begin{cases} \phi_I(\mathbf{x})\Psi(\mathbf{x}) = 0 & \forall \mathbf{x} \in \Omega^{\text{std}} \\ \phi_I(\mathbf{x})\Psi(\mathbf{x}) = \Psi(\mathbf{x}) & \forall \mathbf{x} \in \Omega^{\text{enr}} \\ \phi_I(\mathbf{x})\Psi(\mathbf{x}) \neq \Psi(\mathbf{x}) & \forall \mathbf{x} \in \Omega^{\text{blnd}} \end{cases}\tag{2.9}$$

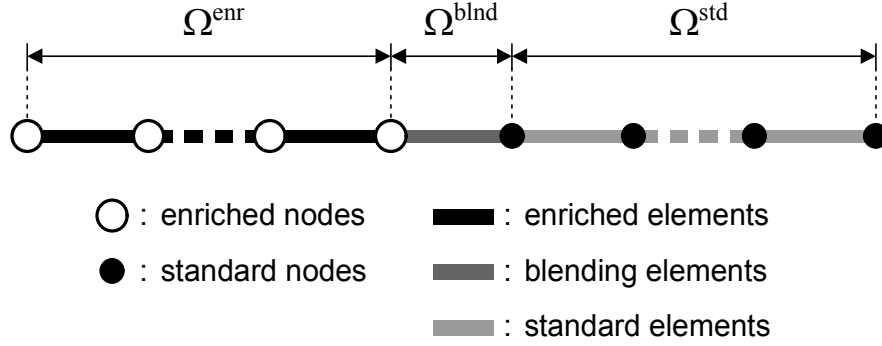
That is, the enrichment function $\Psi(\mathbf{x})$ is absent in the domain Ω^{std} and is reproduced in the domain Ω^{enr} in the enriched part of the approximation $\mathbf{u}^{\text{enr}}(\mathbf{x})$. On the other hand, the enrichment function $\Psi(\mathbf{x})$ cannot be reproduced in the domain Ω^{blnd} , because only some of the nodes in the blending elements in the domain Ω^{blnd} belong the set of nodes

N^{enr} , i.e., the partition of unity $\sum_{I \in N^{enr}} \phi_I(\mathbf{x}) = 1$ is not satisfied.

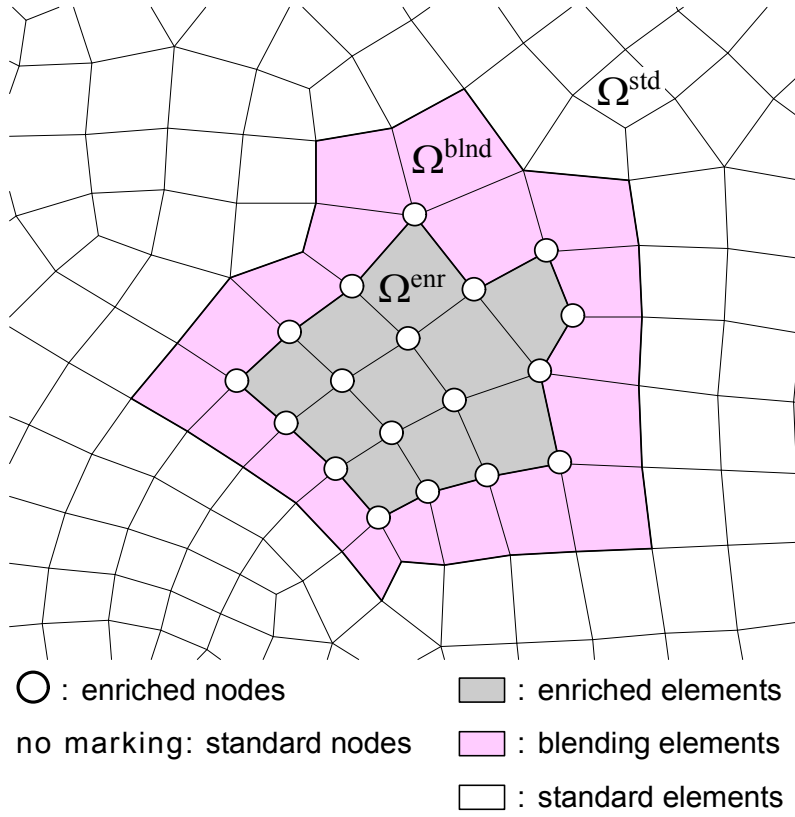
Due to the above lack of the partition of unity, the unwanted ‘parasitic terms’, which cannot be generally compensated by the approximation of the standard finite element part $\mathbf{u}^{std}(\mathbf{x})$, is introduced in the approximation. This can be easily seen in the following example. Consider a blending element with only one elemental node belonging to N^{enr} . The approximation in this element is

$$\begin{aligned} \mathbf{u}_{ap}(\mathbf{x}) &= \mathbf{u}^{std}(\mathbf{x}) + \mathbf{u}^{enr}(\mathbf{x}) \\ &= \sum_{I \in N} \phi_I^{std}(\mathbf{x}) \mathbf{u}_I + \phi_{I'}^{enr}(\mathbf{x}) \Psi(\mathbf{x}) \mathbf{a}_{I'}, \end{aligned} \tag{2.10}$$

where $I' \in N^{enr}$ is the enriched node. If $\mathbf{a}_{I'} \neq \mathbf{0}$, there are the parasitic term $\phi_{I'}^{enr}(\mathbf{x}) \Psi(\mathbf{x}) \mathbf{a}_{I'}$, which cannot be compensated by the approximation of the standard finite element part $\sum_{I \in N} \phi_I(\mathbf{x}) \mathbf{u}_I$ for the general enrichment functions $\Psi(\mathbf{x})$.



(a) In one dimensional problem



(b) In two dimensional problem

Figure 2.1 Definition of blending elements and sub-domains Ω^{std} , Ω^{enr} and Ω^{blnd} in the standard XFEM

2.3 APPLICATION TO LINEAR FRACTURE MECHANICS

The XFEM was originally proposed to be focused on crack analysis as ‘the finite element method for crack growth without remeshing’, by Belytschko and his coworkers (Belytschko & Black 1999, Moës *et al.* 1999). In this section, the application of the standard XFEM to two dimensional linear fracture mechanics is described.

2.3.1 Governing Equations

In this section, the governing equations for the displacement field in an elastic-static analysis are described. The domain of the problem is Ω with boundary Γ , as shown in Fig. 2.2. The boundary Γ is subdivided into two parts, Γ_u and Γ_t . The displacement is prescribed on Γ_u , and the traction is prescribed on Γ_t . In addition to the external boundary, the crack surface presents an additional boundary inside Ω . The crack surface denoted by Γ_c is traction free. The boundary of crack surface Γ_c consists of two coincident surfaces Γ_{c+} and Γ_{c-} . Under the assumption of absence of body forces, the strong form of the equilibrium equations and boundary conditions are

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \quad \text{in } \Omega \quad (2.11)$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u \quad (2.12)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t \quad (2.13)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_c \quad (2.14)$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{u}$ is the displacement vector and $\mathbf{n}$ is the unit normal vector. $\bar{\mathbf{u}}$ and $\bar{\mathbf{t}}$ are the prescribed displacement and traction, respectively.

We consider small strains and displacements, so the strain-displacement relation is

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \quad (2.15)$$

where $\boldsymbol{\varepsilon}$ is the linear strain tensor.

The constitutive relation is given by Hooke's Law,

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} \quad (2.16)$$

where $\mathbf{C}$ is the Hooke tensor.

According to these equations, the weak form of equilibrium equation is expressed as

$$\int_{\Omega} \delta \boldsymbol{\varepsilon} : \boldsymbol{\sigma} d\Omega = \oint_{\Gamma^t} \delta \mathbf{u} \cdot \bar{\mathbf{t}} d\Gamma \quad (2.17)$$

It is shown in the reference (Belytschko & Black 1999) that the weak form of the equilibrium equation of Eq. (2.17) implies the traction-free conditions on the crack surface Γ_c in Eq. (2.14).

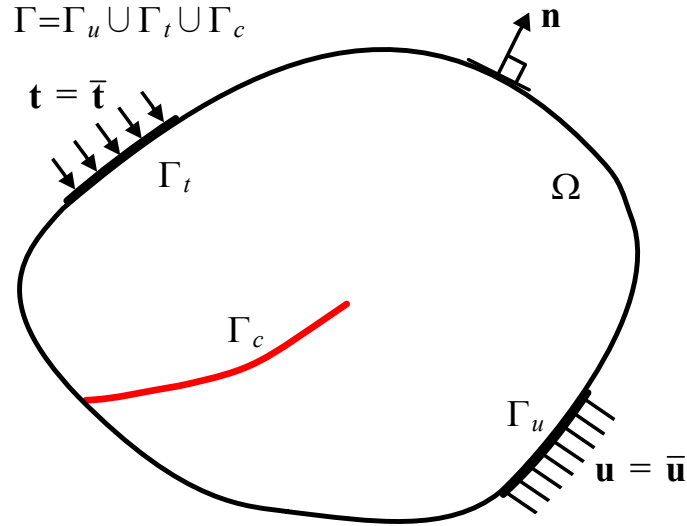


Figure 2.2 Domain and boundaries for two dimensional linear fracture mechanics

2.3.2 Enrichment Functions

In modeling of crack analysis using XFEM in linear fracture mechanics, we use two types of enrichment functions, i.e. the enrich function for strong discontinuities and the ones for fields near crack tips.

a) Strong discontinuities

In the crack analysis of the XFEM, the modeling of the enrichment for the strong discontinuities in the displacement fields is required along the crack surfaces.

In this study, the Heaviside function $H(\mathbf{x})$, which is one of the simplest enrichment functions for the presence of strong discontinuities, is employed. The Heaviside function $H(\mathbf{x})$ is defined as

$$H(\mathbf{x}) = \begin{cases} 1 & \text{on } \Omega_+ \\ -1 & \text{on } \Omega_- \end{cases} \quad (2.18)$$

where Ω_+ and Ω_- are the upper and lower sides of the domain defined on the supports Ω_I of node I containing the enrichment based on the Heaviside function (Moës *et al.* 1999).

This enrichment provides the possibility to avoid the modeling elemental boundaries and double nodes along crack surfaces in the classical FEM.

It is noted that the enrichment based on the above Heaviside function $H(\mathbf{x})$ is special case which does not lead to the problem caused by blending elements in the standard XFEM approximation. This is because the Heaviside enrichment function $H(\mathbf{x})$ is a constant in the blending elements, i.e. $H(\mathbf{x}) = 1$ or -1 in the entire domain in each element. And therefore, the parasitic terms in this case can be compensated by the approximation of the standard finite element part $\mathbf{u}^{\text{std}}$.

b) Fields near crack tips

The stresses and strains are singular at crack tips. In the two dimensional problem, arbitrary deformation near the crack tip is identified as a superposition of the two types

of the deformation modes, i.e., the opening mode (mode I) and the sliding mode (mode II), as illustrated in Fig. 2.3.

The asymptotic near crack tip field in plane deformation composed of mode I and mode II (Irwin 1958, Gdoutos 2005) can be written as

$$\begin{aligned}
 u_x &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \left\{ \cos \frac{\theta}{2} \left(\kappa - 1 + 2 \sin^2 \frac{\theta}{2} \right) \right\} + \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \left\{ \sin \frac{\theta}{2} \left(\kappa + 1 + 2 \cos^2 \frac{\theta}{2} \right) \right\} \\
 &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \left\{ (\kappa - 1) \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \sin \theta \right\} + \frac{K_{II}}{2\mu} \left\{ \sqrt{\frac{r}{2\pi}} (\kappa + 1) \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \sin \theta \right\}
 \end{aligned} \tag{2.19}$$

$$\begin{aligned}
 u_y &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \left\{ \sin \frac{\theta}{2} \left(\kappa + 1 - 2 \cos^2 \frac{\theta}{2} \right) \right\} + \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \left\{ \cos \frac{\theta}{2} \left(-\kappa + 1 + 2 \sin^2 \frac{\theta}{2} \right) \right\} \\
 &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \left\{ (\kappa + 1) \sin \frac{\theta}{2} - \cos \frac{\theta}{2} \sin \theta \right\} + \frac{K_{II}}{2\mu} \left\{ \sqrt{\frac{r}{2\pi}} (-\kappa + 1) \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \sin \theta \right\}
 \end{aligned} \tag{2.20}$$

where r and θ are the local polar co-ordinates at the crack tip as shown in Fig. 2.4. K_I and K_{II} are the mode I and mode II stress intensity factors, respectively. μ corresponds to the shear modulus of elasticity. κ is the material parameter, defined as

$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3 - \nu}{1 + \nu} & \text{plane stress} \end{cases} \tag{2.21}$$

where ν is the Poisson ratio.

Considering above descriptions, the following four enrichment function is used in the fields near crack tips:

$$\gamma_1(\mathbf{x}) = \sqrt{r} \cos \frac{\theta}{2} \quad (2.22)$$

$$\gamma_2(\mathbf{x}) = \sqrt{r} \sin \frac{\theta}{2} \quad (2.23)$$

$$\gamma_3(\mathbf{x}) = \sqrt{r} \cos \frac{\theta}{2} \sin \theta \quad (2.24)$$

$$\gamma_4(\mathbf{x}) = \sqrt{r} \sin \frac{\theta}{2} \sin \theta \quad (2.25)$$

The above enrichment functions $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$) are a basis for the asymptotic near crack tip displacement fields in Eqs. (2.19) and (2.20), developed by Fleming *et al.* (1997) in the framework of the element-free Galerkin method (Belytschko *et al.* 1994). The graphical description of the enrichment functions $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$) are shown in Fig. 2.5.

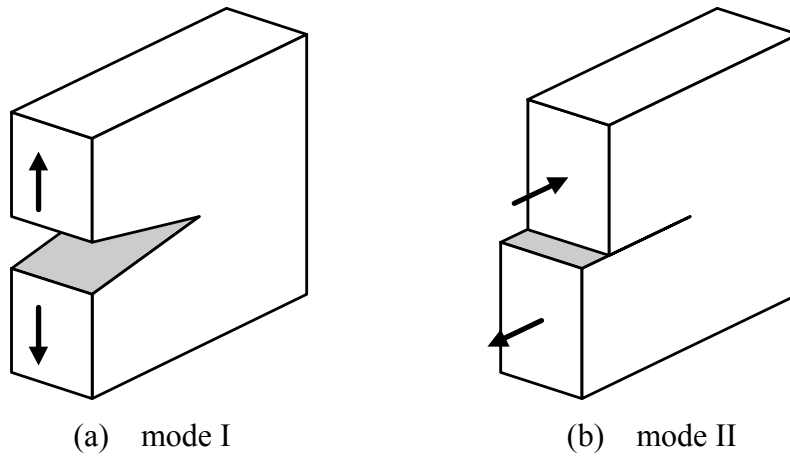


Figure 2.3 Deformation modes near the crack tip in two dimensional problem

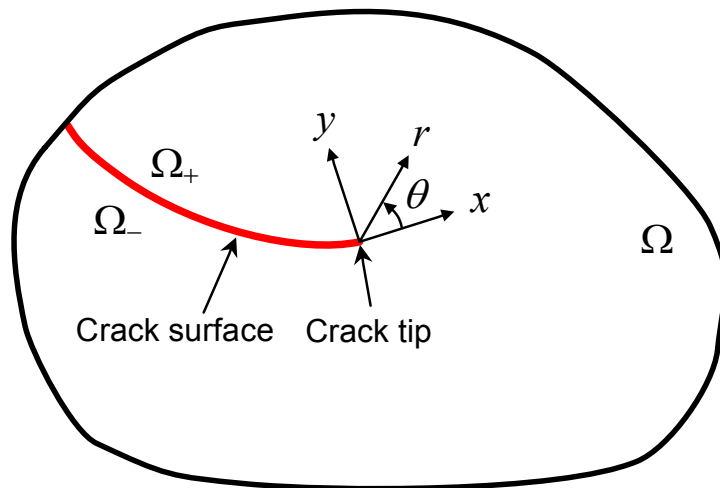


Figure 2.4 Polar coordinates at the crack tip

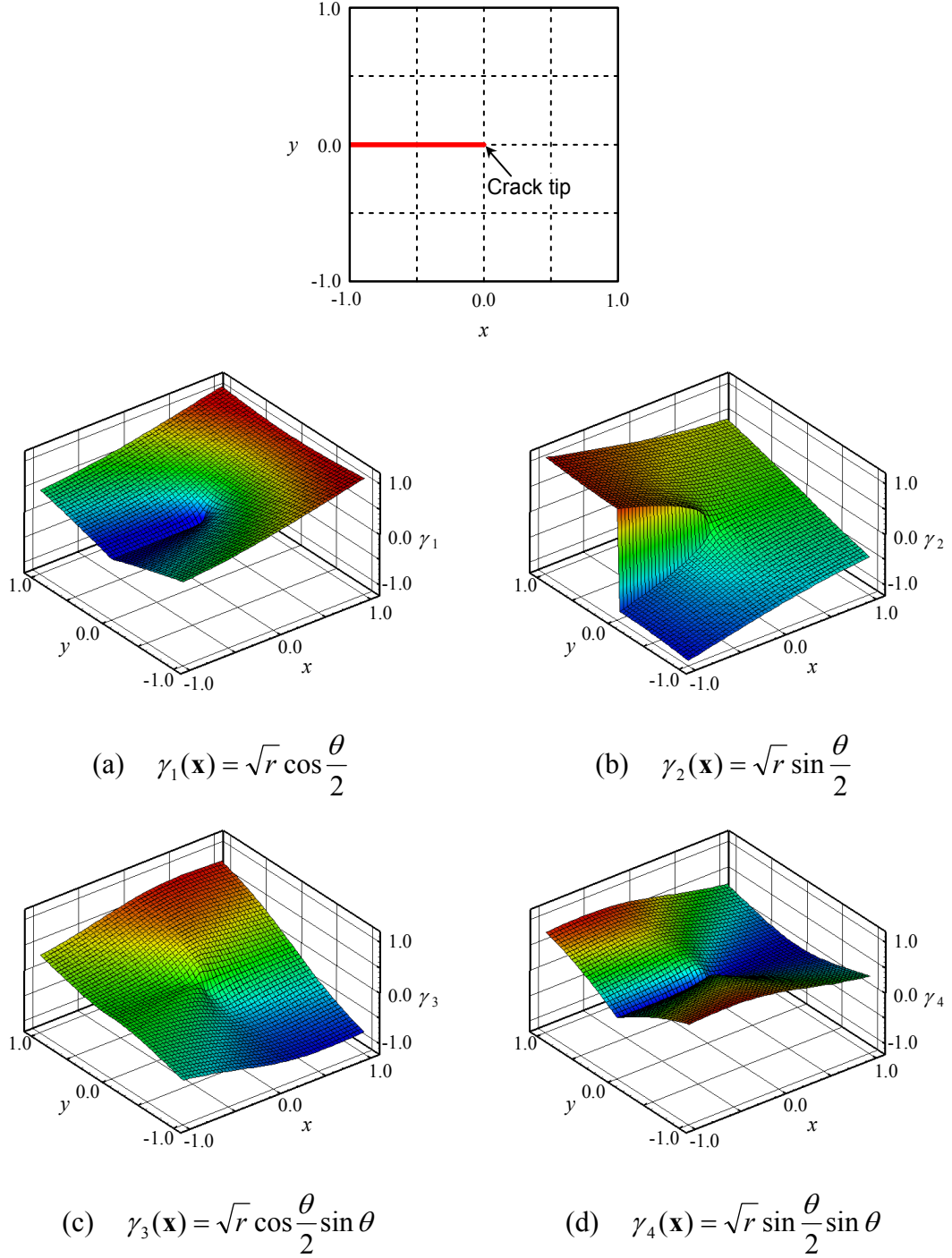


Figure 2.5 Graphical description of the enrichment functions $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$)

2.3.3 Approximation of Displacement Fields

In order to model crack surfaces and tips, the approximation of the displacement fields in the standard XFEM is defined as

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \mathbf{u}^{\text{std}}(\mathbf{x}) + \mathbf{u}^{\text{enr-}J}(\mathbf{x}) + \mathbf{u}^{\text{enr-}C}(\mathbf{x}) \quad (2.26)$$

where $\mathbf{u}^{\text{std}}$ is the approximation of the standard finite element part defined as the same form in Eq. (2.6). $\mathbf{u}^{\text{enr-}J}$ and $\mathbf{u}^{\text{enr-}C}$ are the approximations of the enrichment parts relating to the respective enrichment functions $H(\mathbf{x})$ and $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$), defined as

$$\mathbf{u}^{\text{enr-}J}(\mathbf{x}) = \sum_{I \in J} \phi_I H(\mathbf{x}) \mathbf{b}_I \quad (2.27)$$

$$\mathbf{u}^{\text{enr-}C}(\mathbf{x}) = \sum_{I \in C} \phi_I \sum_{k=1}^4 \gamma_k(\mathbf{x}) \mathbf{c}_I^k \quad (2.28)$$

In Eqs. (2.27) and (2.28), $\mathbf{b}_I$ and $\mathbf{c}_I^k$ ($k = 1, \dots, 4$) are nodal degrees of freedom. J and C are the sets of nodes, respectively.

One of the typical definitions (Laborde *et al.* 2005, Béchet *et al.* 2005, Shibamura & Utsunomiya 2007) of the respective set of the nodes J and C are

$$J = \{ I \in N \mid \Omega_I \cap D \neq \emptyset, I \notin C \} \quad (2.29)$$

$$C = \{ I \in N \mid \|\mathbf{x}_I - \mathbf{x}_{\text{tip}}\| \leq R_C \} \quad (2.30)$$

where $\mathbf{x}_I$ and $\mathbf{x}_{\text{tip}}$ are the coordinates of the node I and the crack tip. Ω_I are the supports of the interpolation functions $\phi_I(\mathbf{x})$. D is the geometry of the crack and R_C is the radius using the definition of the set of nodes C . The geometry of the crack D and the radius R_C are treated as input data. An example of the sets of nodes J and C defined in Eqs. (2.29) and (2.30) near the crack is shown in Fig. 2.6.

2.3.4 Integration

The numerical integration procedure in each element is carried out by employing the standard Gauss quadrature used in the classical FEM except elements that are cut by the crack discontinuity.

In the elements cut by the crack, the sub-division of the domain is required. In this study, the first order finite element approximation is employed as the interpolation function $\phi_I(\mathbf{x})$. A discontinuity of the crack is therefore modeled as one line segment in each element as shown in Fig. 2.7. This modeling enables a simple division of integral domain.

In the case of the element contains a crack tip, the following ‘almost polar integration’ is employed (Laborde *et al.* 2005, Béchet *et al.* 2005). In this integration scheme, the geometric transformation of

$$\tau : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix} \quad (2.31)$$

is firstly defined. This transformation map the reference quadrangular element onto a triangular element as shown Fig. 2.8. Using this transformation, it is possible to build new integration points $\bar{\xi}$ and their weights $\bar{\eta}$ as

$$\bar{\xi} = \tau(\xi) \quad (2.32)$$

$$\bar{\eta} = \eta \det(\nabla \tau) \quad (2.33)$$

where of ξ and η are the integration points and weights in original Gauss quadrature rule.

This method takes account of the character of the solution around crack tip, i.e., the singular behavior dominated by $1/\sqrt{r}$ can be integrated with good accuracy in this integration scheme.

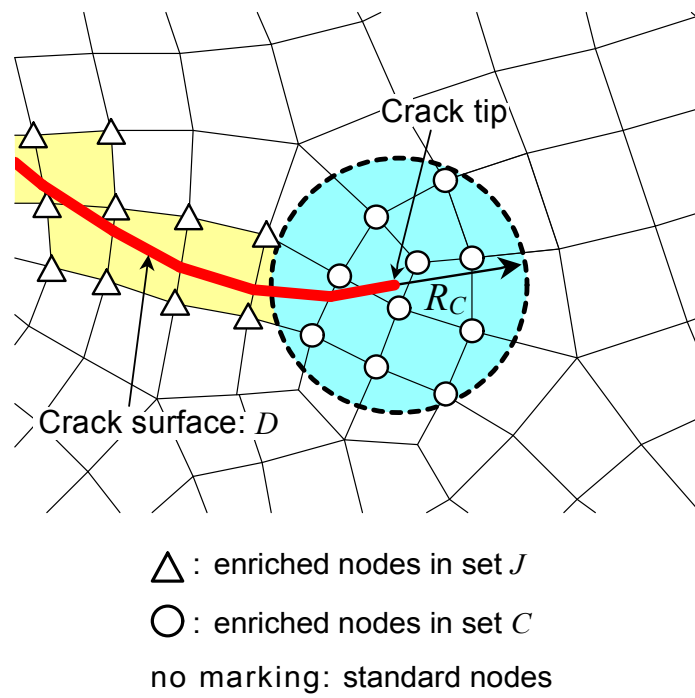


Figure 2.6 Modeling of the sets of nodes J and C in the standard XFEM

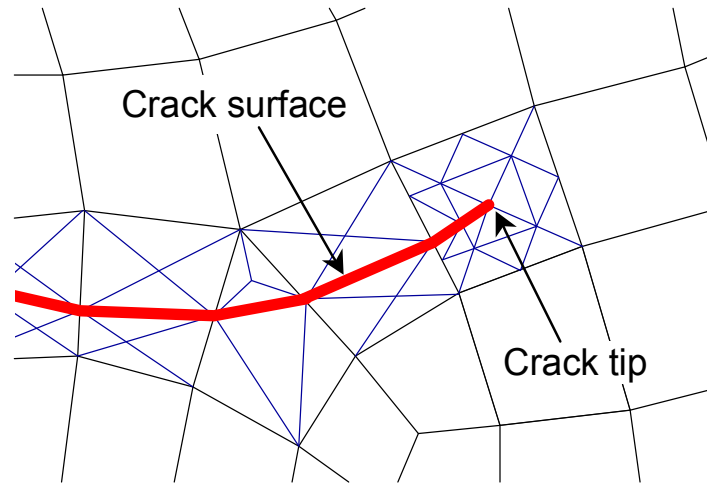


Figure 2.7 Subdivisions of the elements

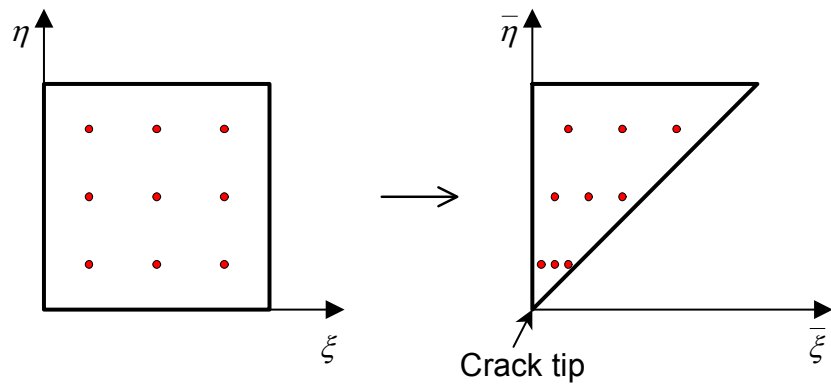


Figure 2.8 Geometric transformation mapping reference quadrangular element onto triangular element

2.3.5 Blending Elements

The enrichment in Eq. (2.27), which is based on the Heaviside function $H(\mathbf{x})$, does not lead to the problem of blending elements, as described in the section 2.3.2 **b)**. The objective blending elements are therefore defined as only those relating to the crack tip enrichment in Eq. (2.28). An example of the blending elements and sub-domains Ω^{std} , Ω^{enr} and Ω^{blnd} in two dimensional crack analysis is shown in Fig. 2.9.

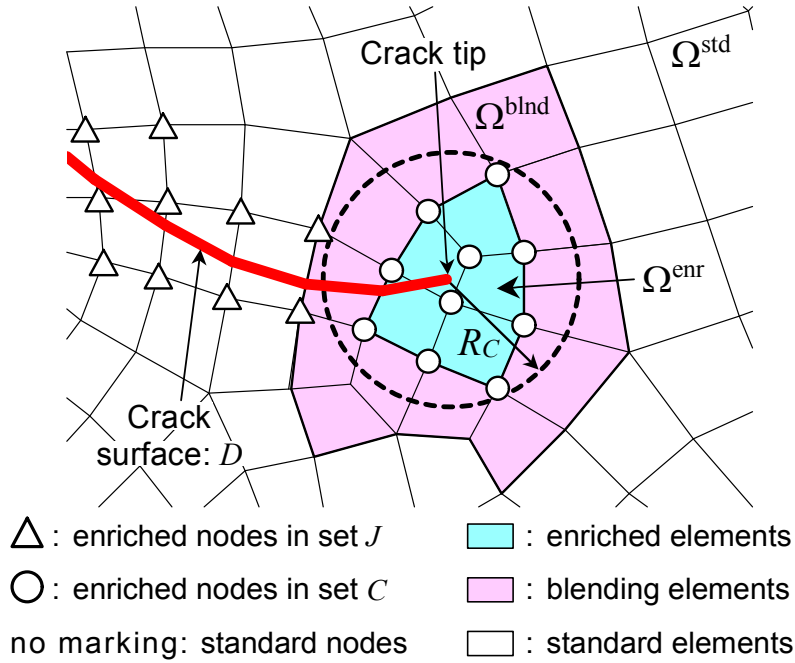


Figure 2.9 Definition of blending elements and sub-domains Ω^{std} , Ω^{enr} and Ω^{blnd} in two dimensional crack analysis

2.3.6 Evaluation of Fracture Mechanics Parameters

In the application of the XFEM to the linear fracture mechanics, the domain integral forms of the well-known J -integral method and the M -integral (interaction integral) method are frequently employed for the evaluation of the fracture mechanics parameters (Belytschko & Black 1999, Zi *et al.* 2004, Meschke & Dumstorff 2007, Wyart *et al.* 2009). In this section, these methods are briefly described.

a) J -integral method

The path independent J -integral is proposed as a criterion for crack propagation in fracture mechanics by Rice (1968).

The problem is considered without the presence of the body force and the tractions along the crack surfaces. The two dimensional form of the J -integral is written as

$$J = \int_{\Gamma} \left(W dy - \mathbf{t} \frac{\partial \mathbf{u}}{\partial x} d\Gamma \right) \quad (2.34)$$

where x (x_1) and y (x_2) are the Cartesian coordinates, Γ is a contour around the crack tip, $\mathbf{u}$ is the displacement vector and $\mathbf{t}$ is the traction vector on the plane defined by the outward normal $\mathbf{n}$, as shown in Fig. 2.10(a). W is strain energy density as

$$W = \sigma_{ij} \varepsilon_{ij} \quad (2.35)$$

in linear elastic materials.

J is equivalent to the fracture energy release rate for linear elastic materials G .

For general mixed mode of I and II, we have the following relationship between the J -integral and the stress intensity factors K_I and K_{II} as

$$J = \frac{1}{E'} \left(K_I^2 + K_{II}^2 \right) \quad (2.36)$$

where E' is the effective Young's modulus defined by using the Young's modulus E and the Poisson's ratio ν , as

$$E' = \begin{cases} \frac{E}{1-\nu^2} & \text{plane strain} \\ E & \text{plane stress} \end{cases} \quad (2.37)$$

b) Equivalent domain integral method

The equivalent domain integral method is proposed by Li *et al.* (1985) as an alternative approach of the contour J -integral defined in Eq. (2.34). In this approach, J -integral can be defined as

$$J = \int_A \left(\sigma_{ij} \frac{\partial u_i}{\partial x_j} - W \delta_{li} \right) \frac{\partial q}{\partial x_i} d\Omega \quad (2.38)$$

where q is an arbitrary smooth function which is equal to unity on the inner path Γ_1 and zero on the outer path Γ_0 around crack tip (see Fig. 2.10(b)). The inner path Γ_1 is often chosen as an infinitesimal contour around the crack tip. Details of the formulation of Eq. (2.38) can be found in the references (Li *et al.* 1985, Moran & Shih 1987).

The values of q within an element can be interpolated as

$$q(\mathbf{x}) = \sum_{I=1}^m \phi_I(\mathbf{x}) q_I \quad (2.39)$$

where m is the number of the elemental nodes. $\phi_I(\mathbf{x})$ are the standard finite element interpolation functions. q_I are the nodal values of q . The nodal values q_I are frequently determined by applying a circle with radius R_F (Moës *et al.* 1999, Nagashima *et al.* 2003, Shibamura & Utsunomiya 2008), as being equal to unity at inner nodes and zero at outer nodes of the circle, as shown in Fig. 2.11. In this modeling, the outer path Γ_0 coincides with the element boundaries. And therefore, the domain integral can be calculated only on the domain Ω_F composed of the elements where the values of $\partial q / \partial x_i$

are non-zero (see Fig. 2.11), as

$$J = \int_{\Omega_F} \left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1i} \right) \frac{\partial q}{\partial x_i} d\Omega \quad (2.40)$$

Therefore, the equivalent domain integral form of the J -integral enables to facilitate the numerical calculations, in comparison with the original contour integral method. In this study, the radius R_F is called as the radius of the J -integral path.

c) M -integral method

The M -integral method (interaction integral method) was introduced in order to evaluate the respective stress intensity factors K_I and K_{II} in the mixed mode I and II (Shi *et al.* 1965, Yau *et al.* 1980).

In this method, auxiliary fields are introduced and superimposed onto the actual fields. The J -integral for the superposition of the two states, i.e., the actual state and the auxiliary state, can be defined in domain form, as

$$\begin{aligned} J &= \int_{\Omega_F} \left[\left(\sigma_{ij}^{\text{act}} + \sigma_{ij}^{\text{aux}} \right) \left(\frac{\partial u_i^{\text{act}}}{\partial x_1} + \frac{\partial u_i^{\text{aux}}}{\partial x_1} \right) - \left(\sigma_{ij}^{\text{act}} + \sigma_{ij}^{\text{aux}} \right) \left(\varepsilon_{ij}^{\text{act}} + \varepsilon_{ij}^{\text{aux}} \right) \delta_{1i} \right] \frac{\partial q}{\partial x_i} d\Omega \\ &= J^{\text{act}} + J^{\text{aux}} + \int_{\Omega_F} \left[\sigma_{ij}^{\text{act}} \frac{\partial u_i^{\text{aux}}}{\partial x_1} + \sigma_{ij}^{\text{aux}} \frac{\partial u_i^{\text{act}}}{\partial x_1} - \sigma_{ij}^{\text{act}} \varepsilon_{ij}^{\text{aux}} \delta_{1j} \right] \frac{\partial q}{\partial x_j} d\Omega \end{aligned} \quad (2.41)$$

where the superscripts of “act” and “aux” represent the actual states and auxiliary states, respectively. In Eq. (2.41), the third term of the right hand side is defined as the M -integral, i.e.,

$$M = \int_{\Omega_F} \left[\sigma_{ij}^{\text{act}} \frac{\partial u_i^{\text{aux}}}{\partial x_1} + \sigma_{ij}^{\text{aux}} \frac{\partial u_i^{\text{act}}}{\partial x_1} - \sigma_{ij}^{\text{act}} \varepsilon_{ij}^{\text{aux}} \delta_{1j} \right] \frac{\partial q}{\partial x_j} d\Omega \quad (2.42)$$

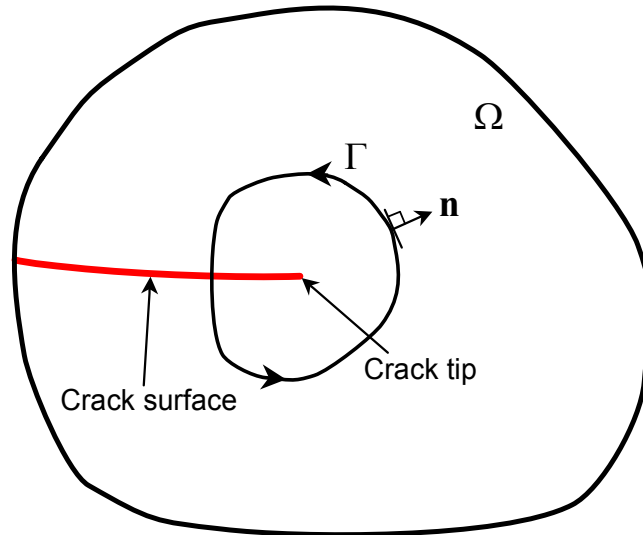
The J -integral for the superposition of the two states can be also defined as

$$\begin{aligned}
 J &= \frac{1}{E'} \left\{ \left(K_I^{\text{act}} + K_I^{\text{aux}} \right)^2 + \left(K_{II}^{\text{act}} + K_{II}^{\text{aux}} \right)^2 \right\} \\
 &= J^{\text{act}} + J^{\text{aux}} + \frac{2}{E'} \left(K_I^{\text{act}} K_I^{\text{aux}} + K_{II}^{\text{act}} K_{II}^{\text{aux}} \right)
 \end{aligned} \tag{2.43}$$

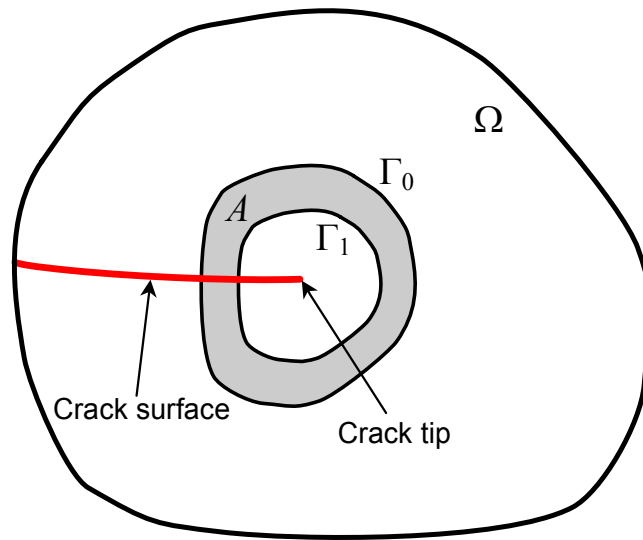
The M -integral can be therefore defined as

$$M = \frac{2}{E'} \left(K_I^{\text{act}} K_I^{\text{aux}} + K_{II}^{\text{act}} K_{II}^{\text{aux}} \right) \tag{2.44}$$

The stress intensity factors of the actual state K_I^{act} and K_{II}^{act} can be evaluated by calculating M -integral in Eqs. (2.42) and (2.44). That is, selecting the auxiliary state as $K_I^{\text{aux}} = 1$ and $K_{II}^{\text{aux}} = 0$ gives the value of K_I^{act} and selecting the auxiliary state as $K_I^{\text{aux}} = 0$ and $K_{II}^{\text{aux}} = 1$ gives the value of K_{II}^{act} .



(a) J -integral path Γ around the crack tip



(b) Equivalent domain A

Figure 2.10 Definition of the J -integral around a crack tip

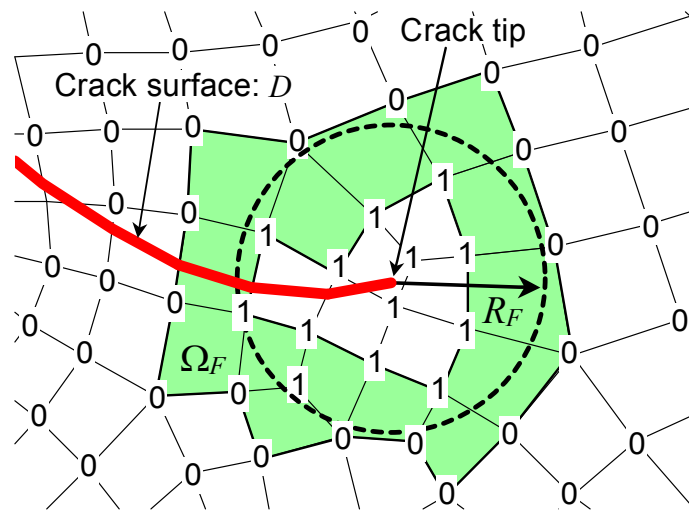


Figure 2.11 Modeling of the domain integral method and nodal values of the function $q(\mathbf{x})$

Chapter 3

Effective Modeling of Crack Tip Enrichment and J -integral in Standard XFEM

3.1 INTRODUCTION

The application to the elastic linear fracture mechanics is the most principal application of the XFEM. In this application, two types of the local enrichment, which are the discontinuity along crack surface and the singular field near crack tip, are introduced in the approximation. The features of the application are described as follows:

- (a) The crack can be modeled independently from the finite element meshes.
- (b) the fracture mechanics parameters can be accurately evaluated without refinement of the meshes or use of the singular elements around the crack tip (Barsoum 1976).
- (c) The complicated remeshing procedures according to the crack propagation can be avoided.
- (d) we can take advantages of the features of the FEM in the modeling of the entire structure.

Thus, there are many investigations in the application of the XFEM to the crack analysis.

It was reported that there are the partially enriched elements called the blending elements causing the problems in numerical accuracy in the XFEM. However, the influence of the problem of the blending elements in the application of the standard XFEM to the linear fracture mechanics is not sufficiently investigated in the past studies. Thus, the influence of the blending elements has reluctantly been ignored in many cases of the recent studies (Elguedj *et al.* 2007, Loehnert & Belytschko 2007, Mohammadi 2008). In addition, the method to effectively avoid the influence of the problem of blending elements in the framework of the standard XFEM will be useful, because there are a lot of studies of the crack analyses by the XFEM are reported based on the standard XFEM approximation.

According to the above statements, we first evaluate the influence of the blending elements, which relate to the enrichment near the crack tip field, in the application of the standard XFEM to the two dimensional linear fracture mechanics. In this evaluation, the reproducibility of the stress fields and the numerical accuracy of the fracture mechanics parameters are verified by using models in the infinite plates. And then, based on the results of the evaluation, we propose the effective method to improve the numerical accuracies of the fracture mechanics parameters in the frameworks of the standard XFEM.

An outline of this chapter is as follows: In Section 3.2, we evaluate the influence of the blending elements in the application of the standard XFEM to the two dimensional linear fracture mechanics. In Section 3.3, we then propose the effective modeling of the crack tip enrichment and J -integral in the frameworks of the standard XFEM. Finally, in Section 3.4, the concluding remarks of the study in this chapter are summarized.

3.2 INFLUENCE OF BLENDING ELEMENTS IN CRACK ANALYSES

In this section, we present basic evaluations of the influence of the problem of blending elements in the standard XFEM approximation to the numerical results in crack analyses. This section contains as follows: In the section 3.2.1, the definition of the blending elements are briefly reviewed. In the section 3.2.2, numerical examples to evaluate the influence of the problem of the blending elements are presented by using the crack models with unit fracture mode I or mode II in the infinite plate.

3.2.1 *Blending Elements in Crack Analyses*

In order to model crack surfaces and tips, the approximation of the displacement fields in the standard XFEM are defined as

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \mathbf{u}^{\text{std}}(\mathbf{x}) + \mathbf{u}^{\text{enr-}J}(\mathbf{x}) + \mathbf{u}^{\text{enr-}C}(\mathbf{x}) \quad (3.1)$$

where

$$\mathbf{u}^{\text{std}}(\mathbf{x}) = \sum_{I \in N} \phi_I(\mathbf{x}) \mathbf{u}_I \quad (3.2)$$

$$\mathbf{u}^{\text{enr-}J}(\mathbf{x}) = \sum_{I \in J} \phi_I H(\mathbf{x}) \mathbf{b}_I \quad (3.3)$$

$$\mathbf{u}^{\text{enr-}C}(\mathbf{x}) = \sum_{I \in C} \phi_I \sum_{k=1}^4 \gamma_k(\mathbf{x}) \mathbf{c}_I^k \quad (3.4)$$

(see details in the section 2.3)

The enrichment in Eq. (3.3), which is based on the Heaviside function $H(\mathbf{x})$, does

not lead to the problem of blending elements in the standard XFEM approximation as described in the section 2.3. The blending elements are therefore defined by only the crack tip enrichment in Eq. (3.4). An example of the blending elements and sub-domains Ω^{std} , Ω^{enr} and Ω^{blnd} in two dimensional crack analyses is shown in Fig. 3.1.

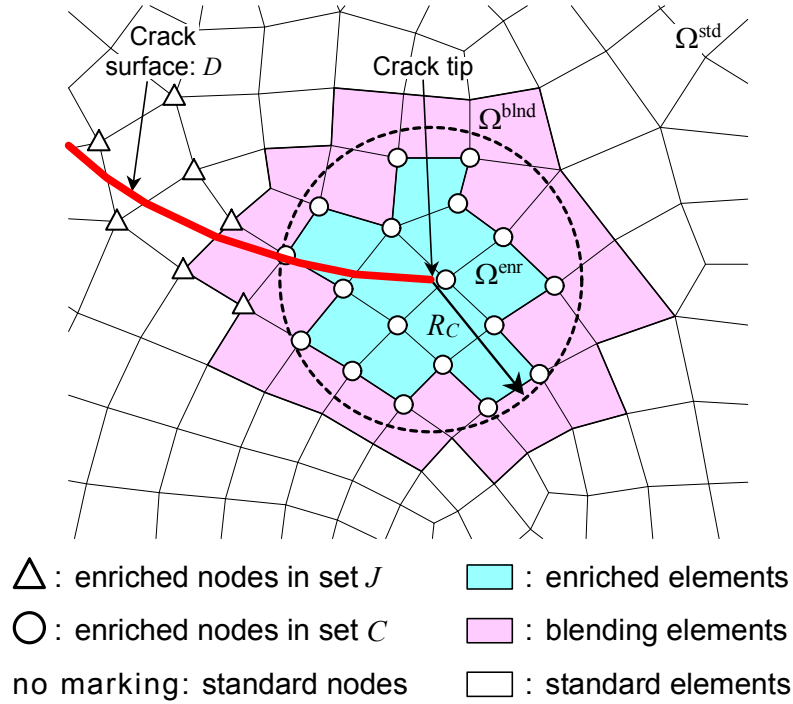


Figure 3.1 Definition of blending elements and sub-domains Ω^{std} , Ω^{enr} and Ω^{blnd} in two dimensional crack analyses

3.2.2 *Influence of Blending Elements*

a) Model descriptions

The numerical analyses in this section are evaluated by using the models which assumes finite domains near the crack tip in the infinite plates. The enrichment function $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$) used in Eq. (3.4) can be reproduced as the exact solution in this model. We evaluate the influence of the blending elements relating to the crack tip enrichment in the following.

The respective unit fracture mode I and II in the infinite plates—including straight crack under the plane strain condition are considered as shown in Figs. 3.2(a) and (b). Then, let us define a local finite square domain A including the crack tip in the center in the respective infinite plates as numerical models. The dimension of the domain A is sufficiently smaller than the crack length $2a$. The size of this numerical domain A is 2×2 as shown in Fig. 3.2(c). The domain A is divided into the elements of 9×9 . The mesh size is therefore $h = 2/9$.

The material constants are the Young's modules $E = 2.0 \times 10^{11}$ and the Poisson's ratio $\nu = 0.3$.

The Gauss quadrature is used in the numerical integration for the calculations of the stiffness matrixes and the J -integral. Based on our preliminary research, the integration order is 5, except the calculations of the elemental stiffness matrixes in the domain Ω^{std} in which the bi-linear finite element approximation is used. The integration order is 2 in the calculations of the elemental stiffness matrixes in the domain Ω^{std} .

The boundary conditions on the boundary Γ_A of the domain A assume the asymptotic solution of displacement field near the crack tip in the respective fracture mode I ($K_I = 1$ and $K_{II} = 0$) and mode II ($K_I = 0$ and $K_{II} = 1$). The nodal displacements in the respective models are therefore directly specified on Γ_A as

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \frac{1}{\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos \frac{\theta}{2} \left(1 - 2\nu + \sin^2 \frac{\theta}{2} \right) \\ \sin \frac{\theta}{2} \left(2 - 2\nu - \cos^2 \frac{\theta}{2} \right) \end{Bmatrix} \quad (\text{in fracture mode I}) \quad (3.5)$$

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \frac{1}{\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin \frac{\theta}{2} \left(2 - 2\nu + \cos^2 \frac{\theta}{2} \right) \\ -\cos \frac{\theta}{2} \left(1 - 2\nu - \cos^2 \frac{\theta}{2} \right) \end{Bmatrix} \quad (\text{in fracture mode II}) \quad (3.6)$$

where μ is the modules of elasticity in shear.

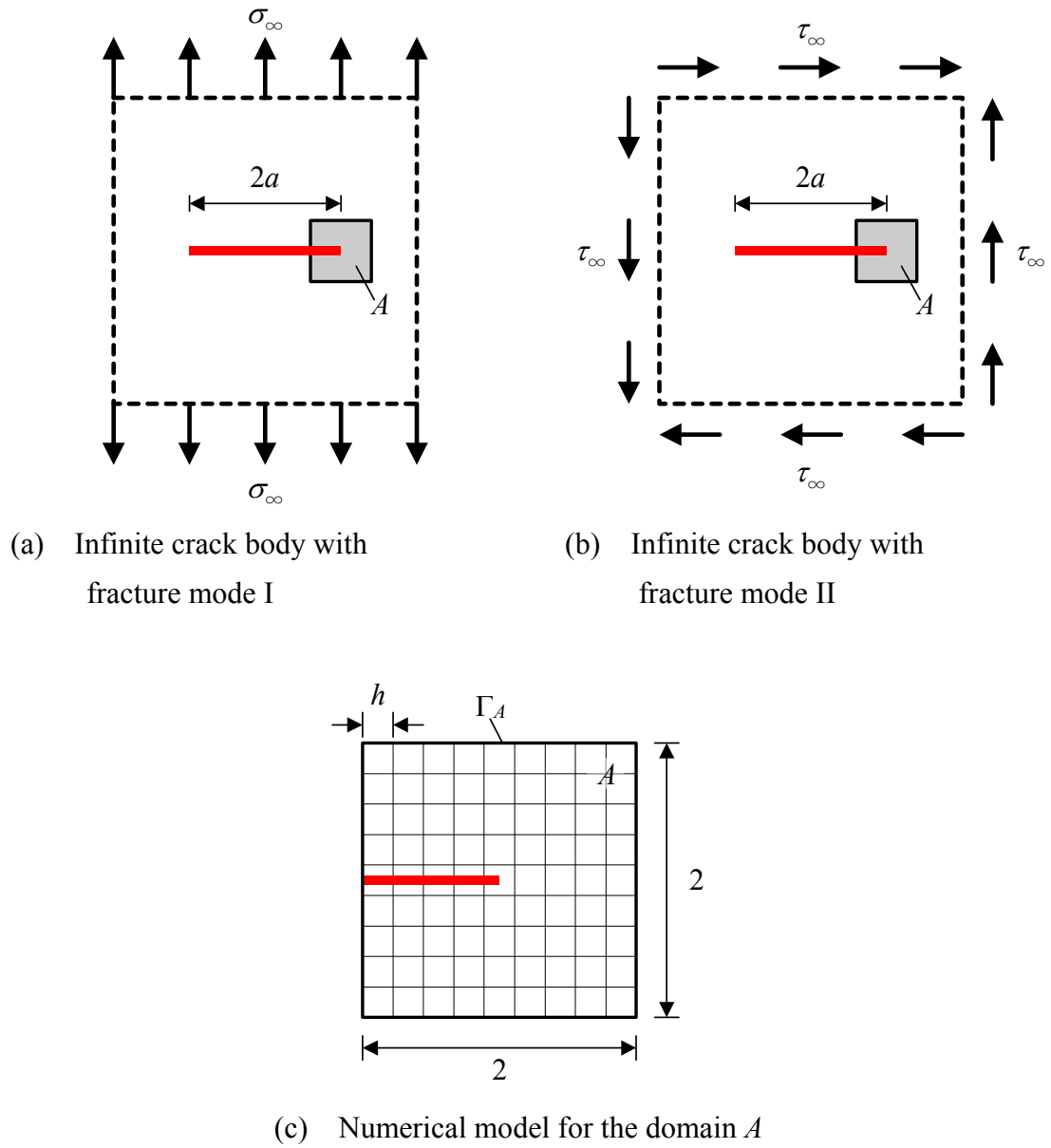


Figure 3.2 Numerical models in infinite plates

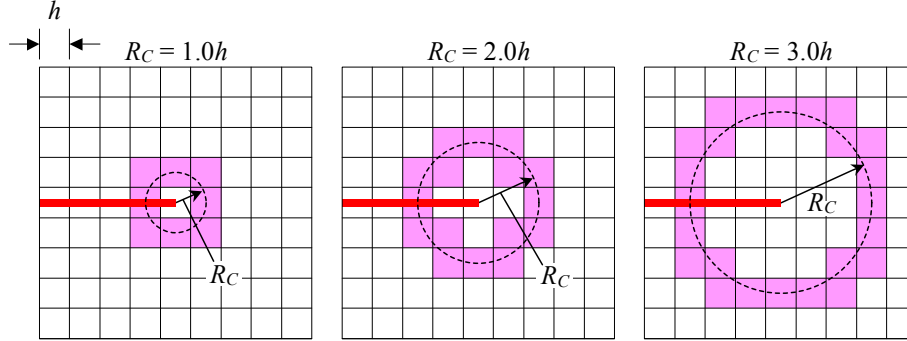


Figure 3.3 Blending domains Ω^{blind} in the numerical model

b) Evaluation of stress fields

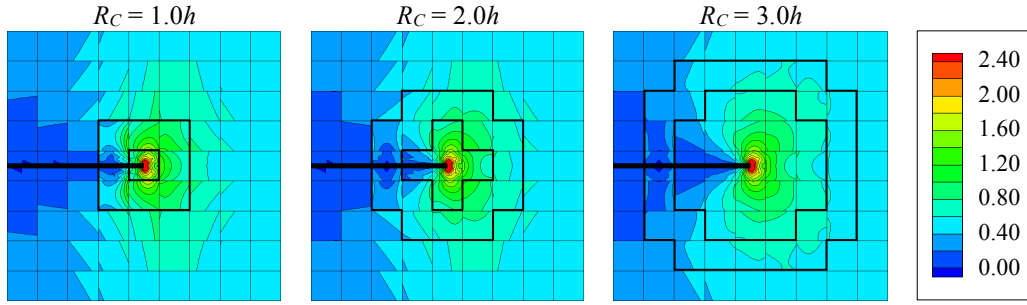
When a certain mesh is assumed in the numerical model, the blending domain Ω^{blind} composed of the blending elements is defined by the radius R_C for the definition of the set of nodes C , which is related to the crack tip enrichment.

The reproducibility of the stress fields is evaluated according to parameter of the radius R_C . In this evaluation, the cases of $R_C = 1.0h$, $2.0h$ and $3.0h$ are considered, corresponding to the mesh size h in the respective models of fracture mode I and II.

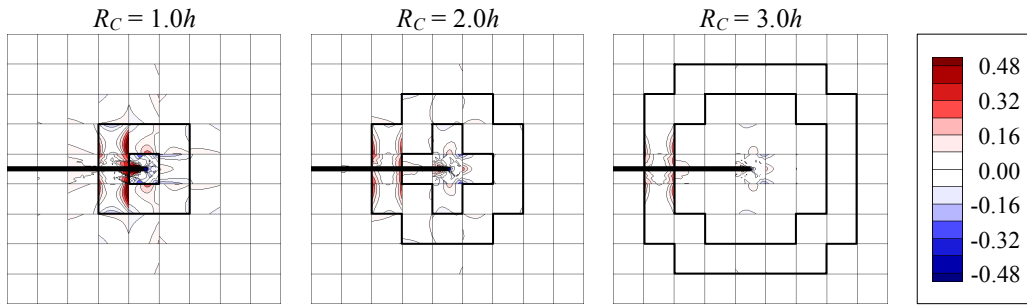
The blending domains Ω^{blind} are shown in Fig. 3.3, corresponding to respective R_C . The maximum principal stress σ_{max} and the Von Mises stress σ_{VM} are evaluated. The numerical results of the stress fields and their errors from the asymptotic solution are shown in Fig. 3.4 and Fig. 3.5, respectively.

We find large errors in the blending elements in the both models. In particular, the most serious errors are found in the cases of $R_C = 1.0h$. It is noted that the definition of the set of nodes C using the radius $R_C = 1.0h$ corresponds to that of the set of nodes whose supports of the interpolation functions containing the crack tip, which is widely employed in exiting studies.

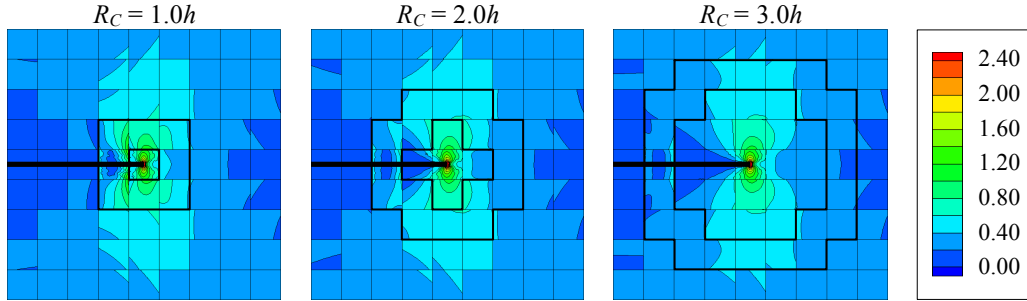
On the other hand, it is shown that enlarging the ranges of the enrichment R_C tend to reduce the errors of the stress fields.



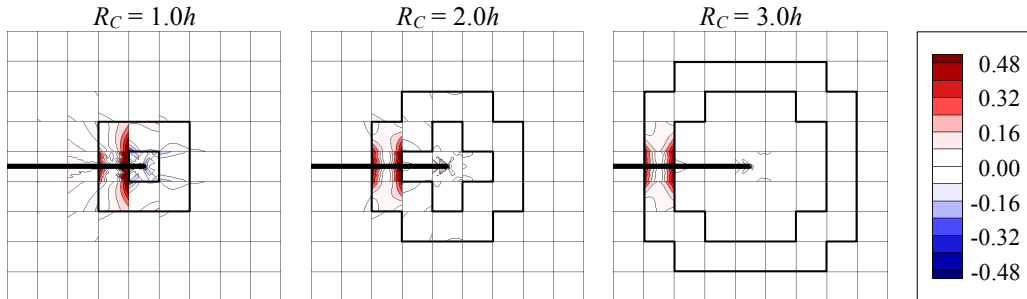
(a) Maximum principal stress $\sigma_{\max}$



(b) Errors of $\sigma_{\max}$ from the asymptotic solution

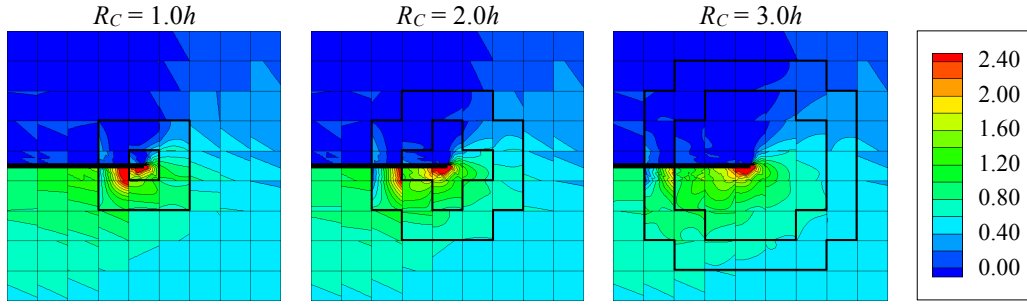


(c) Von Mises stress σ_{VM}

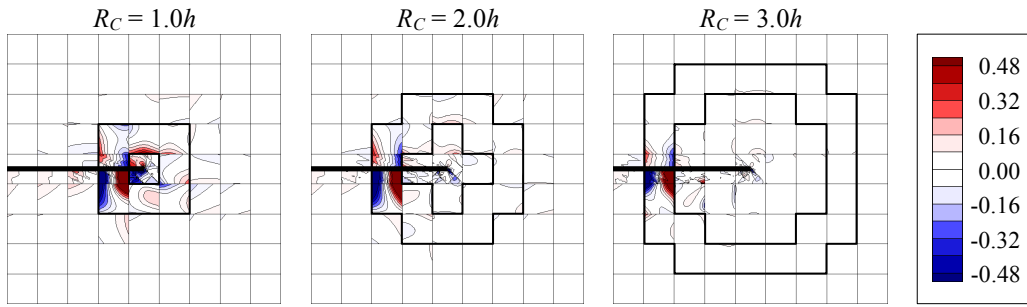


(d) Errors of σ_{VM} from the asymptotic solution

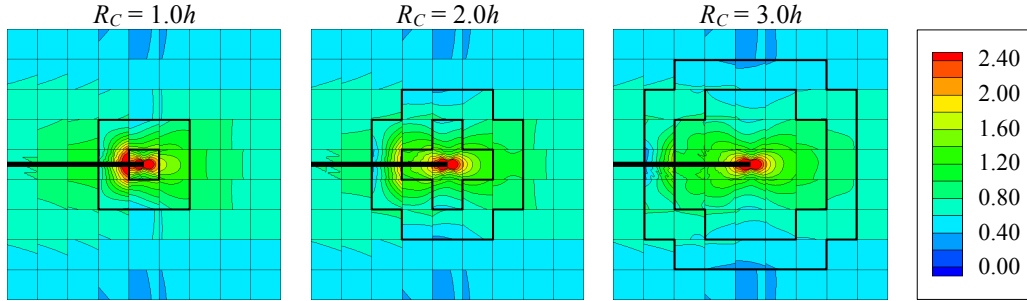
Figure 3.4 Numerical results of the stress in the mode I model ($K_I = 1$, $K_{II} = 0$)



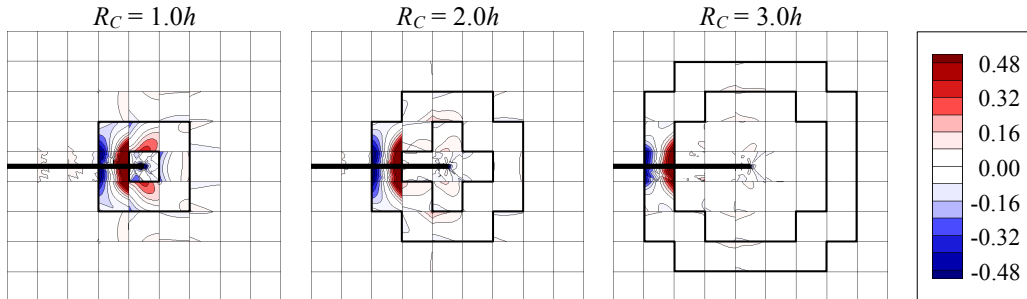
(a) Maximum principal stress $\sigma_{\max}$



(b) Errors of $\sigma_{\max}$ from the asymptotic solution



(c) Von Mises stress σ_{VM}



(d) Errors of σ_{VM} from the asymptotic solution

Figure 3.5 Numerical results of the stress in the mode II model ($K_I = 0$, $K_{II} = 1$)

c) Evaluation of fracture mechanics parameters

In this subsection, the influence of the problem of blending elements to the numerical accuracies of the fracture mechanics parameters is evaluated. The numerical accuracies of the fracture mechanics parameters are especially important in the application of crack analyses for real structures, because the behaviors of the cracks, e.g. crack propagation rate and direction, can be modeled based on the fracture mechanics parameters.

The parameters used for the evaluation are the stress intensity factor K_I or K_{II} and the energy release rate G which are the basic linear fracture mechanics parameters. For the evaluation of the parameters, we employ the J -integral method and the M -integral method described in the section 2.5. These methods are generally used in crack analyses by using the XFEM.

We employ the ratio m_C / m_F as the measure of this evaluation, where m_F is the number of the elements in the integration domain Ω_F of the J -integral and m_C is the number of the elements in the common part of the blending domain Ω^{blend} and J -integral domain Ω_F .

Four cases of $R_C = 1.0h, 2.0h, 3.0h$ and $4.0h$ are adopted as the conditions of ranges of the enrichment related to the nodes of C . In addition, four cases of $R_F = 1.0h, 2.0h, 3.0h$ and $4.0h$ for the respective ranges R_C are also adopted as the integral radiuses for the J -integral.

Numerical results for the evaluation of the fracture mechanics parameters in the respective models of the fracture mode I and II are shown in Figs. 3.6 and 3.7, respectively. In the Figs. 3.6 and 3.7, the ratio m_C / m_F is also described.

It is found that there are significant errors in the numerical results of the stress intensity factor K_I or K_{II} and the energy release rate G where the ratio m_C / m_F is increased in comparison with the other conditions. This tendency is typical in the case of $m_C / m_F = 1$, where the two radiuses of R_C and R_F are the same each other, i.e. $R_C = R_F$. It is particularly found out that the maximum errors of 4.8% for K_I and 13.2% for G in the mode I crack model and those of 5.5% for K_{II} and 18.2% for G in the mode II crack model, respectively. Consequently, the lack of numerical accuracies is commonly caused when the integration domain of the J -integral Ω_F is fully overlapped with the blending domain Ω^{blend} .

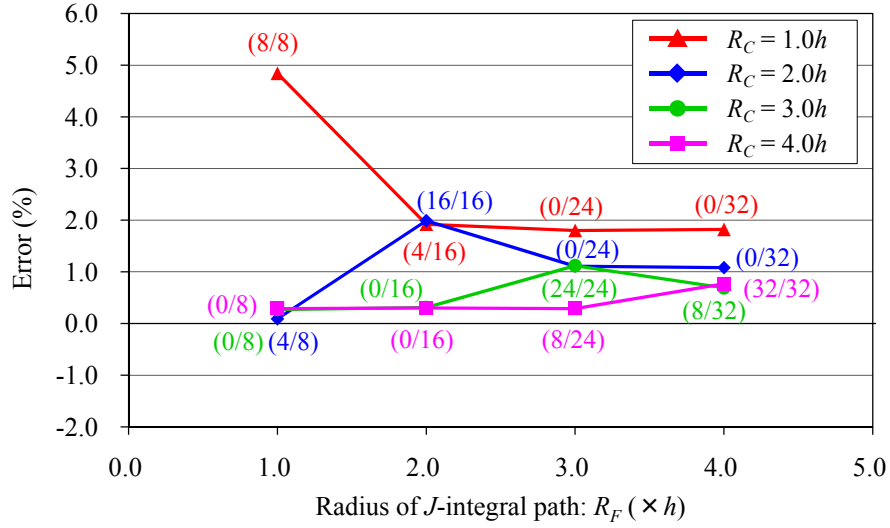
It is therefore verified that the influence of blending elements, which is assumed to

be small enough in many recent studies (Elguedj *et al.* 2007, Loehnert & Belytschko 2007, Mohammadi 2008), is too large to be neglected in the evaluation of the fracture mechanics parameters by using the standard XFEM.

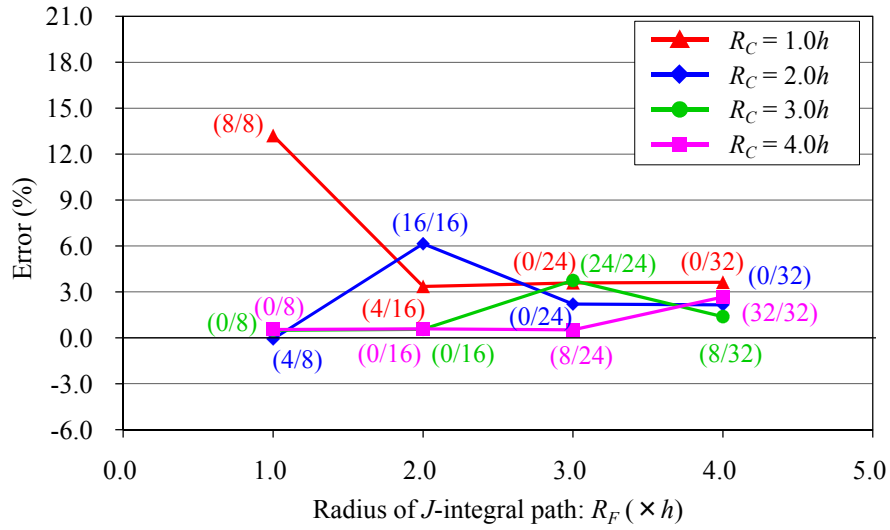
In addition, under the condition satisfying $\Omega_F \subset \Omega^{\text{enr}}$, the numerical results show good accuracies. This condition implies that there are not any common parts between the blending domain Ω^{blnd} and the J -integral domain Ω_F , i.e., the ratio $m_C/m_F = 0$. That is, the errors of the fracture mechanics parameters are less than 0.4% for K_I and less than 0.6% for G in the mode I crack model, and the errors are also less than 0.8% for K_{II} and less than 1.6% for G in the mode II crack model.

According to the above results, the integration domain of the J -integral Ω_F should satisfy the condition to be included in the domain Ω^{enr} , which is composed of the elements whose nodes are fully enriched in the set C , i.e., $\Omega_F \subset \Omega^{\text{enr}}$. This condition also satisfies that the J -integral domain Ω_F is not overlapped with the blending domain Ω^{blnd} at the same time.

3.2 INFLUENCE OF BLENDING ELEMENTS IN CRACK ANALYSES

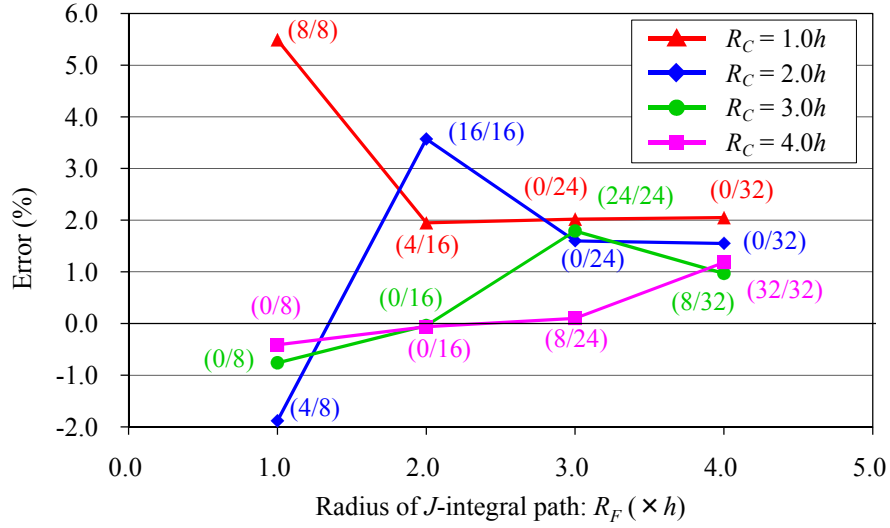


(a) Stress intensity factor: K_I

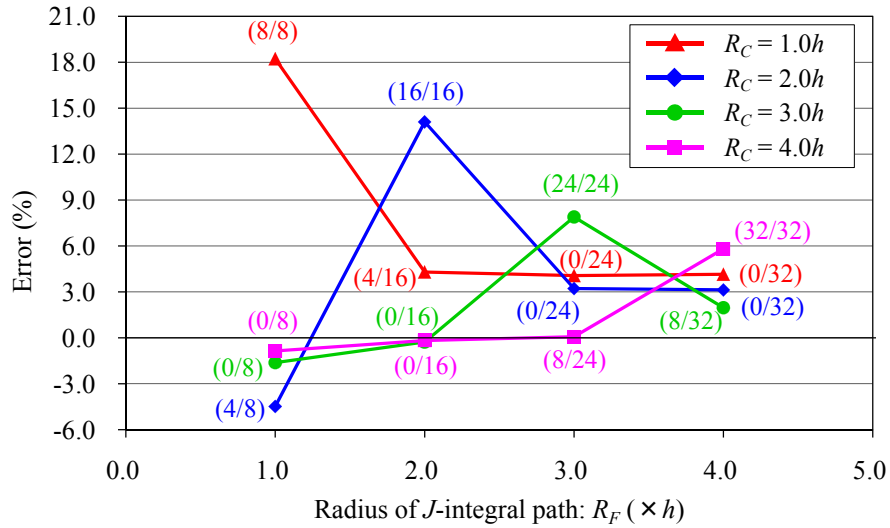


(b) Energy release rate: G

Figure 3.6 Numerical results of fracture mechanics parameters in the mode I model
(figures in the graph show the ratio m_C/m_F)



(a) Stress intensity factor: K_{II}



(b) Energy release rate: G

Figure 3.7 Numerical results of fracture mechanics parameters in the mode II model (figures in the graph show the ratio m_C/m_F)

3.3 IMPROVEMENT OF NUMERICAL ACCURACY OF FRACTURE MECHANICS PARAMETERS

3.3.1 *Modeling of Crack Tip Enrichment*

In the numerical results in the subsection 3.2.2 **a)**, it was found that the definition of the set of nodes C with large range can improve the reproducibility of the stress fields. However, such the definitions cause the limitations of the coarse size of the meshes and significant increase of the nodal degrees of freedom. Therefore, the range of the set of nodes should be minimized in the actual application to the real structures.

In the numerical results in the subsection 3.2.2 **b)**, it was also found that the fracture mechanics parameters can be evaluated with high accuracy in the case that the domain Ω^{enr} , which is composed of the enriched elements, fully covers the J -integral domain Ω_F , i.e., $\Omega_F \subset \Omega^{\text{enr}}$.

Based on the above results, in this section, the definition of the minimum range of the set of nodes C is proposed, based on the standard XFEM approximation. Using the proposed definition of the set of nodes C , we can avoid the influences of the problem of blending elements to accurately evaluate the fracture mechanics parameters according to an arbitrary J -integral path.

We first define a set of the elements E_B in which the summation of the nodal values of the weight function q_I , which is defined in Eq. (2.32) in the section 2.3, is not zero using the radius of the J -integral path R_F as

$$E_B = \left\{ e \in E : \sum_{I \in N} q_I > 0 \right\} \quad (3.7)$$

where

$$q_I = \begin{cases} 1 & (\|\mathbf{x}_I - \mathbf{x}_{\text{tip}}\| \leq R_F) \\ 0 & (\|\mathbf{x}_I - \mathbf{x}_{\text{tip}}\| > R_F) \end{cases} \quad (3.8)$$

E is a set of all elements in the entire domain Ω , and N is a set of nodes composing the element E .

Then, the domain Ω_B is defined to be composed of the elements in the set E_B , as

$$\Omega_B = \bigcup_{e \in E_B} \Omega_e \quad (3.9)$$

An example of the domain Ω_B in this modeling is shown in Fig. 3.8.

In addition, the enriched domain Ω^{enr} is defined in agreement with the domain Ω_B . That is, the set of nodes C is

$$C = \bigcup_{e \in E_B} I \quad (3.10)$$

It is noted that the proposed set of nodes C is defined in Eq. (3.10) without any consideration of the radius R_C previously defined in Eq. (2.30) in the section 2.3.

From the set of nodes C in Eq. (3.10), the enriched domain Ω^{enr} and the blending domain Ω^{blnd} are defined, respectively, as shown in Fig. 3.9.

According to the results in the section 3.2.2, it was found that the fracture mechanics parameters can be evaluated with good accuracies, under the condition of the integration domain of the J -integral Ω_F included in the domain Ω^{enr} , i.e., $\Omega_F \subset \Omega^{\text{enr}}$. Based on Eqs. (3.7) – (3.10), the set of nodes C can be defined as the minimum range according to an arbitrary radius R_F of the J -integral path satisfying the above condition.

In this modeling, the J -integral domain Ω_F is defined as shown in Fig. 3.10, corresponding to the blending domain Ω^{blnd} shown in Fig. 3.9.

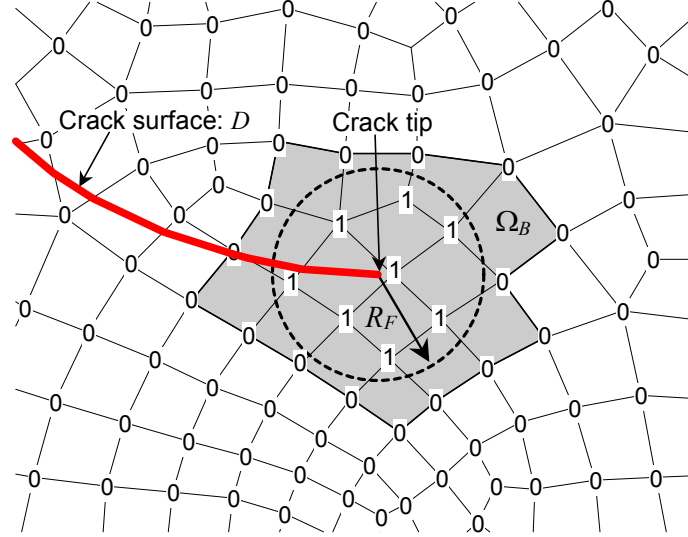


Figure 3.8 Domain Ω_B composing the elements in the set E_C

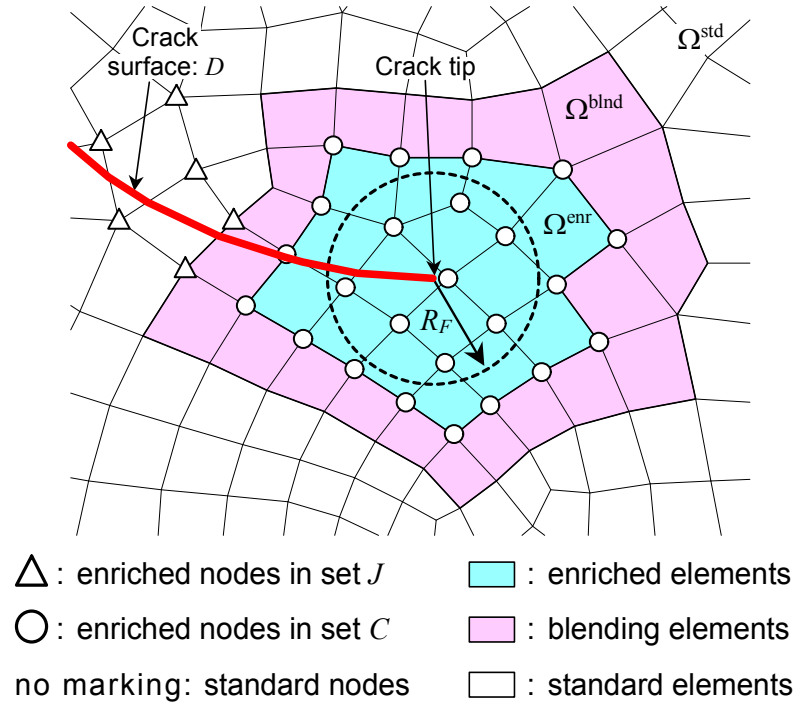


Figure 3.9 Set of node C and blending domain Ω^{blnd} based on J -integral path

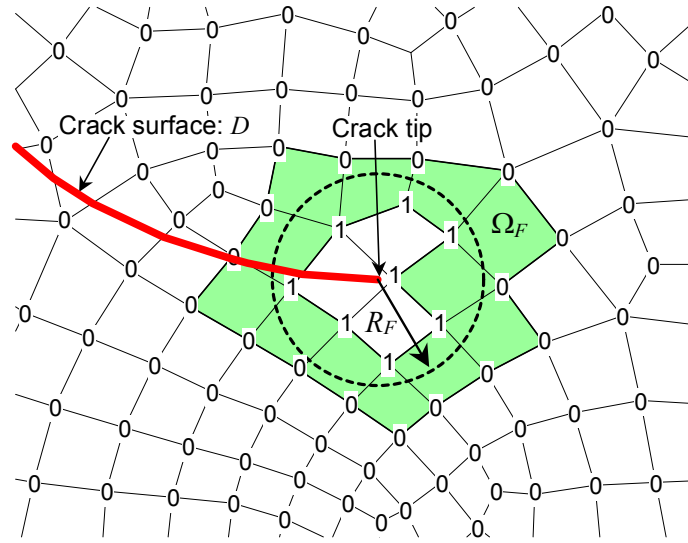


Figure 3.10 J -integral domain Ω_F corresponding to the set of nodes C and the blending domain Ω^{blnd} in Fig. 3.9

3.3.2 *Evaluation of Numerical Results*

In this section, the numerical accuracies on the proposed definition of the set of nodes C based on the radius of J -integral R_F are discussed below.

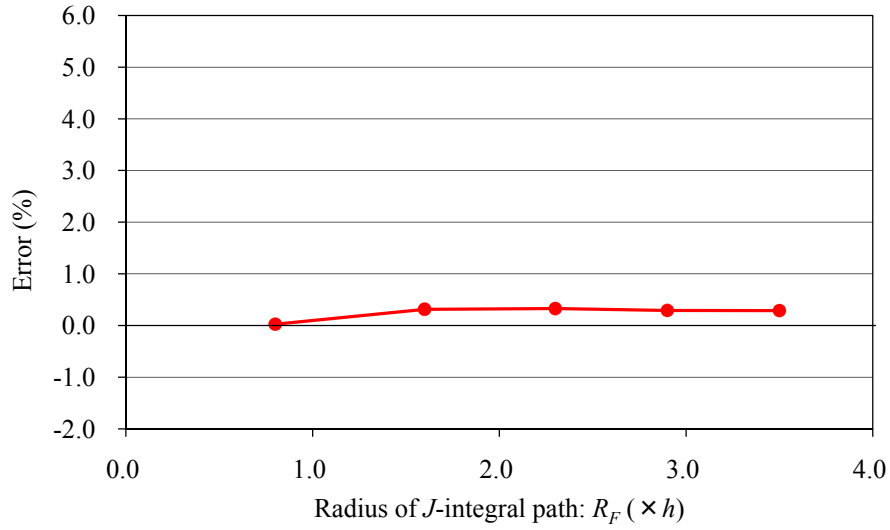
Two numerical models, which are the same as the ones adopted in the section 3.2.2 as shown in Fig. 3.2, are considered for the verification. They are modeled as finite domains near the crack tip in the infinite plates with a unit fracture mode I or II, respectively.

In the proposed method in the section 3.3.1, the set of nodes C and the domain of the J -integral Ω_F are defined at the same time by only one parameter of the radius R_F . The numerical results of the fracture mechanics parameters are therefore evaluated according to parameter of the radius R_F .

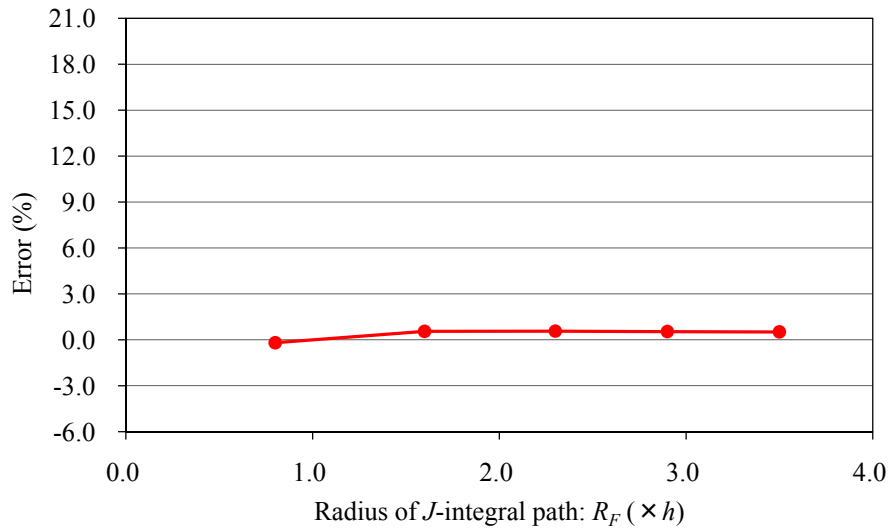
As the evaluated fracture mechanics parameters, the stress intensity factor K_I or K_{II} and the energy release rate G are employed as same as the evaluations in the section 3.2.1.

Numerical results for the evaluation of the fracture mechanics parameters using the proposed modeling are shown in Figs. 3.11 and 3.12, corresponding to the respective numerical models of mode I and II. It is noted that the scales in Figs. 3.11 and 3.12 are the same as those in Figs. 3.6 and 3.7. From these figures, we can see the improvements of the proposed method, compared with the existing modeling of the set of nodes C based on the radius R_C in Figs. 3.6 and 3.7.

In the numerical results, the errors of the respective parameters are 0.4% for K_I and 0.7% for G in the mode I model and 0.2% for K_{II} and 0.4% for G in the mode II model, through the all cases of the radius R_F up to $3.5h$. These results in Figs. 3.11 and 3.12 show stable and sufficient accuracies in the evaluation of the fracture mechanics parameters, in comparison with the results in the existing method of modeling of the nodal set C as shown in Figs. 3.6 and 3.7. This is because the proposed modeling of the set of nodes C based on the radius R_F of the J -integral path enables to avoid the influences of the problem of blending elements in the evaluation of the J -integral.

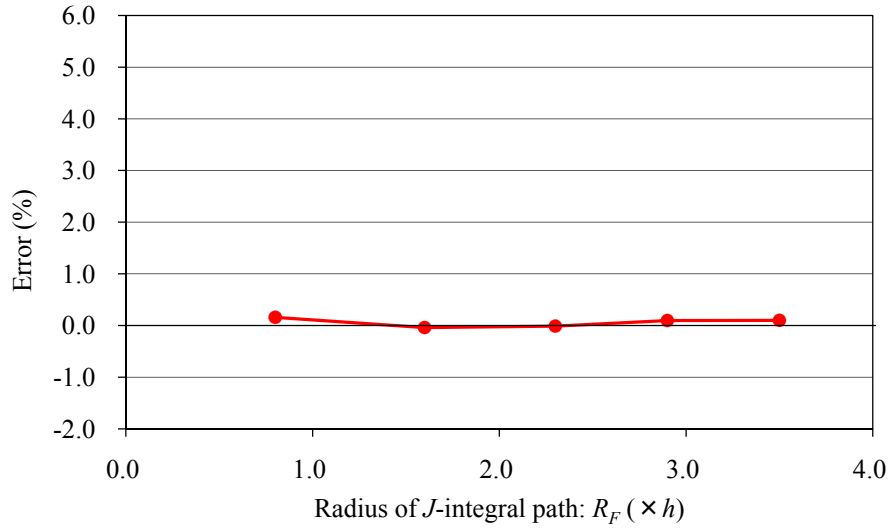


(a) Stress intensity factor: K_I

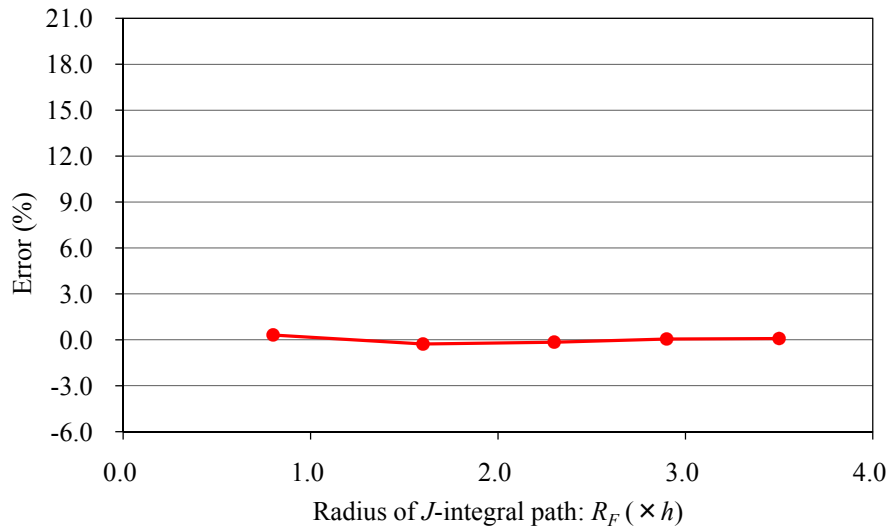


(b) Energy release rate: G

Figure 3.11 Numerical results of fracture mechanics parameters in the mode I model ($K_I = 1$, $K_{II} = 0$) using the proposed definition of the set of nodes C



(a) Stress intensity factor: K_{II}



(b) Energy release rate: G

Figure 3.12 Numerical results of fracture mechanics parameters in the mode II model ($K_I = 0$, $K_{II} = 1$) using the proposed definition of the set of nodes C

3.3.3 Effective Definition of J -integral Domain

In the past studies, the condition of the minimum radius R_F of the J -integral path was reported as $R_F = 2.0h$ in order to accurately evaluate the fracture mechanics parameters (Moës *et al.* 1999, Nagashima *et al.* 2003, Nakasumi *et al.* 2004, Shibamura & Utsunomiya 2007). In the proposed method, it is however shown that the sufficient numerical accuracies can be obtained to the arbitrary ranges of the R_F less than $2.0h$.

According to the above results, the minimum radius R_F of J -integral path can be defined in the proposed method, as

$$R_F = \max_{I \in I'} \|\mathbf{x}_I - \mathbf{x}_{\text{tip}}\| \quad (3.11)$$

where I' is the set of nodes whose supports contain the crack tip.

By the definition of the radius R_F in Eq. (3.11), we can determine the minimum ranges of the set of nodes C and the domain Ω_F without influence of the problem of blending elements under the framework of the standard XFEM.

In addition, this proposal of the minimum range of C and Ω_F is effective for the actual application, because it is reported that a J -integral path with large range causes significant errors in application to a curved or kinked crack.

3.4 CONCLUDING REMARKS

In this chapter, we first evaluated the influence of the blending elements in the two dimensional linear fracture mechanics using basic numerical models. We then proposed the effective modeling of the crack tip enrichment and J -integral in the framework of the standard XFEM. The results obtained are summarized as follows:

- (i) The reproducibility of the stress fields was evaluated according to the parameter of the radius R_C , which is used for the definition of the range of the crack tip enrichment in the existing standard XFEM. We found large errors in the domain of the blending elements. In particular, the most serious errors were found in the cases of the minimum R_C , i.e., $R_C = 1.0h$. On the other hand, it was shown that enlarging the ranges of the enrichment R_C tend to reduce the errors.
- (ii) The influence of the problem of blending elements to the numerical accuracies of the fracture mechanics parameters, which are especially important for the application of crack analyses for real structures, was evaluated. The parameters are calculated by the generally used J -integral method whose integral path is defined by the radius R_F . As a result, it was found that there are significant errors in the numerical results when the two radii of R_C and R_F are the same each other. That is, the lack of numerical accuracies is caused when the integration domain of the J -integral Ω_F is overlapped with the blending domain Ω^{blnd} . It was also found that the numerical results can be evaluated with good accuracies under the condition that the integration domain of the J -integral Ω_F is included in the domain of the fully enriched elements Ω^{enr} , i.e., $\Omega_F \subset \Omega^{\text{enr}}$.
- (iii) Based on the condition for good accuracies in the evaluations on the fracture mechanics parameters, which was found in the results of (ii), the definition of the minimum range of the set of nodes C was proposed within the framework of the standard XFEM approximation. Using this proposal, we can avoid the influences of the problem of blending elements to accurately evaluate the fracture mechanics parameters according to an arbitrary radius R_F of the J -integral path.
- (iv) The numerical accuracies of the fracture mechanics parameters using the proposed definition of the set of nodes C based on the radius of J -integral R_F were verified. The numerical results showed stable and sufficient accuracies in the evaluation of the fracture mechanics parameters, in comparison with the results in the existing method of modeling of the nodal set C . According to these results, the minimum radius R_F of J -integral path was defined in the proposed method.

According to the above summarized results, it was found out that the proposed modeling of the minimum ranges of the set of nodes C and the domain Ω_F is effective in

the framework of the standard XFEM approximation, because the fracture mechanics parameters can be easily evaluated with accuracies in this modeling.

Chapter 4

Reformulation of XFEM Based on PUFEM

4.1 INTRODUCTION

In the XFEM, there are elements with some of their nodes being enriched and the remaining nodes being standard (unenriched) elements in the numerical model. These partially enriched elements are called ‘blending elements’, by which the problems in numerical accuracy were reported (Chessa *et al.* 2003, Gracie *et al.* 2008).

In previous Chapter 3, we proposed the new modeling of the local crack tip enrichment and J -integral in the application to the elastic linear fracture mechanics. Using this proposal, the numerical results showed stable and sufficient accuracies in the evaluation of the fracture mechanics parameters, in comparison with the results in the existing modeling. Therefore, it was found out that the influences of the problem of blending elements to the numerical accuracy of the fracture mechanics parameters can be effectively and easily avoided in the frameworks of the standard XFEM by this proposal. However, this proposed modeling is not a foundational solution of the problem of the blending elements in the XFEM. That is, the application of the proposal is limited to the linear fracture mechanics and any problems except the numerical accuracy of the fracture mechanics parameters cannot be improved by the proposal. In

addition, in this proposal, the enriched domain relating the crack tip enrichment is required to include the domain of the J -integral. However, such relatively large range of the enrichment near the crack tip causes the limitation in the modeling of the actual problem.

On the other hand, Fries (2008) recently proposed ‘the weighted XFEM’ as ‘the corrected XFEM’ employing a new definition of enrichment. The weighted XFEM showed the effective improvement of the reduction of numerical accuracy caused by the blending elements with the various applications. In addition, this method can be implemented by simple revision of the standard XFEM codes. Therefore, this proposal has been an effective method to improve the problem caused by the blending elements. However, this proposal is based on an enforced simple redefinition of enrichment function satisfying to be zero on the boundary between standard elements and blending elements by using a rump function in the framework of the approximation of the standard XFEM. Therefore, the usefulness of this proposal has been unfortunately based only on the results of the inductive evaluations because the theoretical approach was not sufficiently described.

From the above background, a reformulation of the XFEM is required based on the useful theory in order to solve the problem of the blending elements.

The PUFEM was proposed as a generalization of the FEM by Melenk & Babuška (1996). The foundational concept of the PUFEM is the ‘assurance of approximate accuracy’. It is the interesting fact that the XFEM was proposed as one of the developments of the PUFEM, because this fact conflicts the reduction of numerical accuracy caused by the problem of the blending elements in the standard XFEM. In addition, the relationship between the XFEM and the PUFEM is differently defined by the respective recent researches (Bordas & Moran 2006, Wyart *et al.* 2007, Ventura *et al.* 2008). Therefore, as a theoretical approach to solve the problem caused by the blending elements, a reconsideration of the application of the PUFEM to the XFEM will be useful.

In this section, we therefore reformulate the XFEM based on the PUFEM as ‘PU-XFEM’ in order to solve the problem caused by the blending elements. In addition, we try to find out the reason why the weighted XFEM can improve the accuracy reduction caused by the blending elements.

An outline of this chapter is as follows: In Section 4.2, we describe the definition of the existing standard XFEM and its approximation accuracy including the problem of blending elements. In Section 4.3, we propose a reformulation of the XFEM based on concept in the approximation of the PUFEM, which assures the approximation accuracy of the entire numerical domain in order to solve the problem of blending elements. Then, the approximate accuracy of this reformulated XFEM for one dimensional problem is evaluated to clarify the potential ability based on a typical approach using convergence study, including an assessment of the theoretical background of the corrected XFEM. In Section 4.4, an approximation formula of displacement field in two dimensional linear fracture mechanics using the reformulated XFEM is described. In Section 4.5, some numerical results by using the reformulated XFEM in two dimensional linear fracture mechanics are presented. Finally, in Section 4.6, the conclusions of this study are summarized.

4.2 STANDARD XFEM

In this section, we describe the review of the standard XFEM approximation, based on the previous section, and the approximate accuracy of the standard XFEM. This section contains as follows: In the section 4.2.1, the definition of the approximation in the standard XFEM is described. In the section 4.2.2, the definition of blending elements, which affects on the accuracy in the standard XFEM, is described. In the section 4.2.3, the problem in the approximate accuracy of the standard XFEM is evaluated in the one dimensional problem.

4.2.1 *Standard XFEM Approximation*

The standard XFEM approximation $\mathbf{u}_{\text{ap}}(\mathbf{x})$ has the form

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \mathbf{u}^{\text{std}}(\mathbf{x}) + \mathbf{u}^{\text{enr}}(\mathbf{x}) \quad (4.1)$$

where, $\mathbf{u}^{\text{std}}(\mathbf{x})$ is the standard FEM approximation and $\mathbf{u}^{\text{enr}}(\mathbf{x})$ is the approximation using the enrichment.

The standard part of the approximation is

$$\mathbf{u}^{\text{std}}(\mathbf{x}) = \sum_{I \in N} \phi_I(\mathbf{x}) \mathbf{u}_I \quad (4.2)$$

where N is the set of all nodes, $\phi_I(\mathbf{x})$ are the interpolation functions and $\mathbf{u}_I$ are the nodal degrees of freedom associated for standard finite element approximation.

The enriched part of the approximation is

$$\mathbf{u}^{\text{enr}}(\mathbf{x}) = \sum_{I \in N^{\text{enr}}} \phi_I(\mathbf{x}) \Psi(\mathbf{x}) \mathbf{a}_I \quad (4.3)$$

where $\Psi(\mathbf{x})$ is called enrichment function, which is generally a basic function including a priori knowledge of the solution. N^{enr} is the set of nodes enriched by $\Psi(\mathbf{x})$ and $\mathbf{a}_I$ are the nodal degrees of freedom associated for enrichment.

In this section 4, we consider that $\phi_I(\mathbf{x})$ are first order interpolation functions for simplification as well as those in the previous sections.

4.2.2 Blending elements

In this section, the definition of the blending elements is briefly reviewed based on the description in the section 2.2.2.

In the approximation of the XFEM, three sets of elements are defined. The first type is a set of elements E^{std} , in which none of the nodes is enriched. The second type is a set of elements E^{enr} , in which all of the nodes are enriched. The third type is a set of

elements E^{blnd} , in which only some of the nodes are enriched. The sub-domain composed of each set of elements is defined as

$$\begin{aligned}\Omega^{\text{std}} &= \bigcup_{e \in E^{\text{std}}} \Omega_e \\ \Omega^{\text{enr}} &= \bigcup_{e \in E^{\text{enr}}} \Omega_e \\ \Omega^{\text{blnd}} &= \bigcup_{e \in E^{\text{blnd}}} \Omega_e\end{aligned}\tag{4.4}$$

where Ω_e is the domain composed of the element e . The elements composing Ω^{blnd} are called ‘blending elements’. They blend the approximation in the enriched elements in Ω^{enr} with that in the standard elements in Ω^{std} .

In the component i of the approximation $\mathbf{u}^{\text{enr}}(\mathbf{x})$ in Eq. (4.3), if $a_{Ii} = 1$ then

$$u_i^{\text{enr}}(\mathbf{x}) = \sum_{I \in \mathcal{N}^{\text{enr}}} \begin{cases} \phi_I(\mathbf{x})\Psi(\mathbf{x}) = 0 & \forall \mathbf{x} \in \Omega^{\text{std}} \\ \phi_I(\mathbf{x})\Psi(\mathbf{x}) = \Psi(\mathbf{x}) & \forall \mathbf{x} \in \Omega^{\text{enr}} \\ \phi_I(\mathbf{x})\Psi(\mathbf{x}) \neq \Psi(\mathbf{x}) & \forall \mathbf{x} \in \Omega^{\text{blnd}} \end{cases}\tag{4.5}$$

The influence of the blending elements on the approximate accuracy in the standard XFEM will be evaluated in the next section 4.2.3 using the convergence study for a one dimensional problem.

4.2.3 Approximate accuracy of the standard XFEM

The approximate accuracy in the standard XFEM for a one dimensional problem is evaluated below based on the L_2 norm of the interpolation error.

The entire domain is assumed as $\Omega = \{x \mid 0 \leq x \leq 1\}$, then each sub-domain defined in Eq. (4.4) is also assumed as $\Omega^{\text{enr}} = \{x \mid 0 \leq x \leq (M-1)h\}$, $\Omega^{\text{blnd}} = \{x \mid (M-1)h \leq x \leq Mh\}$ and $\Omega^{\text{std}} = \{x \mid Mh \leq x \leq 1\}$ as shown in Fig. 4.1, where h is the element size related to the number of elements N , defined as $h = 1 / N$. The blending element is

composed of nodes M and $(M + 1)$. If there are singularities in this problem, their locations are supposed to be in Ω^{enr} .

The interpolation functions which are used in the element n , composed of nodes n and $(n + 1)$, are

$$\begin{aligned}\phi_n(x) &= 1 - \frac{x - (n-1)h}{h} \\ \phi_{n+1}(x) &= \frac{x - (n-1)h}{h}\end{aligned}\tag{4.6}$$

The interpolation error is defined as

$$\Delta(x) \equiv u_{\text{ap}}(x) - u(x)\tag{4.7}$$

The approximate accuracy in the element composing each sub-domain is evaluated as follows.

The approximation of the displacement field in the element n ($< M$), which compose the sub-domain Ω^{enr} is

$$u_{\text{ap}}(x) = \sum_{l=n}^{n+1} \{ \phi_l(x) u_l + \phi_l(x) \Psi(x) a_l \} \quad \text{on } \Omega^{\text{enr}}\tag{4.8}$$

If each nodal degree of freedom in Eq. (4.8) is assumed as $u_n = u_{n+1} = 0$, $a_n = a_{n+1} = 1$, the approximation $u_{\text{ap}}(x)$ is obtained as an exact solution $\Psi(x)$. However, this can only be true in special cases. It is difficult to reproduce the exact solution in the sub-domain Ω^{enr} in actual analyses due to the restriction on the consistency of the interpolation on the boundary between the sub-domains Ω^{enr} and Ω^{blind} . Thus, the following inequality is defined using a constant p (> 0) as

$$\int_{(n-1)h}^{nh} |\Delta(x)|^2 dx \leq Ch^p \quad \text{on } \Omega^{\text{enr}}\tag{4.9}$$

where C is a generic constant independently from h and p .

The approximation of the displacement field in the blending element $n (= M)$, which composes the sub-domain Ω^{blnd} , is

$$u_{\text{ap}}(x) = \sum_{I=M}^{M+1} \{\phi_I(x)u_I\} + \phi_M(x)\Psi(x)a_M \quad \text{on } \Omega^{\text{blnd}} \quad (4.10)$$

An approximation of the displacement field in the blending element in Eq. (4.10) is shown in Fig. 4.2.

The evaluation for the approximate accuracy in Ω^{blnd} can be expressed using as a reference the approach presented by Chessa *et al.* (2003) as follows.

We let y be a point of $(M-1)h \leq y \leq Mh$. The exact solution $\Psi(x)$ is expanded in a Taylor series about $x = y$ to obtain

$$\begin{aligned} \Psi(x) &= \sum_{m=0}^{\infty} \Psi^{(m)}(y) \frac{(x-y)^m}{m!} \\ &= \Psi(y) + \Psi'(y)(x-y) + O(h^2) \\ &= \Psi(y) + \Psi'(\tilde{y})(x-y) \end{aligned} \quad (4.11)$$

where $(M-1)h \leq \tilde{y} \leq Mh$. Substituting the right hand side in Eq. (4.11) into Eq. (4.10), we obtain

$$u_{\text{ap}}(x) = \sum_{I=M}^{M+1} \{\phi_I(x)u_I\} + \phi_M(x)a_M \{\Psi(y) + \Psi'(\tilde{y})(x-y)\} \quad \text{on } \Omega^{\text{blnd}} \quad (4.12)$$

The second order derivative of the approximation $u_{\text{ap}}(x)$ is obtained by Eqs. (4.6) and (4.12), as

$$u_{\text{ap}}''(x) = -\frac{2a_M}{h} \Psi'(\tilde{y}) \quad \text{on } \Omega^{\text{blnd}} \quad (4.13)$$

Thus

$$\Delta''(x) = u''(x) + \frac{2a_M}{h} \Psi'(\tilde{y}) \quad \text{on } \Omega^{\text{blnd}} \quad (4.14)$$

In the approximation, the maximum interpolation error occurs at the point $\bar{x}$ where $\Delta'(\bar{x}) = 0$. Then a Taylor series expansion of the interpolation error $\Delta(x)$ about $x = \bar{x}$ gives

$$\begin{aligned} \Delta(x) &= \sum_{m=0}^{\infty} \frac{\Delta^{(m)}(\bar{x})}{m!} (x - \bar{x})^m \\ &= \Delta(\bar{x}) + \Delta'(\bar{x})(x - \bar{x}) + \frac{1}{2} \Delta''(\bar{x})(x - \bar{x})^2 + O(h^3) \quad \text{on } \Omega^{\text{blnd}} \quad (4.15) \\ &= \Delta(\bar{x}) + \frac{1}{2} \Delta''(\tilde{x})(x - \bar{x})^2 \end{aligned}$$

where $(M-1)h \leq \tilde{x} \leq Mh$.

If we let $x = (M-1)h$, then $\Delta((M-1)h) = 0$ since $u_{\text{ap}}(x)$ is the finite element interpolation of $u(x)$. Therefore, we obtain

$$\begin{aligned} \Delta(\bar{x}) &= \frac{1}{2} \{(M-1)h - \bar{x}\}^2 \Delta''(\tilde{x}) \\ &\leq \frac{1}{2} \{(M-1)h - \bar{x}\}^2 \max |\Delta''(x)| \\ &\leq \frac{1}{2} \{(M-1)h - \bar{x}\}^2 \max \left| u''(x) + \frac{2a_M}{h} \Psi'(\tilde{y}) \right| \quad \text{on } \Omega^{\text{blnd}} \quad (4.16) \\ &\leq \frac{1}{2} h^2 \max \left| u''(x) + \frac{2a_M}{h} \Psi'(\tilde{y}) \right| \end{aligned}$$

and it follows that

$$\Delta(x) \leq \Delta(\bar{x}) \leq \frac{1}{2} h^2 \max \left| u''(x) + \frac{2a_M}{h} \Psi'(\tilde{y}) \right| \quad \text{on } \Omega^{\text{blnd}} \quad (4.17)$$

Thus, the interpolation error in the sub-domain Ω^{blnd} has the order h (Chessa *et al.* 2003). This leads to

$$\int_{(M-1)h}^{Mh} |\Delta(x)|^2 dx \leq Ch^3 \quad \text{on } \Omega^{\text{blnd}} \quad (4.18)$$

where C is a generic constant independently from h .

The error of the approximation of the displacement field in the element n ($> M$), which composes the sub-domain Ω^{std} corresponds to one of the standard finite element approximation as

$$u_{\text{ap}}(x) = \sum_{I=n}^{n+1} \phi_I(x) u_I \quad \text{on } \Omega^{\text{std}} \quad (4.19)$$

If the approximation $u_{\text{ap}}(x)$ in Eq. (4.19) corresponds to the exact solution $\Psi(x)$ at the nodes as the evaluation in the sub-domain Ω^{blnd} , the nodal degrees of freedom are determined as $u_n = \Psi((n-1)h)$, and $u_{n+1} = \Psi(nh)$, respectively. Thus, the approximation $u_{\text{ap}}(x)$ on Ω^{std} in Eq. (4.19) can be rewritten as

$$u_{\text{ap}}(x) = \phi_n(x) \Psi((n-1)h) + \phi_{n+1}(x) \Psi(nh) \quad \text{on } \Omega^{\text{std}} \quad (4.20)$$

The interpolation error on Ω^{std} can be given by using the expansions of $\Psi(nh)$ and $\Psi(x)$ in a Taylor series expansion around $x = (n-1)h$, as

$$\begin{aligned}
 \Delta(x) &= \{\phi_n(x)\Psi((n-1)h) + \phi_{n+1}(x)\Psi(nh)\} - \Psi(x) \\
 &= \left\{ \phi_n(x)\Psi((n-1)h) + \phi_{n+1}(x) \left(\sum_{m=0}^{\infty} \Psi^{(m)}((n-1)h) \frac{(nh - (n-1)h)^m}{m!} \right) \right\} \\
 &\quad - \left\{ \sum_{m=0}^{\infty} \Psi^{(m)}((n-1)h) \frac{(x - (n-1)h)^m}{m!} \right\} \quad \text{on } \Omega^{\text{std}} \\
 &= \frac{1}{2} \Psi''((n-1)h) (x - (n-1)h)(nh - x) + O(h^3)
 \end{aligned} \tag{4.21}$$

(see Reference (Reddy 2006) for details the procedure in Eq. (4.21))

Thus, the interpolation error in the sub-domain Ω^{std} has the order h^2 . This leads to

$$\int_{(n-1)h}^{nh} |\Delta(x)|^2 dx \leq Ch^5 \quad \text{on } \Omega^{\text{std}} \tag{4.22}$$

where C is a generic constant, which is independent from h .

Due to Eqs. (4.9), (4.18) and (4.22), the L_2 norm on the entire domain Ω by using the standard XFEM is evaluated as

$$\begin{aligned}
 \|u_{\text{ap}}(x) - u(x)\|_{L^2(\Omega)} &= \left\{ \int_{\Omega} |u_{\text{ap}}(x) - u(x)|^2 dx \right\}^{1/2} \\
 &= \left\{ \int_{\Omega^{\text{enr}}} |u_{\text{ap}}(x) - u(x)|^2 dx + \int_{\Omega^{\text{blind}}} |u_{\text{ap}}(x) - u(x)|^2 dx \right. \\
 &\quad \left. + \int_{\Omega^{\text{std}}} |u_{\text{ap}}(x) - u(x)|^2 dx \right\}^{1/2} \\
 &\leq C \left\{ (M-1) \cdot h^p + h^3 + (N-M)h^5 \right\}^{1/2}
 \end{aligned} \tag{4.23}$$

Thus, we obtain

$$\|u_{\text{ap}}(x) - u(x)\|_{L_2(\Omega)} \leq \max\{Ch^{p/2}, Ch^{3/2}\} \quad (4.24)$$

In the case of $p \geq 3$, the convergence rate of the L_2 norm in Eq. (4.24) is determined as $3/2$ due to the approximate accuracy in the sub-domain Ω^{blind} . On the other hand, in the case of $p < 3$, the convergence rate is determined as $p/2$ ($< 3/2$) due to the approximate accuracy in the sub-domain Ω^{enr} . Therefore, there is obviously no case that the convergence rate is more than $3/2$.

The convergence rate on the L_2 norm of the interpolation error in the classical finite element approximation is 2. However, the convergence rate on the L_2 norm of the interpolation error in the standard XFEM as shown in Eq. (4.24) is lower than that of the classical finite element approximation by $1/2$ at least.

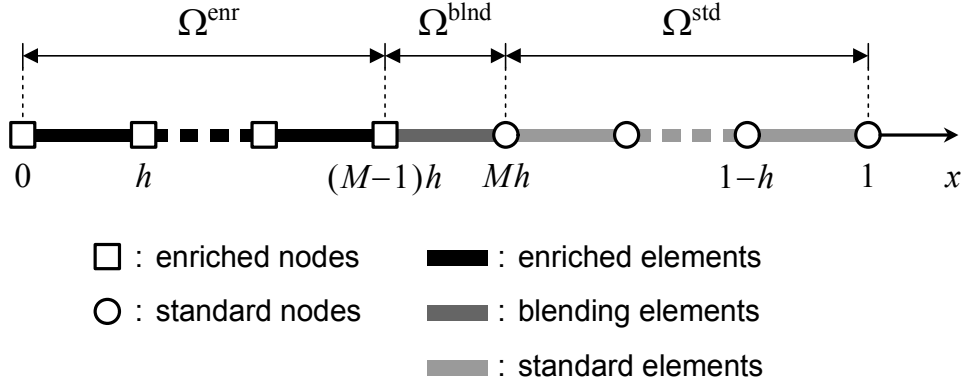


Figure 4.1 Discretized domain for one dimensional problem in the standard XFEM

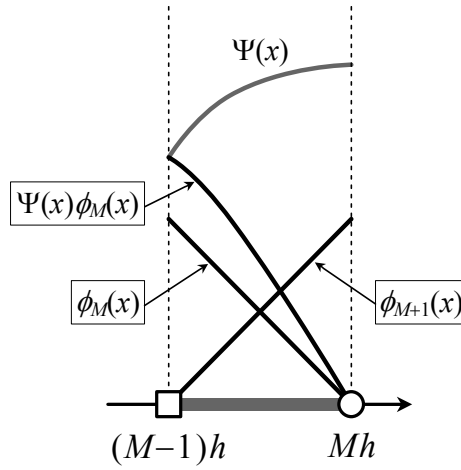


Figure 4.2 Approximation on a blending element for one dimensional problem in the standard XFEM

4.3 REFORMULATION OF XFEM BASED ON PUFEM

In the section 4.2, the limitation in the approximate accuracy of the standard XFEM was described. It was therefore necessary to reconsider the approximation of the XFEM to solve the problem of the standard XFEM.

It is known that the approximate accuracy based on the PUFEM is assured in the entire numerical domain (Melenk & Babuska 1996). Therefore, we propose a reformulation of the approximation of the XFEM based on the PUFEM so as to solve the lack of accuracy in the blending elements.

The contents of this section are as follows: In the section 4.3.1, the approximation of the PUFEM is described, based on the work of Melenk & Babuska (1996). In the section 4.3.2, the approximation of the XFEM including enrichment is reformulated based on the concept of the approximation in the PUFEM, as ‘PU-XFEM’. In the section 4.3.3, the approximation accuracy for the one dimensional problem in the PU-XFEM is evaluated. In addition, in the section 4.3.4, the consideration for the weighted XFEM is discussed.

4.3.1 *PUFEM Approximation*

In this section, the approximation in the PUFEM is described, based on the work of Melenk & Babuska (1996).

Assume the functions φ_j , satisfying

$$\sum_j \varphi_j = 1 \quad \text{on } \Omega \quad (4.25)$$

$$\text{card}\{j \mid \mathbf{x} \in \Omega_j\} \leq N_0 \quad (4.26)$$

$$\|\varphi_j\|_{L_{\infty}, \Omega} \leq C_0 \quad (4.27)$$

where Ω_j are the support of φ_j and C_0 is a constant. Note that φ_j is PU but their definitions are different from the interpolation functions used in the classical finite element method.

Using local approximation spaces V_j^{ap} on Ω_j , the PUFEM space V_{ap} is defined as

$$V_{\text{ap}} \equiv \sum_j \varphi_j V_j^{\text{ap}} = \left\{ \sum_j \varphi_j v_j^{\text{ap}} \mid v_j^{\text{ap}} \in V_j^{\text{ap}} \right\} \quad (4.28)$$

Assume that the local approximate functions v_j^{ap} on Ω_j have the following approximation properties for the exact solution u .

$$\|v_j^{\text{ap}} - u\|_{L_2(\Omega_j \cap \Omega)} \leq \varepsilon_j \quad (4.29)$$

Then, the approximation u_{ap} in the PUFEM space V_{ap} on Ω is defined as

$$u_{\text{ap}} = \sum_j \varphi_j v_j^{\text{ap}} \in V_{\text{ap}} \quad (4.30)$$

Due to Eq. (4.29), the L_2 norm on the entire domain Ω by using the PUFEM is evaluated as

$$\begin{aligned} \|u_{\text{ap}} - u\|_{L_2(\Omega)} &= \left\{ \left\| \sum_j \varphi_j (v_j^{\text{ap}} - u) \right\|_{L_2(\Omega)}^2 \right\}^{1/2} \\ &\leq \left\{ N_0 \sum_j \|\varphi_j (v_j^{\text{ap}} - u)\|_{L_2(\Omega)}^2 \right\}^{1/2} \\ &\leq \left\{ N_0 \sum_j \|\varphi_j (v_j^{\text{ap}} - u)\|_{L_2(\Omega_j \cap \Omega)}^2 \right\}^{1/2} \\ &\leq \sqrt{N_0} C_0 \left\{ \sum_j \varepsilon_j^2 \right\}^{1/2} \end{aligned} \quad (4.31)$$

Due to Eq. (4.31) in the approximation based on the PUFEM, it is shown that the approximate accuracy is assured in the entire domain Ω according to the local approximate accuracy on Ω_j .

See the reference (Melenk & Babuska 1996) for details of the concept of the PUFEM approximation.

4.3.2 Formulation of XFEM based on PUFEM

As shown in the section 4.3.1, the approximate accuracy in the PUFEM, which is different from the standard XFEM, is assured in the entire numerical domain. Then, in this section, an approximation of the XFEM with enrichment is reformulated based on the concept of the approximation of the PUFEM, as ‘PU-XFEM’.

Let us redefine two new domains Ω_0 and Ω_1 by using sub-domains Ω^{enr} , Ω^{blnd} and Ω^{std} , which are defined in Eq. (4.4), as

$$\begin{aligned}\Omega_0 &= \Omega^{\text{std}} \cup \Omega^{\text{blnd}} \\ \Omega_1 &= \Omega^{\text{enr}} \cup \Omega^{\text{blnd}}\end{aligned}\tag{4.32}$$

Corresponding to the local approximate function v_j^{ap} in the PUFEM described in the section 4.3.1, the approximate functions $\mathbf{v}_0^{\text{ap}}(\mathbf{x})$ on Ω_0 and $\mathbf{v}_1^{\text{ap}}(\mathbf{x})$ on Ω_1 , are respectively defined as

$$\mathbf{v}_0^{\text{ap}}(x) = \sum_I \phi_I(\mathbf{x}) \mathbf{u}_I \tag{4.33}$$

$$\mathbf{v}_1^{\text{ap}}(x) = \sum_I \phi_I(\mathbf{x}) (\mathbf{w}_I + \Psi(\mathbf{x}) \mathbf{a}_I) \tag{4.34}$$

where $\mathbf{u}_I$, $\mathbf{w}_I$ and $\mathbf{a}_I$ are nodal degrees of freedom.

The functions $\phi_0(\mathbf{x})$ and $\phi_1(\mathbf{x})$ are defined using the node set N^{enr} , which is composing the enriched domain Ω^{enr} , as

$$\varphi_1(\mathbf{x}) = \sum_{I \in N^{\text{enr}}} \phi_I(\mathbf{x}) \quad (4.35)$$

$$\varphi_0(\mathbf{x}) = 1 - \varphi_1(\mathbf{x}) \quad (4.36)$$

where the functions $\varphi_0(\mathbf{x})$ and $\varphi_1(\mathbf{x})$ are PU, i.e., the following is satisfied

$$\varphi_0(\mathbf{x}) + \varphi_1(\mathbf{x}) = 1 \quad (4.37)$$

The respective PU of $\varphi_0(\mathbf{x})$ and $\varphi_1(\mathbf{x})$ are active on Ω_0 and Ω_1 .

By using Eqs. (4.33) – (4.37), a reformulated approximation of $\mathbf{u}_{\text{ap}}(\mathbf{x})$ in the XFEM based on the PUFEM (PU-XFEM) on the entire domain Ω is proposed as

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \varphi_0(\mathbf{x}) \mathbf{v}_0^{\text{ap}}(\mathbf{x}) + \varphi_1(\mathbf{x}) \mathbf{v}_1^{\text{ap}}(\mathbf{x}) \quad (4.38)$$

In this approximation of the PU-XFEM, local approximations $\mathbf{v}_0^{\text{ap}}(\mathbf{x})$ and $\mathbf{v}_1^{\text{ap}}(\mathbf{x})$ are both active on Ω^{blnd} . That is, the approximation in the blending elements in the PU-XFEM is redefined in the common part of the domains Ω_0 and Ω_1 using PU based on the PUFEM approximation.

4.3.3 Approximate accuracy of the PU-XFEM

Let us consider the one dimensional problem in the PU-XFEM corresponding to the case of the standard XFEM in the section 4.2.3. Thus, the entire domain is assumed as $\Omega = \{x \mid 0 \leq x \leq 1\}$, then each sub-domain is defined as $\Omega^{\text{enr}} = \{x \mid 0 \leq x \leq (M-1)h\}$, $\Omega^{\text{blnd}} = \{x \mid (M-1)h \leq x \leq Mh\}$ and $\Omega^{\text{std}} = \{x \mid Mh \leq x \leq 1\}$. If there are singularities in this problem, their locations are supposed to be in Ω^{enr} . Then two new domains Ω_0 and Ω_1 defined in Eq. (4.32) can be illustrated as shown in Fig. 4.3.

In this case, the PU $\varphi_0(x)$ and $\varphi_1(x)$, defined in the general form of Eqs. (4.35) and

(4.36), are redefined respectively as

$$\varphi_0(x) = \begin{cases} 0 & \text{on } \Omega^{\text{enr}} \\ \frac{x - (M-1)h}{h} & \text{on } \Omega^{\text{blnd}} \\ 1 & \text{on } \Omega^{\text{std}} \end{cases} \quad (4.39)$$

$$\varphi_1(x) = \begin{cases} 1 & \text{on } \Omega^{\text{enr}} \\ 1 - \frac{x - (M-1)h}{h} & \text{on } \Omega^{\text{blnd}} \\ 0 & \text{on } \Omega^{\text{std}} \end{cases} \quad (4.40)$$

so the functions satisfies $\varphi_0(x) + \varphi_1(x) = 1$. These functions $\varphi_0(x)$ and $\varphi_1(x)$ are illustrated in Fig. 4.4. The cardinal number in Eq. (4.26) is therefore $N_0 = 2$ in this case.

Then the interpolations in the domains Ω_0 and Ω_1 using the approximation functions $v_0^{\text{ap}}(x)$ and $v_1^{\text{ap}}(x)$ are considered, respectively.

When the approximation in the domain Ω_0 consists of only one function $v_0^{\text{ap}}(x)$ defined in Eq. (4.33), the interpolation error in the domain Ω_0 is the same as that of the classical finite element approximation, as shown in the following expression corresponding to Eq. (4.19).

$$\|v_0^{\text{ap}} - u\|_{L_2(\Omega_0 \cap \Omega)} \leq Ch^2 \quad (= \varepsilon_0) \quad (4.41)$$

When the approximation in the domain Ω_1 consists of only one function $v_1^{\text{ap}}(x)$ defined in Eq. (4.34), the interpolation error in the domain Ω_1 can be also defined using a constant $q (> 0)$ as

$$\|v_1^{\text{ap}} - u\|_{L_2(\Omega_1 \cap \Omega)} \leq Ch^q \quad (= \varepsilon_1) \quad (4.42)$$

In the approximation $v_1^{\text{ap}}(x)$, it is possible to obtain the interpolation taking into account the ‘*priori* knowledge’ of the solution, by using the enrichment function $\Psi(x)$

without any problems in the standard XFEM approximation. Therefore, the convergence order of q in Eq. (4.42) can be made more than that in the classical finite element approximation in Eq. (4.41). In particular, in the case that the enrich function $\Psi(x)$ corresponds to the exact solution, it is possible to obtain the interpolation without error. Thus, we can assume to be $q > 2$.

In the blending element composing the domain Ω^{blend} , the local approximations $v_0^{\text{ap}}(x)$ and $v_1^{\text{ap}}(x)$ are both active. The approximation in the blending elements in the PU-XFEM can be illustrated as shown in Fig. 4.5. It is therefore noted that the approximation in the PU-XFEM is different from that in the standard XFEM, as shown in Fig. 4.2.

Based on the above discussion, the approximate accuracy of the PU-XFEM, which is defined in the general form of Eq. (4.38), on the one dimensional problem is evaluated using the L_2 norm of the interpolation error. That is, the L_2 norm of the interpolation error on the entire domain Ω by using the PU-XFEM approximation is evaluated due to Eqs. (4.31), (4.41) and (4.42), as

$$\begin{aligned} \|u_{\text{ap}} - u\|_{L_2(\Omega)} &\leq \sqrt{N_0} C_0 \left\{ \sum_j \varepsilon_j^2 \right\}^{1/2} \\ &= \sqrt{2} C_0 \left\{ (Ch^2)^2 + (Ch^q)^2 \right\}^{1/2} \end{aligned} \quad (4.43)$$

The convergence rate q can be assumed to be $q > 2$, as described in Eq. (4.42). Thus, we obtain

$$\|u_{\text{ap}} - u\|_{L_2(\Omega)} \leq Ch^2 \quad (4.44)$$

Due to Eq. (4.44), it is shown that the insufficient convergence rate on the L_2 norm of the interpolation error in the standard XFEM is overcome in the PU-XFEM. It is therefore found that the reformulated approximation of the PU-XFEM in Eq. (4.38) provides a potential ability to solve the problem of the lack of accuracy for blending elements in the standard XFEM.

According to above discussion, the XFEM has been reformulated on the basis of the PUFEM, by establishing the relationship between the XFEM and the PUFEM. It is therefore found out that the approximation of the standard XFEM is *not* exactly based on that of the PUFEM. That is, the right XFEM can be led by the PU-XFEM.

4.3.4 *Consideration for the weighted XFEM*

It was recently reported that the problem of blending elements is effectively improved by employing ‘the weighted XFEM’ as ‘the corrected XFEM’ proposed by Fries (2008). However, this proposal was based on a simple redefinition of enrichment function satisfying to be zero on the boundary between standard elements and blending elements by using a rump function in the framework of the standard XFEM approximation. That is, the theoretical approach of the weighted XFEM was not sufficiently described. The usefulness of this proposal was thus unfortunately based only on the results of the inductive evaluations. Therefore, we try to clarify the reason why the weighted XFEM can improve the accuracy reduction caused by the blending elements, based on the above discussion on the reformulation of the PU-XFEM.

The weighted XFEM was based on a simple redefinition of enrichment function satisfying zero in the boundary between Ω^{blnd} and Ω^{std} by using a rump function in the framework of the approximation in the standard XFEM. The form of the weighted XFEM is

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) \mathbf{u}_I + \varphi(\mathbf{x}) \sum_I \phi_I(\mathbf{x}) \Psi(\mathbf{x}) \mathbf{a}_I \quad (4.45)$$

where $\varphi(\mathbf{x})$ is the weighted function, which looks to be the PU $\varphi_1(\mathbf{x})$ in Eq. (4.38) used in the formulation of the PU-XFEM.

That is, the weighted XFEM approximation of Eq. (4.45) can be obtained as a result by simplifying the two kinds of the degrees of freedom $\mathbf{u}_I$ in Eq. (4.33) and $\mathbf{w}_I$ in Eq. (4.34) into the only one $\mathbf{u}_I$ in the approximation of the PU-XFEM. Therefore, it is presumed that the numerical accuracy in the weighted XFEM can be improved because

the appearance of the approximation is similar to that of the PU-XFEM.

However, the approximate accuracy evaluated in Eqs. (4.41) and (4.42) can be assured by only independently defining $\mathbf{u}_I$ and $\mathbf{w}_I$. Therefore, the incompleteness of the proposed form of the weighted XFEM was clarified as the solution of the problem of the blending elements.

According to the above discussion, it is concluded that the proposed formulation of the PU-XFEM is led based on the theoretical background and the PU-XFEM is essentially different from that of the weighted XFEM.

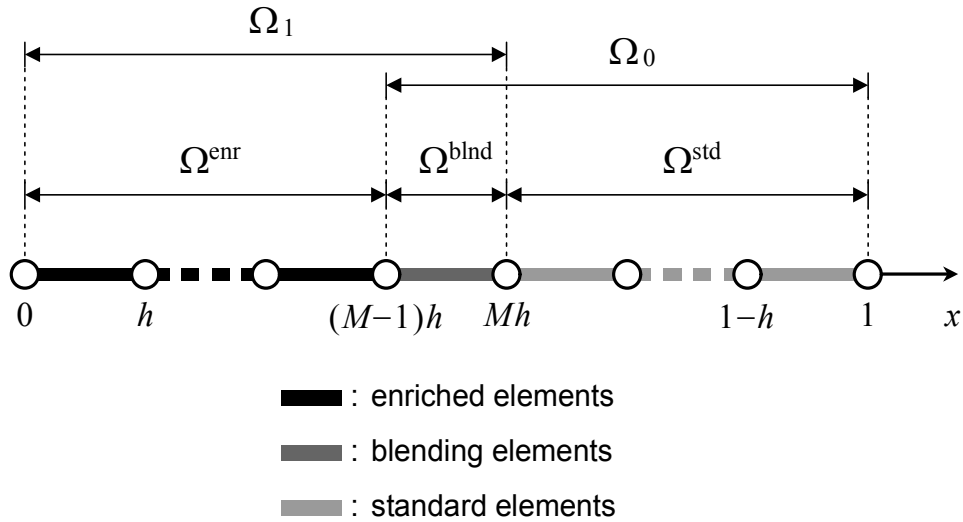


Figure 4.3 Discretized domain for one dimensional problem in the PU-XFEM

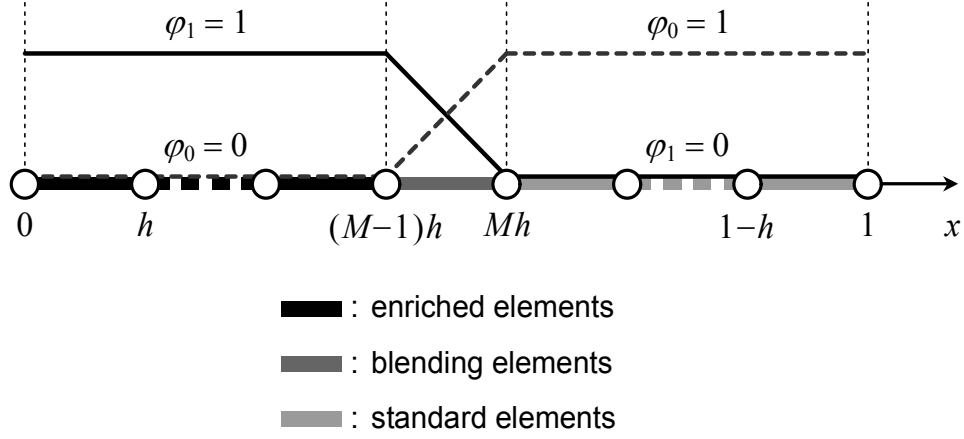
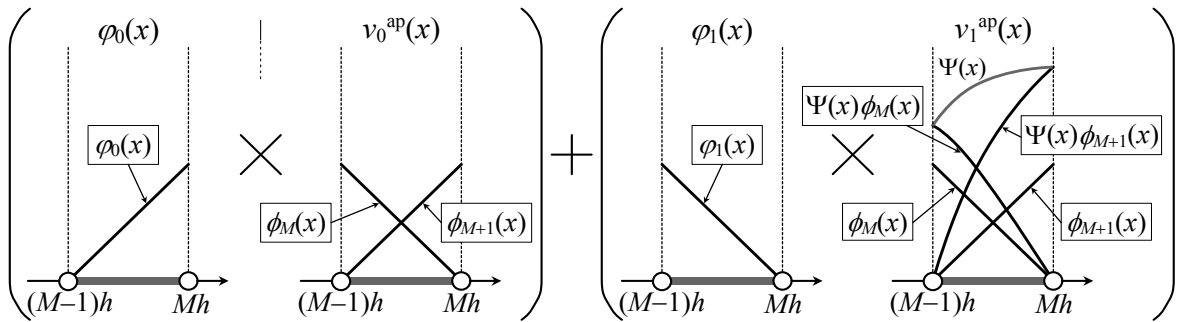

 Figure 4.4 Schematic description of the functions $\varphi_0(x)$ and $\varphi_1(x)$


Figure 4.5 Approximation on a blending element for one dimensional problem in the PU-XFEM

4.4 APPLICATION TO LINEAR FRACTURE MECHANICS

In this section, the approximation of displacement field in two dimensional linear fracture mechanics by the PU-XFEM is described.

The governing equations in the problem are the same as those described in the section 2.3.1. Therefore, we describe only the approximation of displacement field based on the PU-XFEM in this section.

In modeling of crack analyses of the PU-XFEM, we use the two types of enrichment functions, which are the same as those in the standard XFEM approximation, as

$$H(\mathbf{x}) = \begin{cases} 1 & \text{on } \Omega_+ \\ -1 & \text{on } \Omega_- \end{cases} \quad (4.46)$$

$$\gamma_1(\mathbf{x}) = \sqrt{r} \cos \frac{\theta}{2} \quad (4.47)$$

$$\gamma_2(\mathbf{x}) = \sqrt{r} \sin \frac{\theta}{2} \quad (4.48)$$

$$\gamma_3(\mathbf{x}) = \sqrt{r} \cos \frac{\theta}{2} \sin \theta \quad (4.49)$$

$$\gamma_4(\mathbf{x}) = \sqrt{r} \sin \frac{\theta}{2} \sin \theta \quad (4.50)$$

where $H(\mathbf{x})$ is the Heaviside function used for the enrichment in the presence of strong discontinuities. Ω_+ and Ω_- are the upper and lower sides of the domain defined on the supports Ω_I of node I containing the Heaviside enrichment (Moës *et al.* 1999).

It is noted that this Heaviside enrichment dose not lead to the problem caused by blending elements even in the standard XFEM. This is because the Heaviside enrichment function is a constant in the blending elements and is not assumed as an

exact solution described in the section 4.3. The detailed discussion of this topic is found in the section 2.3.2. Therefore, the form of the Heaviside enrichment in the standard XFEM is useful for the reduction of number of degrees of freedom.

$\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$) are the set of singular enrichment functions, introduced by Fleming *et al.* (1997) in the framework of the element-free Galerkin method (Belytschko *et al.* 1994), based on the near-field asymptotic solution. (r, θ) are the local polar co-ordinates at the crack tip.

By using the above enrichment functions, the approximate functions $\mathbf{v}_0^{\text{ap}}(\mathbf{x})$ and $\mathbf{v}_C^{\text{ap}}(\mathbf{x})$ of the displacement field are defined as

$$\mathbf{v}_0^{\text{ap}}(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) \mathbf{u}_I + \sum_{I \in J} \phi_I(\mathbf{x}) H(\mathbf{x}) \mathbf{b}_I \quad (4.51)$$

$$\mathbf{v}_C^{\text{ap}}(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) \left(\mathbf{w}_I + \sum_{k=1}^4 \gamma_k(\mathbf{x}) \mathbf{c}_I^k \right) \quad (4.52)$$

where $\mathbf{u}_I, \mathbf{w}_I$, are the nodal degree of freedoms related to the classical finite element approximation in the respective approximate functions $\mathbf{v}_0^{\text{ap}}(\mathbf{x})$ and $\mathbf{v}_C^{\text{ap}}(\mathbf{x})$. $\mathbf{b}_I$ and $\mathbf{c}_I^k$ ($k = 1, \dots, 4$) are those related to the enrichment function $H(\mathbf{x})$ and $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$), respectively. J is a set of nodes related to the enrichment function $H(\mathbf{x})$.

In the application of the PU-XFEM to a two dimensional linear fracture mechanics, the PU $\varphi_j(\mathbf{x})$ is defined as

$$\varphi_C(\mathbf{x}) = \sum_{I \in C} \phi_I(\mathbf{x}) \quad (4.53)$$

$$\varphi_0(\mathbf{x}) = 1 - \varphi_C(\mathbf{x}) \quad (4.54)$$

where C is a set of nodes near the crack tip.

The functions $\varphi_0(\mathbf{x})$ and $\varphi_C(\mathbf{x})$ are PU, satisfying

$$\varphi_0(\mathbf{x}) + \varphi_C(\mathbf{x}) = 1 \quad (4.55)$$

Note that the approximate functions $\mathbf{v}_0^{\text{ap}}(\mathbf{x})$ and $\mathbf{v}_C^{\text{ap}}(\mathbf{x})$ in Eqs. (4.51) and (4.52) are

defined on the domains Ω_0 as the support of $\varphi_0(\mathbf{x})$ and on the domain Ω_C as the support of $\varphi_C(\mathbf{x})$, respectively.

From Eqs. (4.51) – (4.54), the approximation of displacement field $\mathbf{u}_{\text{ap}}(\mathbf{x})$ on Ω in two dimensional linear fracture mechanics by using the PU-XFEM is defined as

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \varphi_0(\mathbf{x})\mathbf{v}_0^{\text{ap}}(\mathbf{x}) + \varphi_C(\mathbf{x})\mathbf{v}_C^{\text{ap}}(\mathbf{x}) \quad (4.56)$$

The first and second terms of the right hand side in Eq. (4.56) are active on the domain Ω_0 and on the domain Ω_C , respectively.

In order to provide the ranges of the two supports of the PU Ω_0 and Ω_C and to determine the respective nodal degrees of freedom, it is required to define the following sets of nodes.

First, the set of nodes C in Eq. (4.53) is defined as

$$C = \left\{ I \in N \mid \left| \mathbf{x}_I - \mathbf{x}_{\text{tip}} \right| \leq R_C \right\} \quad (4.57)$$

where R_C is the radius using the definition of the nodal set C . The set of nodes C defined by the radius R_C provides the ranges of the two supports Ω_0 and Ω_C , and their common part Ω^{blnd} , as described in Figs. 4.6 – 4.8, at the same time.

In the PU-XFEM approximation in Eq. (4.56), the nodal degrees of freedom $\mathbf{u}_I$ and $\mathbf{b}_I$ corresponding to the approximate function $\mathbf{v}_0^{\text{ap}}(\mathbf{x})$ are required to be defined at all the nodes on the support Ω_0 . In addition, the nodal degrees of freedom $\mathbf{w}_I$ and $\mathbf{c}^k_I$ ($k = 1, \dots, 4$) corresponding to the approximate function $\mathbf{v}_C^{\text{ap}}(\mathbf{x})$ are required to be defined at all the nodes on the support Ω_C . Thus, it is useful to define the following additional sets of nodes B_0 and B_C in the blending domain Ω^{blnd} , which is defined as a common part of the supports Ω_0 and Ω_C , as

$$B_0 = \left\{ I \in N \mid \Omega_I \cap \Omega^{\text{blnd}} \neq \emptyset, I \notin C \right\} \quad (4.58)$$

$$B_C = \left\{ I \in N \mid \Omega_I \cap \Omega^{\text{blnd}} \neq \emptyset, I \in C \right\} \quad (4.59)$$

Then, an additional set of nodes C' is defined as a complementary set of B_C in C , i.e.,

$$C' = C \setminus B_C \quad (4.60)$$

The definition of the set of nodes J in Eq. (4.51) in the PU-XFEM includes a small difference from that of the standard XFEM, as

$$J = \left\{ I \in N \mid \Omega_I \cap (D \cap \Omega_0) \neq \emptyset, I \notin C' \right\} \quad (4.61)$$

where D is the geometry of the crack.

In the above modeling of the set of nodes in the PUXFEM, the geometry of the crack D and the radius R_C are treated as input data as well as those in the standard XFEM.

An example of the above defined sets of nodes in Eqs. (4.57) – (4.61) near the crack is shown in Fig. 4.9, which corresponds to the Fig. 4.6 – 4.8.

The active nodal degrees of freedom corresponding to the respective set of nodes and their numbers are shown in Table 4.1. It is noted that the number of nodal degrees of freedom in the PU-XFEM increases than that of the standard XFEM in case of the same range of the enrichment near the crack tip, due to the reason that the nodal degrees of freedom of all nodes constituting the blending elements in the PU-XFEM are increased as compared with that in the standard XFEM approximation, which was described in the section 2.3.

The numerical integration procedure in each element is carried out by using the standard Gauss quadrature in the FEM except elements that are cut by the crack discontinuity. In the elements cut by the crack, the sub-division of the domain is

required. In particular, in the element contains a crack tip, the almost polar integration is employed (Laborde *et al.* 2005, Béchet *et al.* 2005). In the modeling in the present study, the first order finite element approximation is employed as the interpolation function $\phi_I(\mathbf{x})$. A discontinuity of the crack is therefore modeled as one line segment in each element. This modeling enables a simple division of integral domain. Above integration procedure is the same as that of the standard XFEM, and its detail can be seen in Section 2.3.4.

Table 4.1 Number of nodal degree of freedom corresponding to the respective set of nodes in two dimensional problem

Set of nodes	Active NDF	Num. of NDF
J	$\mathbf{u}_I, \mathbf{b}_I$	4
C'	$\mathbf{w}_I, \mathbf{c}_I^k (k = 1, \dots, 4)$	10
B_0	$\mathbf{u}_I, \mathbf{w}_I, \mathbf{c}_I^k (k = 1, \dots, 4)$	12
B_C	$\mathbf{u}_I, \mathbf{w}_I, \mathbf{c}_I^k (k = 1, \dots, 4)$	12
J and B_0	$\mathbf{u}_I, \mathbf{b}_I, \mathbf{w}_I, \mathbf{c}_I^k (k = 1, \dots, 4)$	14
J and B_C	$\mathbf{u}_I, \mathbf{b}_I, \mathbf{w}_I, \mathbf{c}_I^k (k = 1, \dots, 4)$	14
(standard node)	$\mathbf{u}_I$	2

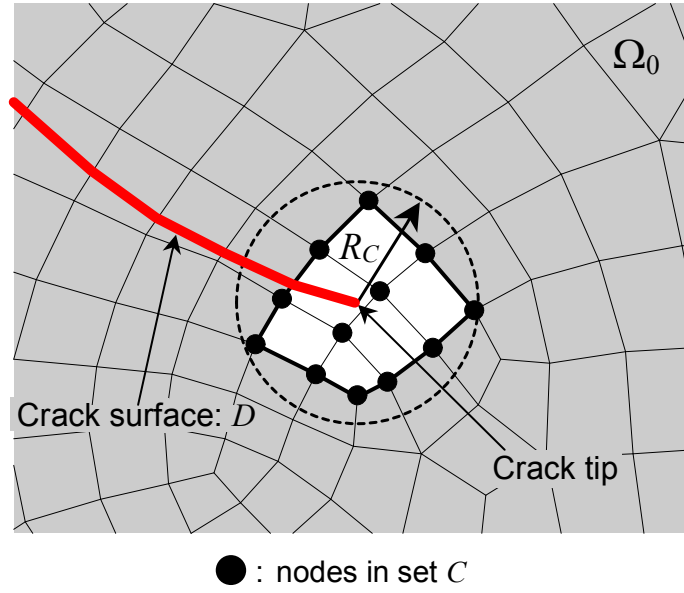


Figure 4.6 Support Ω_0 defined by set of nodes C

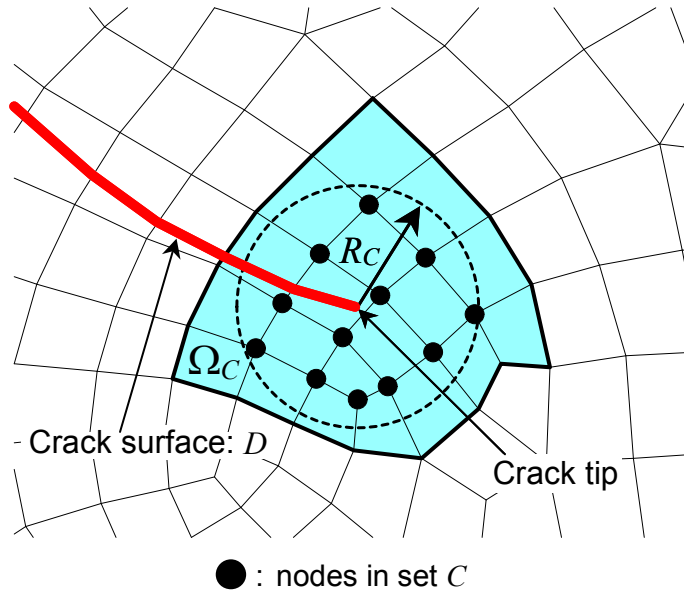


Figure 4.7 Support Ω_C defined by set of nodes C

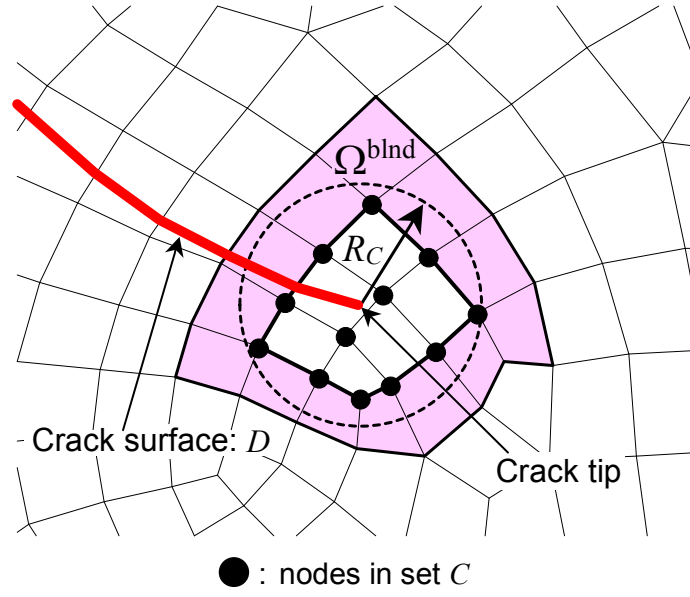


Figure 4.8 Blending domain Ω^{blnd} defined by set of nodes C

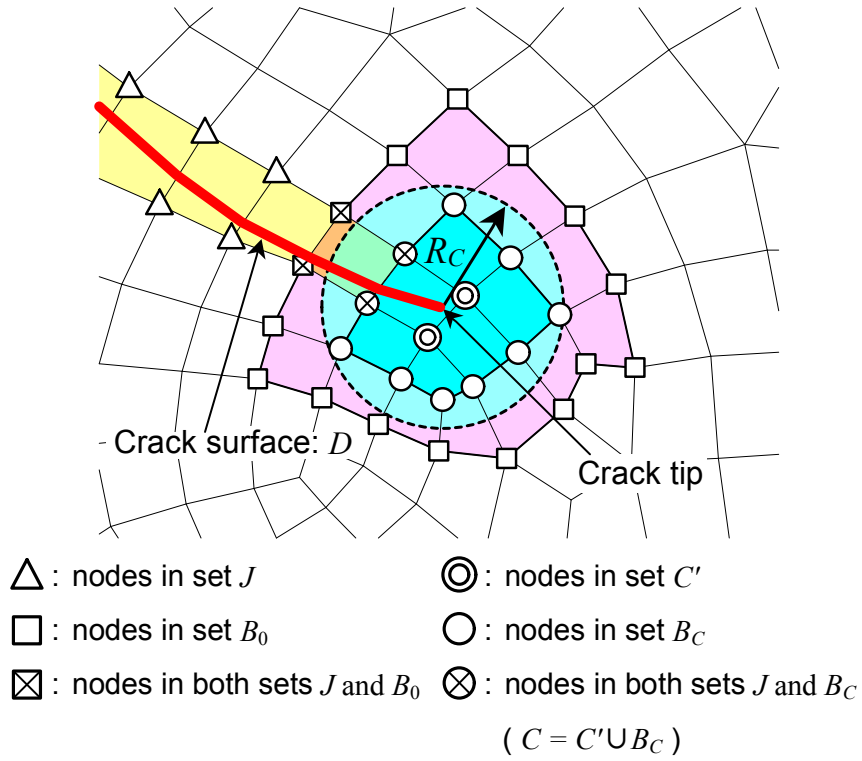


Figure 4.9 Modeling of the sets of nodes in the PU-XFEM

4.5 NUMERICAL EXAMPLES

In this section, we present the evaluations of numerical results in the linear fracture mechanics to reconfirm the usefulness of the proposed PU-XFEM by comparisons with the results of the standard XFEM. In these evaluations, the following models including cracks are used;

Model 1: Fracture mode I crack model in an infinite plate

Model 2: Fracture mode II crack model in an infinite plate

Model 3: Edge crack model under tension in a finite plate

Model 4: Edge crack model under shear in a finite plate

The evaluations on the infinite plate models of the Model 1 and the Model 2 are described in the section 4.5.1. Those on the finite plate models of the Model 3 and the Model 4 are described in the section 4.5.2.

Through all examples, the plane stain conditions are assumed and their material constants are the Young's modules $E = 2.0 \times 10^{11}$ and the Poisson's ratio $\nu = 0.3$. In addition, except the evaluations in the section 4.5.1 **b)**, the radius of definition of the node set C , which is related to the range of the crack tip enrichment is defined to the minimum, i.e., $R_C = 1.0h$, corresponding to the mesh size h , because such minimum ranges of crack tip enrichment is the most effective for the actual application.

4.5.1 Numerical Models in infinite plates

In this section, we investigate the basic evaluations of the numerical accuracy of the PU-XFEM using the respective Fracture mode I and mode II crack models in the infinite plates (Model 1 & Model 2). In particular, the reproduction of the *priori* knowledge introduced as enrichment is evaluated, because the feature of the XFEM is to directly employ an advanced approximation using the *priori* knowledge of the solution

in the framework of the FEM by employing the enrichment in a local area. In the evaluations in this section, the comparison with the weighted XFEM proposed by Fries (2008) is included.

The contents of this section are as follows: In the subsection **a)**, we firstly describe the numerical models used in this section. In the subsection **b)**, we present a convergence study using L_2 norm of the interpolation error. In the subsection **c)**, the evaluations on the reproduction of the *priori* knowledge introduced as the crack tip enrichment are presented. In the subsection **d)**, we evaluate the numerical accuracy of the asymptotic stress fields around the crack tip. In the subsection **e)**, we evaluate the numerical accuracies of the linear fracture mechanics parameters.

a) Model descriptions

The numerical analyses in this section are evaluated by using the models which assume the finite domains near the crack tip in the infinite plates. These models are also used in the chapter 3. The enrichment function $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$) in Eqs. (4.47) – (4.50) can be reproduced as the exact solution in this model.

The respective unit fracture mode I and II in the infinite plates including straight crack are considered as shown in Figs. 4.10(a) and (b). Then, let us define a local finite square domain A including the crack tip in the center in the respective infinite plates as the numerical models. The dimension of the domain A is sufficiently smaller than the crack length $2a$. The size of this numerical domain A is 2×2 as shown in Fig. 4.10 (c). The domain A is divided into the N elements in the both directions of x and y , respectively. The mesh size is therefore $h = 2/N$.

The Gauss quadrature is used in the numerical integration for the calculations of the stiffness matrixes and the J -integral. Based on our preliminary research, the integration order is 5, except the calculations of the elemental stiffness matrixes in the domain Ω^{std} in which the bi-linear finite element approximation is used. The integration order is 2 in the calculations of the elemental stiffness matrixes in the domain Ω^{std} .

The boundary conditions on the boundary Γ_A of the domain A assume the asymptotic solution of displacement field near the crack tip in the respective fracture mode I ($K_I = 1$ and $K_{II} = 0$) and mode II ($K_I = 0$ and $K_{II} = 1$). The nodal displacements in the respective models are therefore directly specified on Γ_A as

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \frac{1}{\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos \frac{\theta}{2} \left(1 - 2\nu + \sin^2 \frac{\theta}{2} \right) \\ \sin \frac{\theta}{2} \left(2 - 2\nu - \cos^2 \frac{\theta}{2} \right) \end{Bmatrix} \quad (\text{in fracture mode I}) \quad (4.62)$$

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \frac{1}{\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin \frac{\theta}{2} \left(2 - 2\nu + \cos^2 \frac{\theta}{2} \right) \\ -\cos \frac{\theta}{2} \left(1 - 2\nu - \cos^2 \frac{\theta}{2} \right) \end{Bmatrix} \quad (\text{in fracture mode II}) \quad (4.63)$$

where μ is the modules of elasticity in shear.

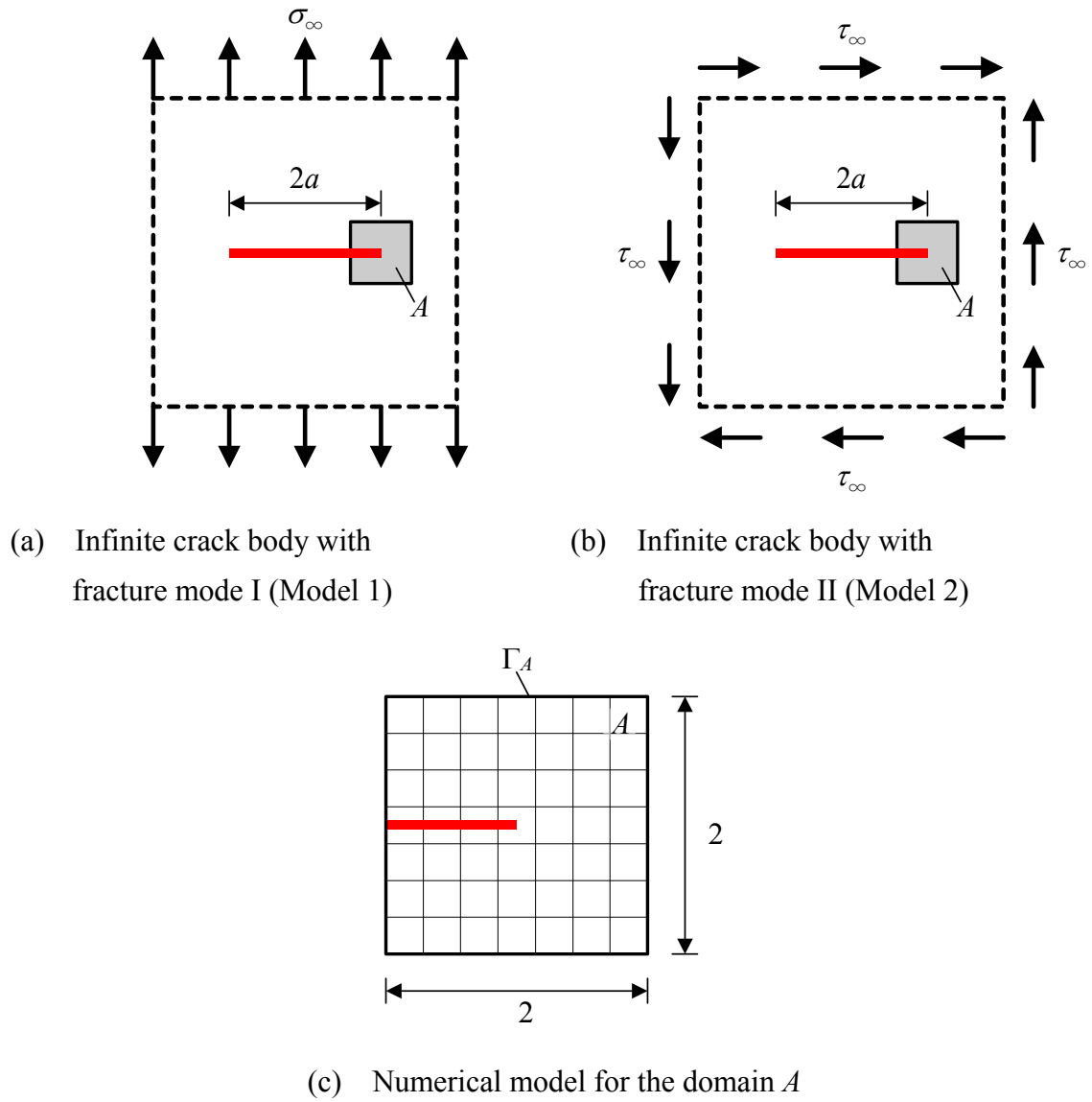


Figure 4.10 Fracture mode I and mode II crack models in the infinite plates (Model 1 & Model 2)

b) Convergence study using L_2 norm

In this section, we investigate the convergence study using L_2 norm of the interpolation error, which is one of the most generally basic evaluations of numerical approximation.

This evaluation is made by using the following relative L_2 norm e_2 .

$$e_2 = \frac{\|\mathbf{u}_{\text{ap}} - \mathbf{u}\|_{L_2(A)}}{\|\mathbf{u}\|_{L_2(A)}} = \left\{ \frac{\int_A |\mathbf{u}_{\text{ap}} - \mathbf{u}|^2 dA}{\int_A |\mathbf{u}|^2 dA} \right\}^{1/2} \quad (4.64)$$

In the Model 1 and the Model 2, the following respective asymptotic solutions equal their exact solutions.

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \frac{1}{\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos \frac{\theta}{2} \left(1 - 2\nu + \sin^2 \frac{\theta}{2} \right) \\ \sin \frac{\theta}{2} \left(2 - 2\nu - \cos^2 \frac{\theta}{2} \right) \end{Bmatrix} \quad (\text{in fracture mode I}) \quad (4.65)$$

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \frac{1}{\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin \frac{\theta}{2} \left(2 - 2\nu + \cos^2 \frac{\theta}{2} \right) \\ -\cos \frac{\theta}{2} \left(1 - 2\nu - \cos^2 \frac{\theta}{2} \right) \end{Bmatrix} \quad (\text{in fracture mode II}) \quad (4.66)$$

In the evaluation of relative L_2 norm, the numerical integration order is 5 on the entire domain, taking into account the asymptotic solution including the singularity of $\sqrt{r}$.

The numerical results in the convergence study of L_2 norm are shown in Figs. 4.11 and 4.12.

In the both cases of the evaluations using the Model 1 and the Model 2, it is found out that the PU-XFEM provides the best results in comparison with the standard XFEM and the weighted XFEM in the convergence rate as well as the improved numerical accuracy.

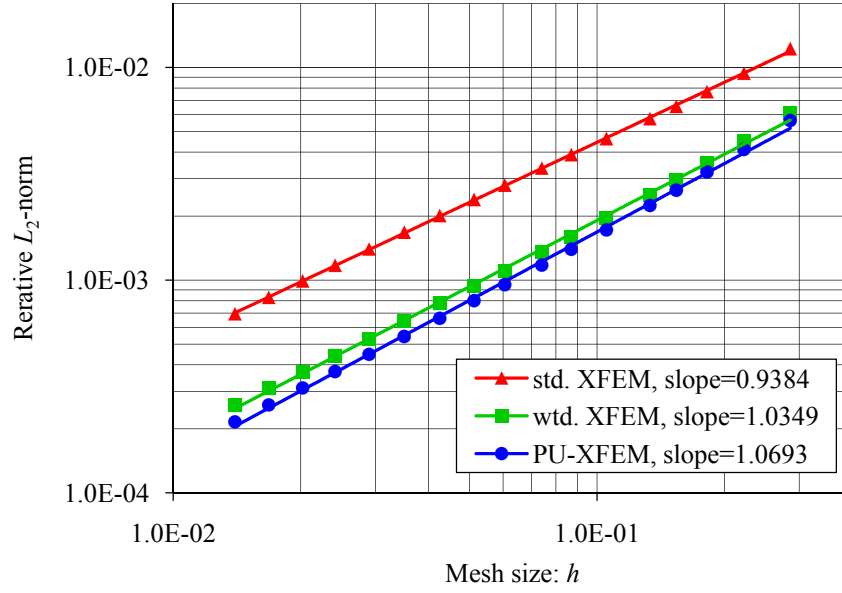


Figure 4.11 Convergence of relative L_2 norm of Model 1 (fracture mode I)

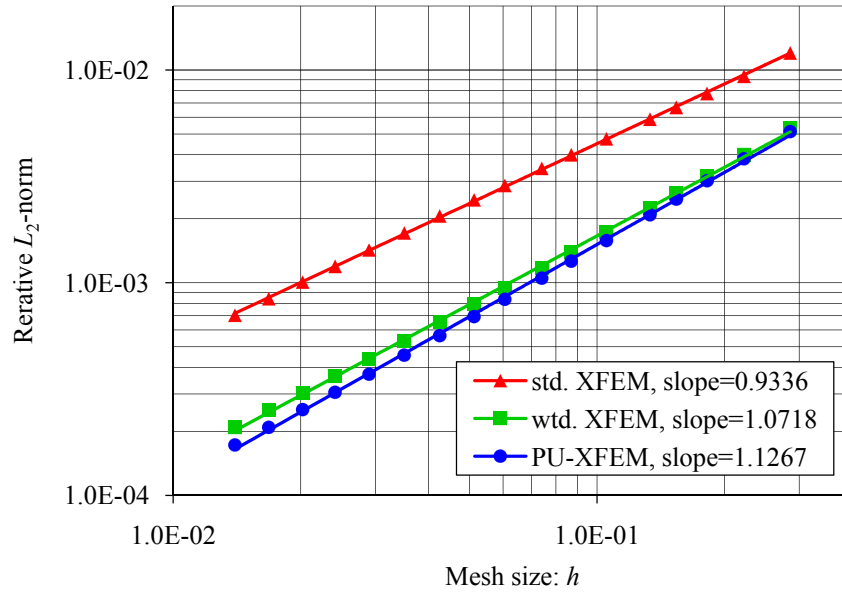


Figure 4.12 Convergence of relative L_2 norm of Model 2 (fracture mode II)

c) Dominant ratio of enrichment in displacement field

The exact solutions in the Model 1 and the Model 2 correspond to the asymptotic solutions of the fracture mode I and mode II, respectively. Thus, the enrichment functions $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$) in Eqs. (4.47) – (4.50) can absolutely constitute the exact solution in the sub-domain Ω^{enr} near the crack tip. In this case, the numerical results have a possibility of the exact solution when the ‘dominant ratio’ of the enrichment relating to the function $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$) satisfies 100%. That is, the numerical results will contain a certain errors when the dominant ratio of the enrichment is lower than 100%. A proposal of the dominant ratio of the enrichment is therefore useful for the evaluation of the XFEM approximation using the *priori* knowledge of solution.

Thus, the dominant ratio of enrichment in the displacement field is evaluated in this section.

The dominant ratio P_I of the enrichment in the displacement field related to node I is defined in the standard XFEM as

$$P_I = \frac{\left| \sum_{k=1}^4 \gamma_k(\mathbf{x}_I) \mathbf{c}_I^k \right|}{\left| \mathbf{u}_{\text{ap}}(\mathbf{x}_I) \right|} \quad (4.67)$$

and in the weighted XFEM or the PU-XFEM as

$$P_I = \frac{\left| \varphi_C(\mathbf{x}_I) \sum_{k=1}^4 \gamma_k(\mathbf{x}_I) \mathbf{c}_I^k \right|}{\left| \mathbf{u}_{\text{ap}}(\mathbf{x}_I) \right|} \quad (4.68)$$

The numerical models in this evaluation have elements of 7×7 .

The radiuses R_C for the definition of the set of nodes C , which is related to the crack tip enrichment, are given as $R_C = 1.0h$, $2.0h$ and $3.0h$, respectively.

The numerical results of the dominant ratio P of the singular enrichment near the crack tip in the displacement field are shown in Figs. 4.13 – 4.15. The boundaries of the

sub-domains Ω^{std} , Ω^{blend} and Ω^{enr} are shown using the solid line in red. The distribution of the dominant ratio P is calculated from the nodal value P_I and the standard interpolation function in the classical FEM.

In the case of the standard XFEM, the convergence rate of approximation of displacement field in the blending elements is lower than that of the classical FEM approximation as described in the section 4.2. It is therefore shown that the dominant ratio of the enrichment in the displacement field is gradually decreasing according to the distance from the crack surface in Ω^{enr} , where all nodes that constitute the element are enriched in the standard XFEM. Consequently, it is found that the asymptotic solution cannot be reflected on the numerical results in the standard XFEM.

In the case of the weighted XFEM, the better results are obtained than those in the standard XFEM. However, the dominant ratio of the enrichment in the displacement field is gradually decreasing according to the distance from the crack tip in Ω^{enr} , as well as those of the standard XFEM. That is, it is shown that the enrichment near the crack tip is not able to sufficiently reproduce the exact solutions.

On the other hand, in the case of the PU-XFEM, there is no lack of convergence rate in the approximation of the displacement field in the blending elements, as described in the section 4.3. The numerical results therefore show dominant ratio of the enrichment is maintained as almost 100% in the entire area of Ω^{enr} for any enrichment radius R_C .

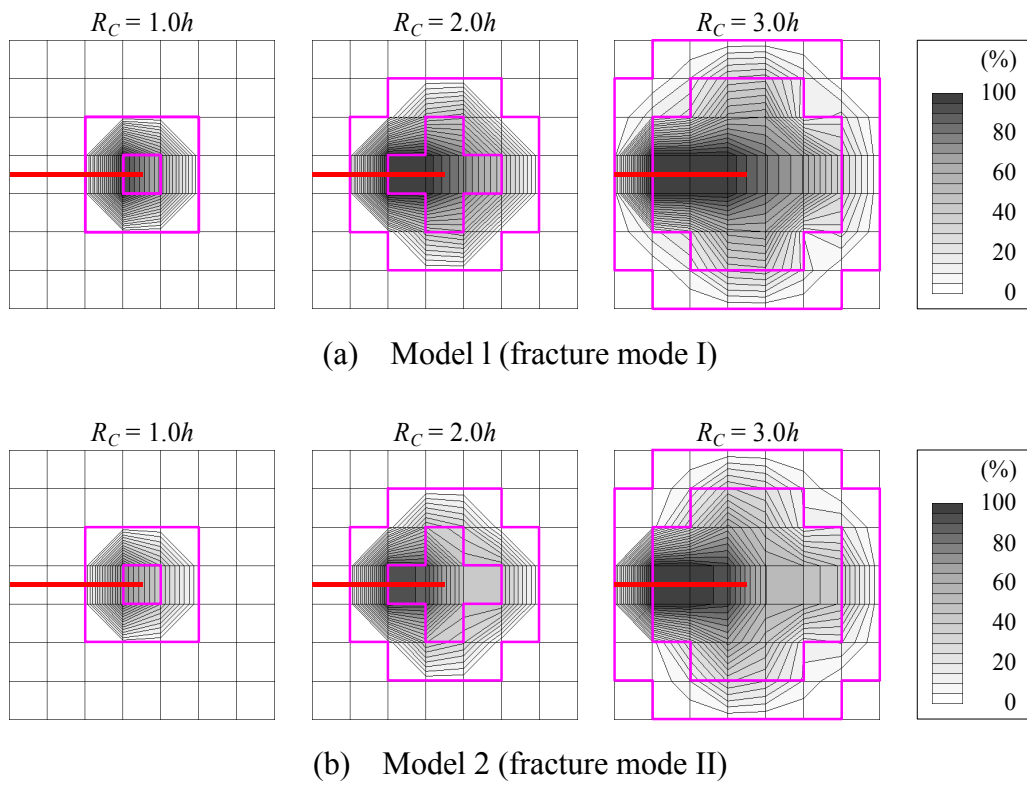


Figure 4.13 Dominant ratio of enrichment in displacement field using the standard XFEM

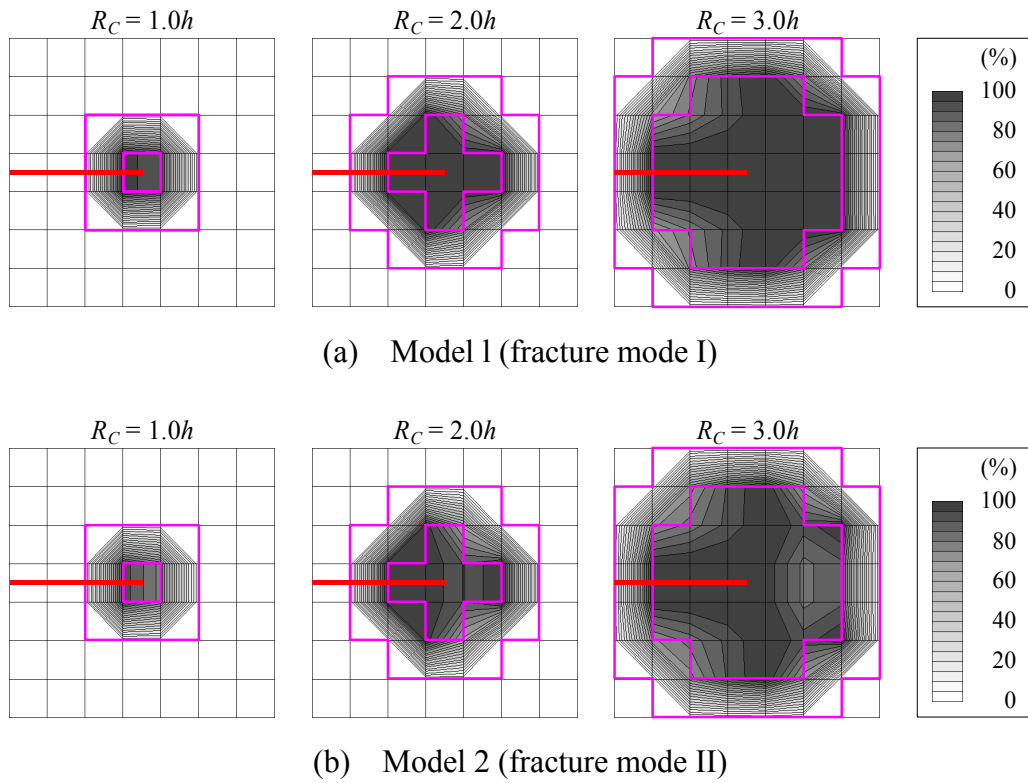
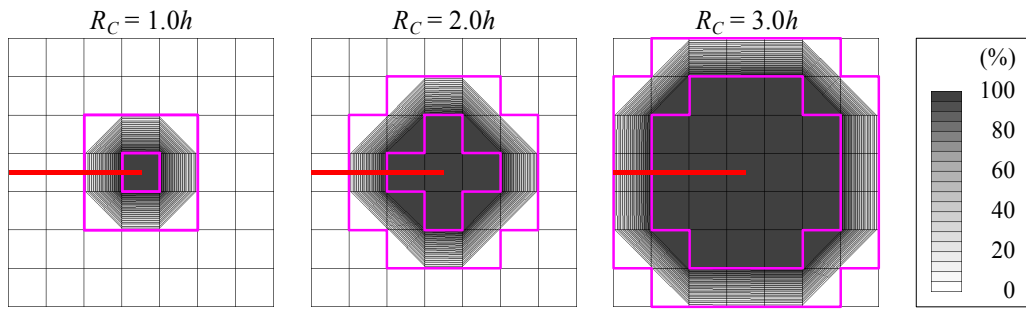
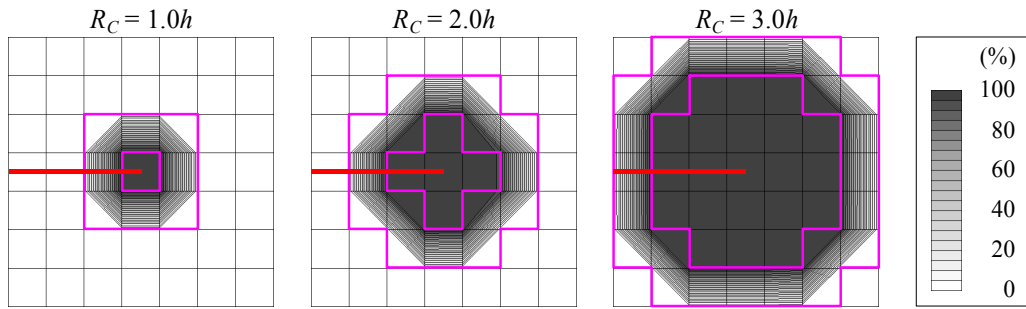


Figure 4.14 Dominant ratio of enrichment in displacement field using the weighted XFEM



(a) Model 1 (fracture mode I)



(b) Model 2 (fracture mode II)

Figure 4.15 Dominant ratio of enrichment in displacement field using the PU-XFEM

d) Evaluation of stress fields

In the crack analyses using the XFEM, the methods based on the J integral method are generally used for evaluation on the fracture mechanics parameters. The numerical results of the stress fields around the crack tip are used as the integrand of the J integral.

In this section, the reproducibility of the maximum principal stress $\sigma_{\max}$ and the Von Mises stress σ_{VM} are evaluated by comparing with the asymptotic solutions.

The numerical models in this evaluation have elements of 7×7 .

The numerical results of the stress fields and their errors from the asymptotic solutions using the Model 1 and the Model 2 are shown in Figs. 4.16 and 4.17, respectively. The boundaries of the sub-domains Ω^{std} , Ω^{blnd} and Ω^{enr} are shown using the bold lines.

In the case of the standard XFEM, we can see serious errors in the both sub-domains Ω^{blnd} and Ω^{enr} , as shown in the section 3.2.2. In particular, there are large errors on an enriched element containing the crack tip, even though all the nodes in this element are fully enriched. Thus, it is shown that the asymptotic solution for the stress field cannot be reproduced in the standard XFEM.

In the case of the weighted XFEM, the results show an improvement of accuracy than that in the standard XFEM. However, the considerable errors are observed near the boundary of the sub-domains Ω^{blnd} and Ω^{enr} in the blending elements. These results show the lack of accuracy caused by the incompleteness of the formulation of the weighted XFEM.

On the other hand, in the case of the PU-XFEM, it is shown that the errors found in the standard XFEM and the weighted XFEM is vanished. It is therefore found that only the PU-XFEM accurately reproduces the singular stress fields near the crack tips in the enriched element. In addition, in the blending elements, the approximation with enrichment is constructed on the boundary of the sub-domains Ω^{blnd} and Ω^{enr} , and the classical finite element approximation is constructed on the boundary of the sub-domains Ω^{std} and Ω^{blnd} . The numerical results in the blending elements show the smooth transition of the two approximations.

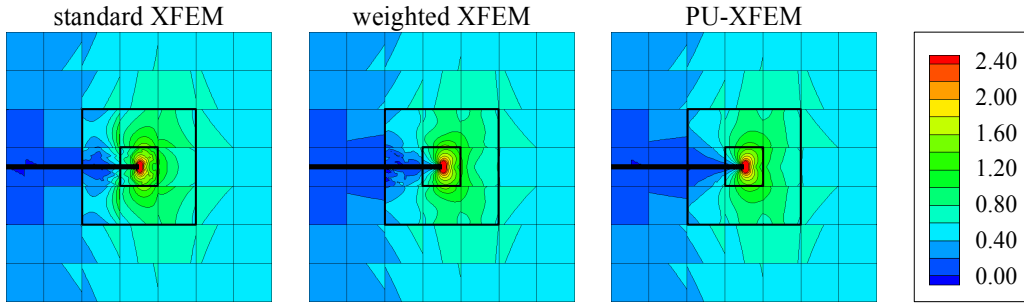
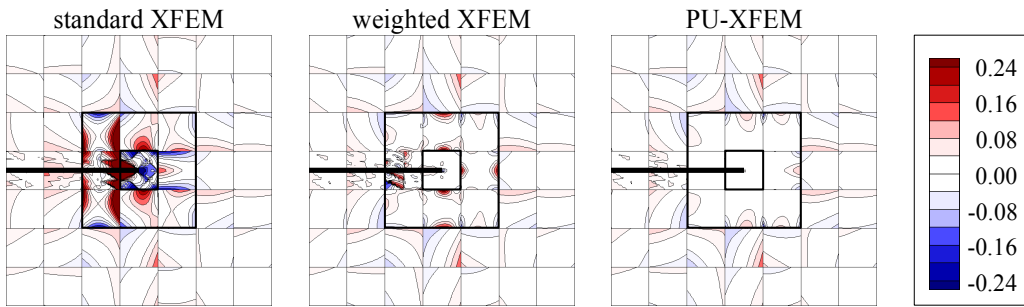
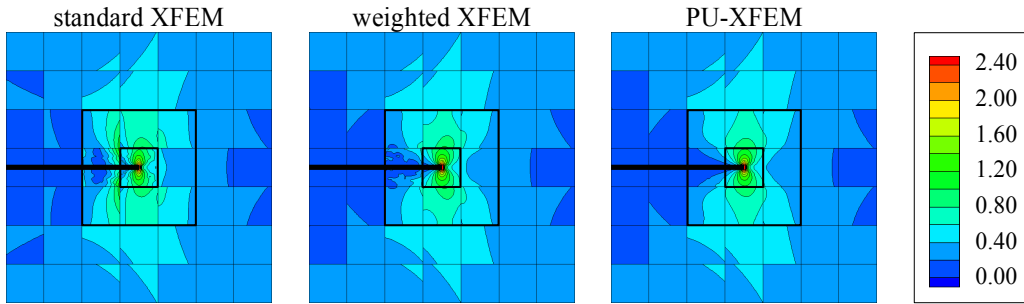
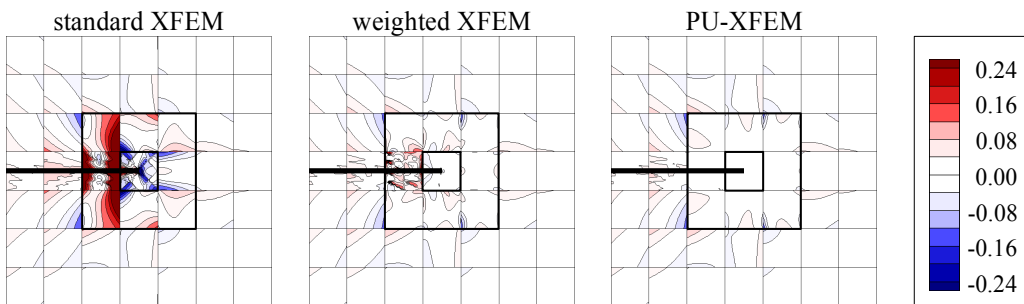
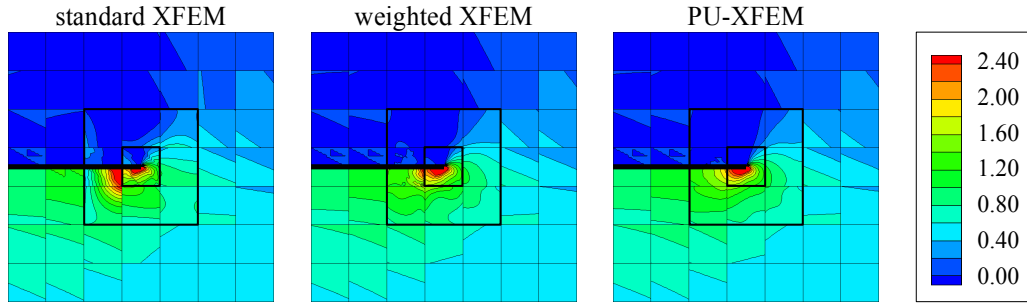
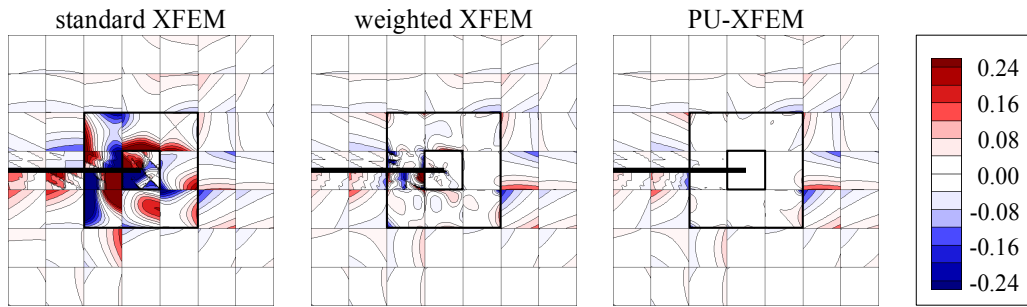

 (a) Maximum principal stress $\sigma_{\max}$

 (b) Errors of $\sigma_{\max}$ from the asymptotic solution

 (c) Von Mises stress σ_{VM}

 (d) Errors of σ_{VM} from the asymptotic solution

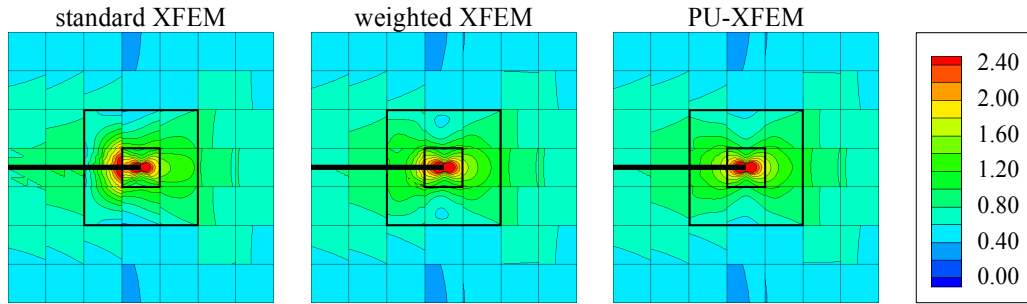
Figure 4.16 Numerical results of the stress in Model 1 (fracture mode I)



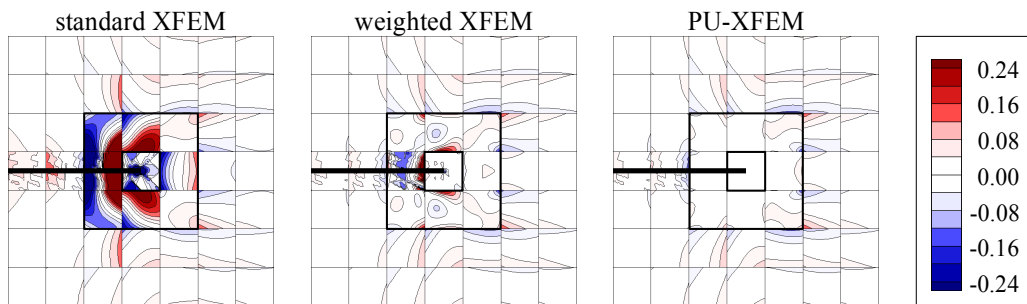
(a) Maximum principal stress $\sigma_{\max}$



(b) Errors of $\sigma_{\max}$ from the asymptotic solution



(c) Von Mises stress σ_{VM}



(d) Errors of σ_{VM} from the asymptotic solution

Figure 4.17 Numerical results of the stress in Model 2 (fracture mode II)

e) Evaluation on linear fracture mechanics parameters

In this section, we evaluate the numerical accuracy of the fracture mechanics parameters, including the path independency of the J -integral.

The parameters used for the evaluation are the stress intensity factors K_I or K_{II} and the energy release rate G , which are the basic linear fracture mechanics parameters.

For the evaluation of these parameters, the J -integral method and the M -integral method are employed. These methods are generally used in the crack analyses by using the XFEM. The procedure of the J -integral method in the PU-XFEM is the same as that of the standard XFEM. Therefore, the detailed formulation of the method can be seen in the section 2.3 of the chapter 2.

The both numerical models of the Model 1 and the Model 2 for the evaluations have the elements of 7×7 . The conditions of the integral radiuses for the J -integral path are considered as three cases of $R_F = 1.0h$, $2.0h$ and $3.0h$, corresponding to that of the ranges of the enrichment related to the nodes of C in the case of $R_C = 1.0h$.

It is noted that the minimum J -integral path of $R_F = 1.0h$ is the most important because the larger range of the J -integral path causes the limitations to the actual applications such as the coarse size of the meshes for simplification of the numerical models. That is, the numerical results in the cases of the $R_F = 2.0h$ and $3.0h$ are considered as the comparison values in order to evaluate the path independency of the J -integral in this evaluations.

The numerical results for the evaluations of the stress intensity factors K_I or K_{II} and the energy release rate G using the Model 1 and the Model 2 are shown in Figs. 4.18 and 4.19, respectively.

In the cases of the standard XFEM, we can see large errors in the case of the minimum J -integral path of $R_F = 1.0h$. These errors are 5.3% for K_I and 14.2% for G in the Model 1, and 6.1% for K_{II} and 19.6% for G in the Model 2, respectively. In addition, the explicit lack of the path independency can be found.

In the cases of the weighted XFEM, the effective improvement of the numerical results can be seen compared with that of the standard XFEM. However, larger errors are found in the case of the minimum J -integral path of $R_F = 1.0h$ than those in the case of $R_F = 2.0h$ and $3.0h$.

On the other hand, in the cases of the PU-XFEM, much better results of the fracture

mechanics parameters are provided in comparison with those of the standard XFEM and the weighted XFEM. That is, the errors are less than 0.8% for K_I and less than 1.6% for G in the Model 1, and the errors are also less than 0.6% for K_{II} and less than 1.2% for G in the Model 2, even if the J -integral is calculated in the blending elements, i.e. under the condition of $R_C = R_F = 1.0h$. In addition, the numerical results of the fracture mechanics parameters are constantly accurate in the various range of J -integral path. That is, the path independencies of the J -integral is assured in the numerical results in the PU-XFEM

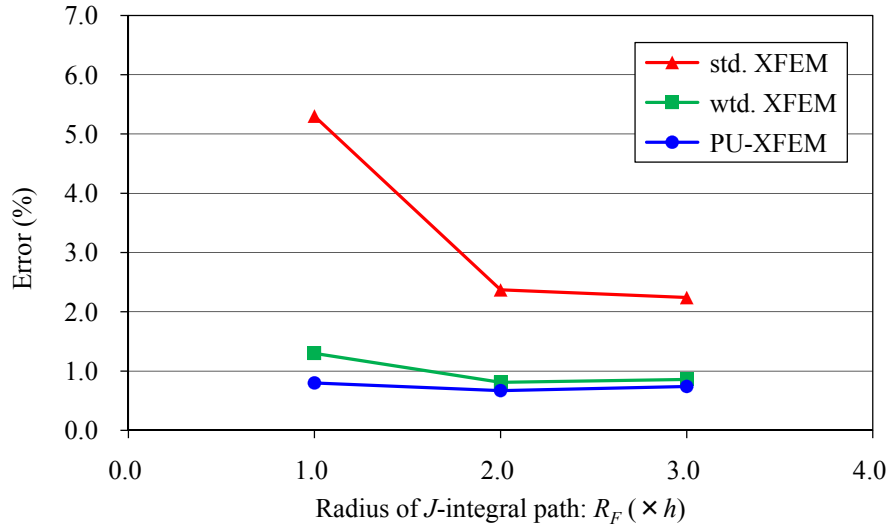
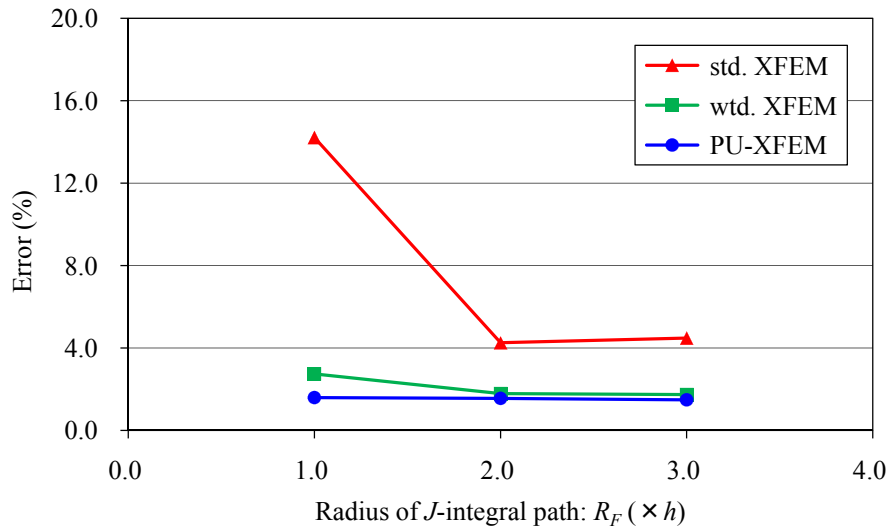
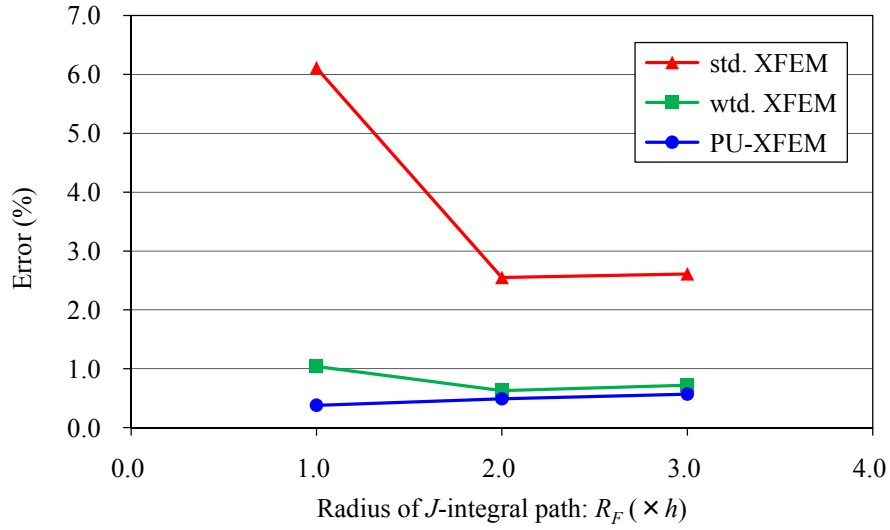
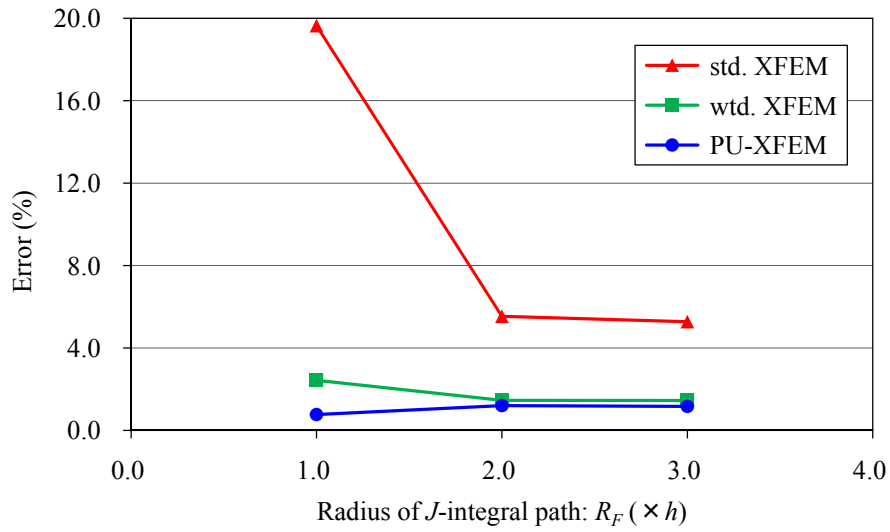
(a) Stress intensity factor: K_I (b) Energy release rate: G

Figure 4.18 Numerical results of fracture mechanics parameters in Model 1
(fracture mode I)



(a) Stress intensity factor: K_{II}



(b) Energy release rate: G

Figure 4.19 Numerical results of fracture mechanics parameters in Model 2 (fracture mode II)

4.5.2 Numerical Models of finite plates

In this section, we investigate the effectiveness of the PU-XFEM in the actual application, using the edge crack model under tension in the finite plate (Model 3) and the edge crack model under shear in the finite plate (Model 4). The evaluations are carried out by verifying the accuracy of the linear fracture mechanics parameters.

The contents of this section are as follows: In the subsection **a)**, we firstly describe the numerical models used in this section. In the subsection **b)**, we then evaluate the convergence of the numerical results of the linear fracture mechanics parameters.

a) Model descriptions

The numerical analyses in this section are evaluated by the following two models: the edge crack model under tension in the finite plate (Model 3) and the edge crack model under shear in the finite plate (Model 4).

We consider a plate with finite width W which is loaded under tension from afar as the Model 3. The numerical model has a length $L = 5$, width $W = 1$, and crack length $a = 0.5$ as shown in Fig. 4.20. It is noted that the length $L = 5$ of the numerical model is assumed to be sufficiently long in this evaluation, because the numerical results of the fracture mechanics parameters show the difference of less than 0.1% in comparison with the length $L = 10$ in the our preliminary research. In this model, constant C for correction of the stress intensity factor considering the finite geometry is given by Okamura (1976), as

$$C = 1.12 - 0.231 \left(\frac{a}{W} \right) + 10.55 \left(\frac{a}{W} \right)^2 - 21.72 \left(\frac{a}{W} \right)^3 + 30.39 \left(\frac{a}{W} \right)^4 \quad (4.69)$$

Under the above condition, the applied stress $\sigma = 1$ gives the stress intensity factor of

$$K_I = C \sigma \sqrt{\pi a} = 3.54 \quad (4.70)$$

and corresponding energy release rate of

$$G = \frac{K_I^2}{E'} = 5.66 \times 10^{-11} \quad (4.71)$$

Then, an edge crack in a finite plate subjected to the distributed nominal shear stress is considered as the Model 4. The model has a length $L = 16$, width $W = 7$, and crack length is $a = 3.5$ as shown in Fig. 4.21. This model is fixed at the bottom and subjected to a uniform shear stress $\tau = 1$ applied at the top. The references of the mixed mode stress intensity factors (Yau *et al.* 1980) are

$$K_I = 34.0 \quad (4.72)$$

$$K_{II} = 4.55 \quad (4.73)$$

In addition, the corresponding energy release rate is

$$G = \frac{K_I^2 + K_{II}^2}{E'} = 5.35 \times 10^{-9} \quad (4.74)$$

The both numerical models are discretized by using the square elements except those on the model edges. In the discretization along the model edges, suitable rectangular elements are used. The inner mesh size of the model is employed as a reference parameter of h in the following evaluations.

The numerical integration procedures are the same as those in the previous section 4.5.1 a).

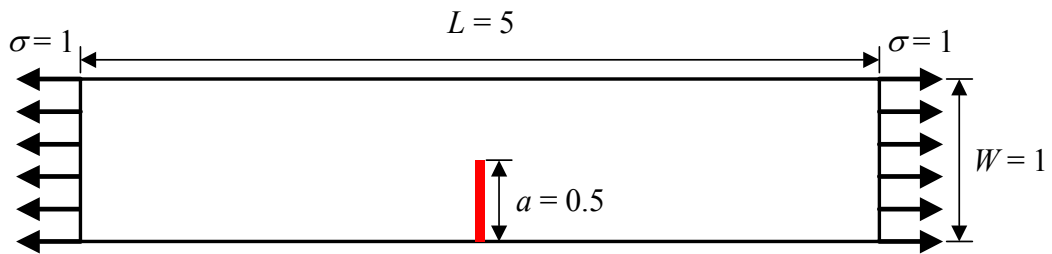


Figure 4.20 Model with edge crack under tension in a finite plate (Model 3)

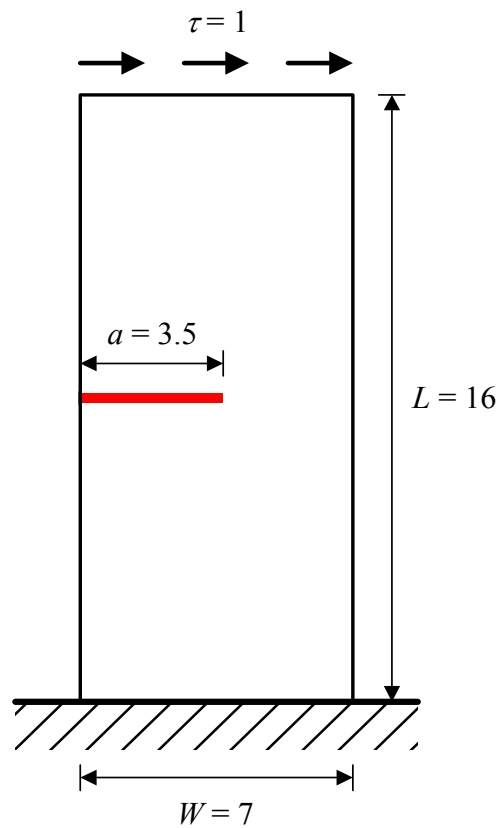


Figure 4.21 Model with edge crack under shear in finite plate (Model 4)

b) Evaluation on linear fracture mechanics parameters

The numerical accuracy of the fracture mechanics parameters are evaluated using the finite plate models of the Model 3 and Model 4.

The parameters used for the evaluation are the stress intensity factors K_I and K_{II} and the energy release rate G , which are the basic linear fracture mechanics parameters. The J -integral method and the M -integral method are used for the evaluation of these parameters as well as the evaluations in previous section 4.5.1 e).

Two types of the evaluations are carried out. In the first, the convergence of the fracture mechanics parameters is evaluated according to the mesh size, which is normalized by the crack length. In the second, the path independency of J -integral in the case of the coarse mesh is evaluated.

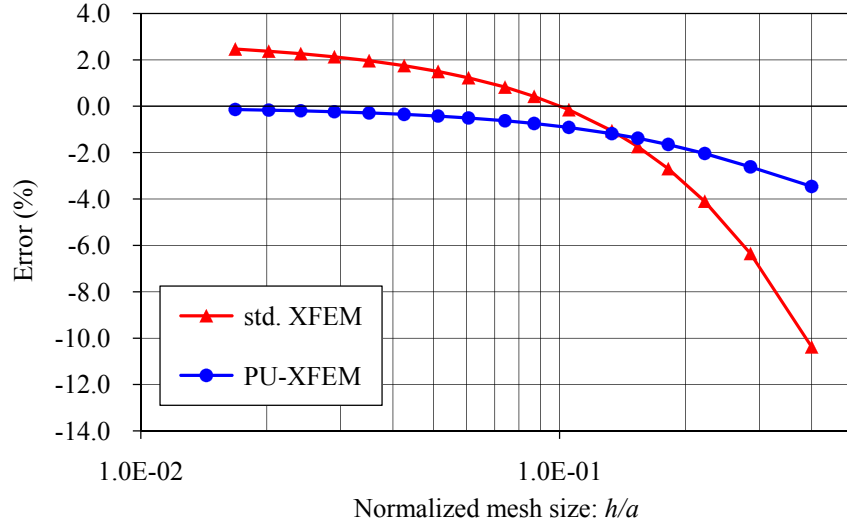
The numerical results of the respective parameters K_I and G in the Model 3 are shown in Figs. 4.22 and 4.23. The numerical results of the respective parameters K_I , K_{II} and G in the Model 4 are also shown in Figs. 4.24 – 4.26.

When the standard XFEM is used, the numerical results of the fracture mechanics parameters are undervalued in the cases of the coarse mesh, and those are overvalued in the cases of the fine mesh. It is shown that there is a serious problem in the stability of the numerical accuracies in the standard XFEM. In the cases of the coarse mesh of $h / a = 0.4$, the errors are -10.4% for K_I and -15.6% for G in the Model 3, and -12.6% for K_I and -17.2% for G in the Model 4, respectively. It can be estimated that the results of finer subdivision of meshes are asymptotically equivalent to those of the infinite plate models of Model 1 or Model 2 evaluated in the section 4.5.1. That is, the numerical results in the standard XFEM cannot converge to the exact solution of the problem. In addition, it is shown that the path independency of the J -integral is impaired in the cases of the standard XFEM.

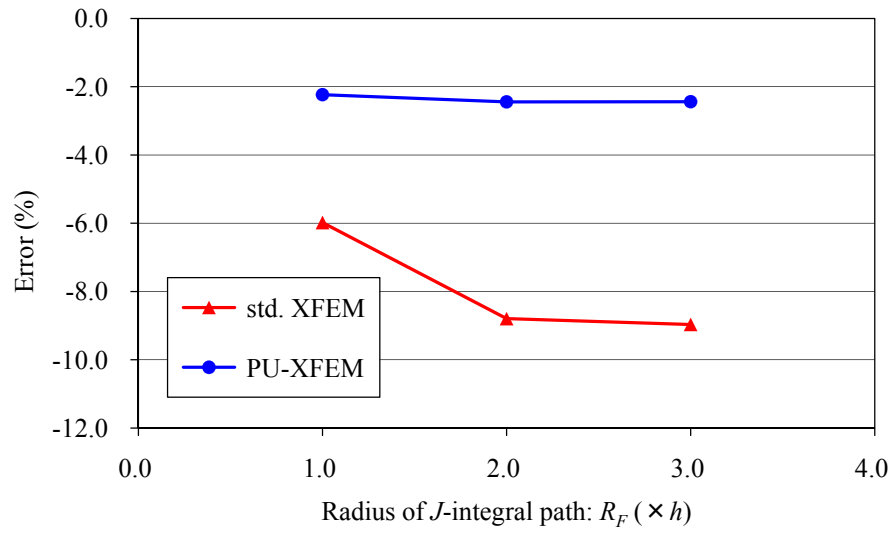
On the other hand, when the PU-XFEM is used, the numerical results of the fracture mechanics parameters show a significant improvement of the accuracy, compared to those in the standard XFEM. That is, the errors in the cases of the coarse mesh of $h / a = 0.4$ are drastically reduced to -3.4% from -10.4% for K_I and -6.7% from -15.6% for G in the Model 3, and -4.7% from -12.6% for K_I and -8.2% from -17.2% for G in the Model 4, respectively. According to the results of the convergence studies, in the cases of the PU-XFEM, we can find the convergence of the numerical results to the exact solutions,

differently from those in the standard XFEM. These convergence behaviors to the exact solutions in the numerical results are important for the establishment of the numerical method. In addition, we can see the improvements in the path independency of the J -integral, because numerical results of the fracture mechanics parameters can be constantly obtained through any ranges of the radius of the J -integral R_F . It is therefore found out that the PU-XFEM is able to provide the remarkably stable and accurate analyses in comparison with the standard XFEM.

It is concluded that the effectiveness of the proposed PU-XFEM is clarified in the actual applications by evaluations using the numerical models of the finite plate in this section.

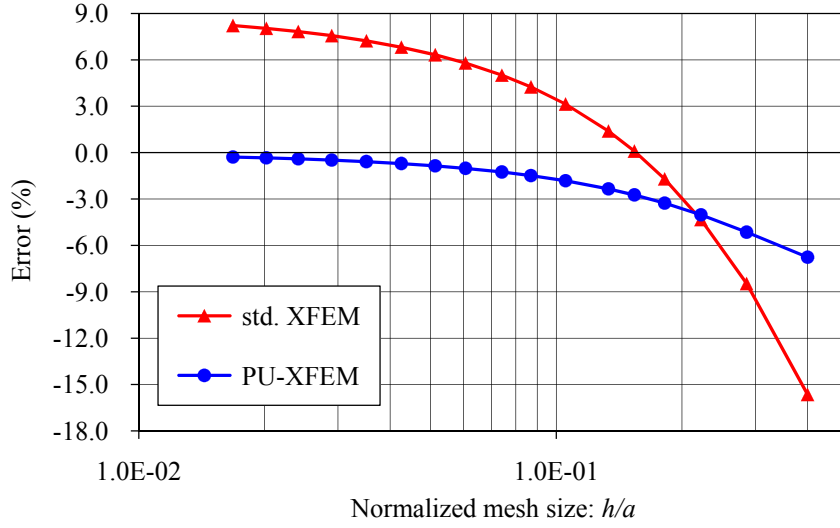


(a) Convergence results ($R_C = R_F = 1.0h$)

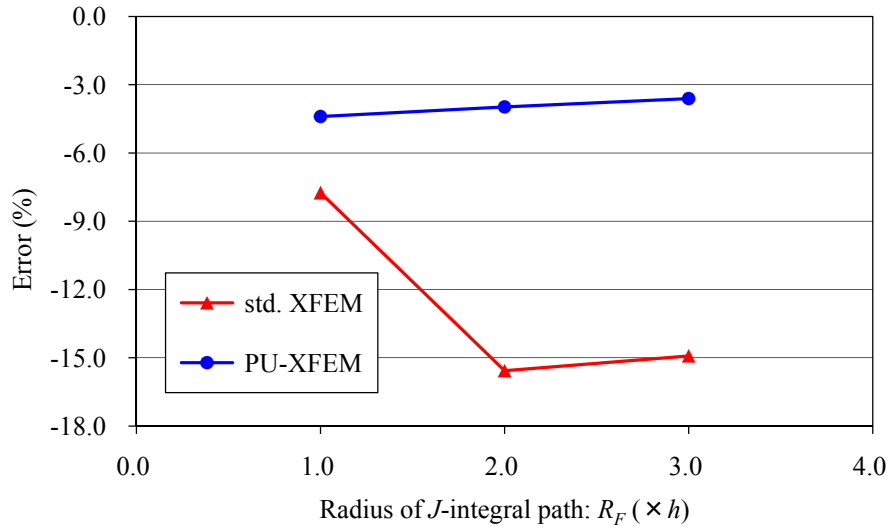


(b) Path independency (7×35 elements, $R_C = 1.0h$)

Figure 4.22 Numerical results for the stress intensity factor K_I in Model 3

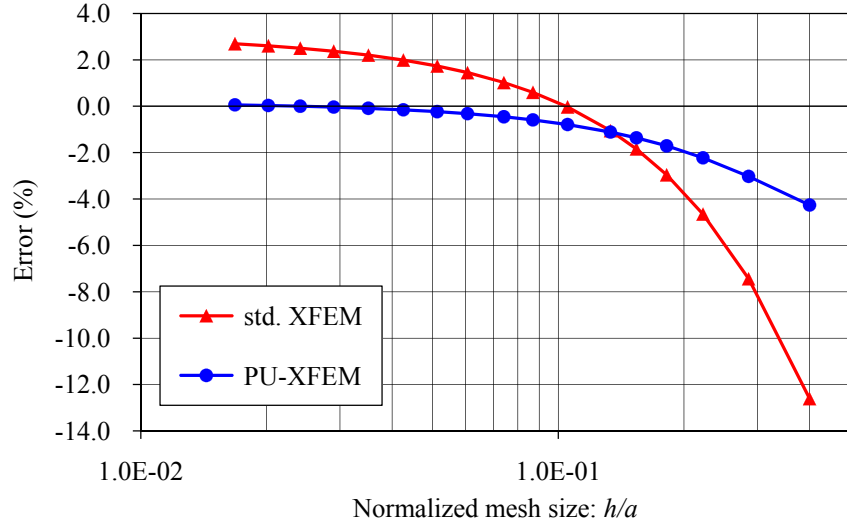


(a) Convergence results ($R_C = R_F = 1.0h$)

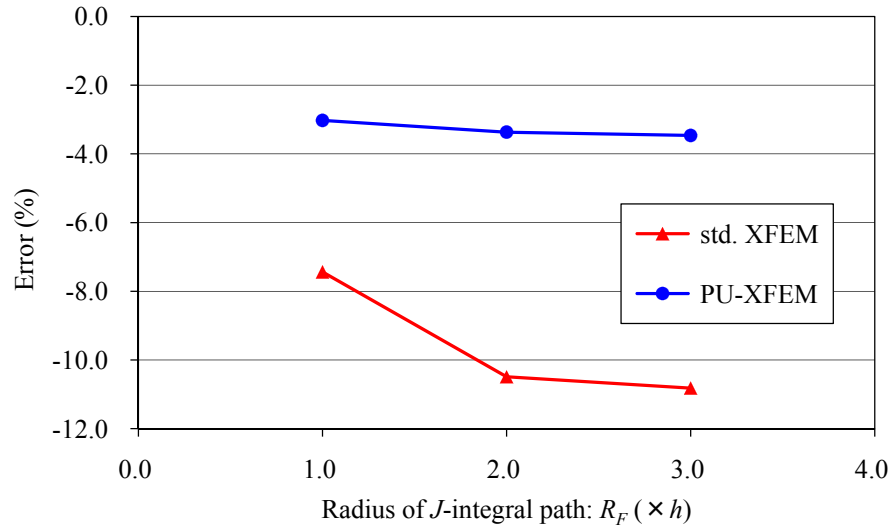


(b) Path independency (7×35 elements, $R_C = 1.0h$)

Figure 4.23 Numerical results for the energy release rate G in Model 3

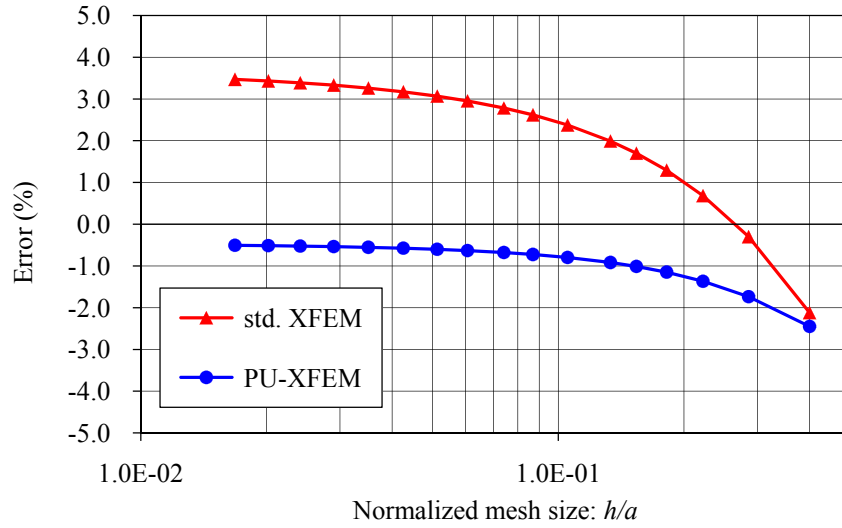


(a) Convergence results ($R_C = R_F = 1.0h$)

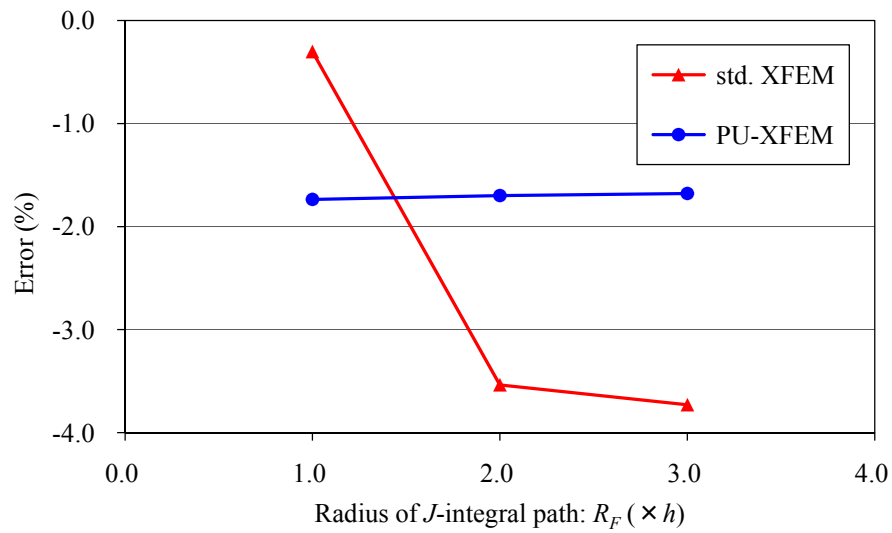


(b) Path independency (7×16 elements, $R_C = 1.0h$)

Figure 4.24 Numerical results for the stress intensity factor K_I in Model 4

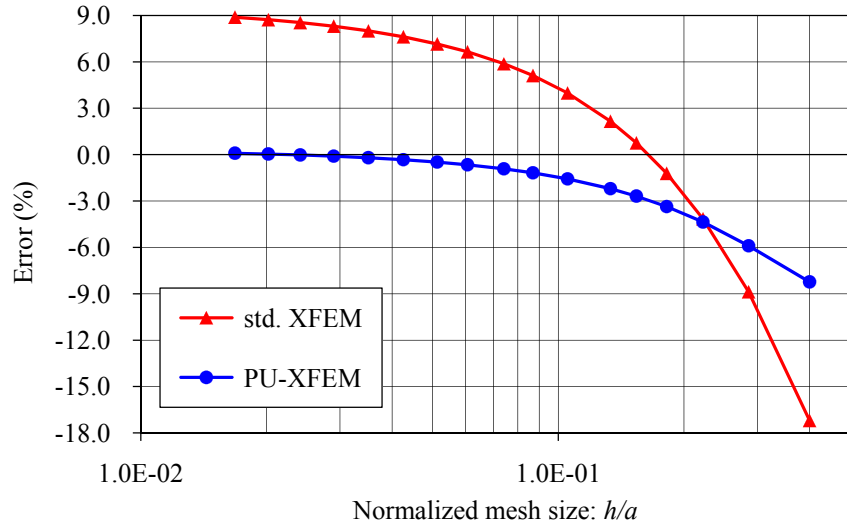


(a) Convergence results ($R_C = R_F = 1.0h$)

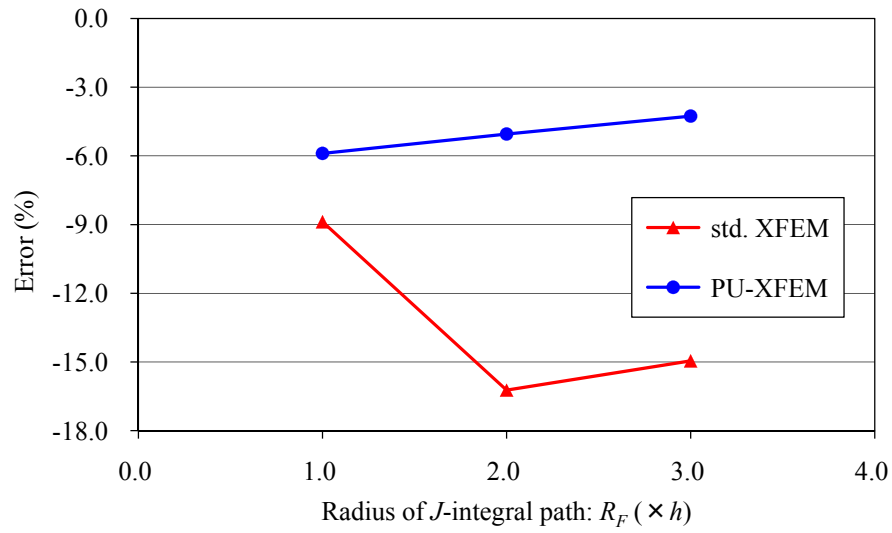


(b) Path independency (7×16 elements, $R_C = 1.0h$)

Figure 4.25 Numerical results for the stress intensity factor K_{II} in Model 4



(a) Convergence results ($R_C = R_F = 1.0h$)



(b) Path independency (7×16 elements, $R_C = 1.0h$)

Figure 4.26 Numerical results for the energy release rate G in Model 4

4.6 CONCLUDING REMARKS

In the past studies, it was pointed out that the approximation of the standard XFEM has the problems caused by the blending elements lowering the convergence rate.

In order to solve the problem of the blending elements, we have reformulated an approximation of the XFEM based on the concept of the PUFEM approximation as ‘PU-XFEM’. The PU-XFEM approximation $\mathbf{u}_{\text{ap}}(\mathbf{x})$ has the form of

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \varphi_0(\mathbf{x})\mathbf{v}_0^{\text{ap}}(\mathbf{x}) + \varphi_1(\mathbf{x})\mathbf{v}_1^{\text{ap}}(\mathbf{x})$$

where the sets of $\varphi_0(\mathbf{x})$ and $\mathbf{v}_0^{\text{ap}}(\mathbf{x})$, and $\varphi_1(\mathbf{x})$ and $\mathbf{v}_1^{\text{ap}}(\mathbf{x})$ are the PU and the approximate functions, relating to the classical finite element approximation and the approximation including enrichment considering the *priori* knowledge of the solution, respectively. In particular, the PU $\varphi_1(\mathbf{x})$ relating the approximation with enrichment is active only in the local area. The blending elements have been redefined as the elements in the common part of the supports of the PU $\varphi_0(\mathbf{x})$ and $\varphi_1(\mathbf{x})$ in the PU-XFEM approximation.

In the evaluation of the approximation accuracies for a one dimensional problem, it is shown that the insufficient convergence rate in the standard XFEM has been overcome in the proposed PU-XFEM. It was therefore found out that the PU-XFEM provides a potential ability to solve the problem of blending elements in the standard XFEM, by showing the relationship between the XFEM and the PUFEM.

The weighted XFEM recently proposed by Fries (2008) showed the effective improvement of the accuracy reduction caused by the blending elements. The usefulness of this method has however been based on the results of the inductive evaluations, because the theoretical background was not sufficiently described in his proposal. We showed that the weighted XFEM approximation can be obtained as a result by simplifying the two kinds of the degrees of freedom into the only one in the PU-XFEM approximation. This fact led the presumption that the improvement in the weighted XFEM is based on the deceptively similar form to the PU-XFEM. At the same time, the

incompleteness in the proposed form of the weighted XFEM as the solution of the problem of the blending elements was clarified.

The PU-XFEM has been applied to the two dimensional linear fracture mechanics and the effectiveness is investigated by the basic evaluations of the numerical accuracy using the models in the infinite plates and the models of the finite plates, respectively. In these investigations, the following results of the PU-XFEM were obtained in comparison with the standard XFEM and the weighted XFEM.

- (a) In the convergence study using the relative L_2 norm on numerical errors, we found the significant improvement on the convergence rate as well as the accuracy.
- (b) The dominant ratio of the enrichment near the crack tip is maintained as almost 100% in the enriched domain in the displacement field. It means that the numerical results have a possibility of the exact solution.
- (c) The stress field around the crack tip is exactly reproduced in the enriched domain.
- (d) The numerical results of the linear fracture mechanics parameters have shown a significant improvement of the accuracy in the cases of coarse mesh. The numerical results are converged to the exact solutions according to the mesh refinement, differently from those in the standard XFEM. These convergence behaviors in numerical results are important for the establishment of the numerical method. In addition, the path independency of the J -integral is assured.
- (e) The numerical results by the weighted XFEM showed the effective improvements compared to those by the standard XFEM. However, these results show the lack of accuracy caused by the incompleteness of the formulation of the weighted XFEM, in comparison with the results by the PU-XFEM.

Therefore, the effectiveness of the proposed PU-XFEM has been verified in the actual application to the crack analyses.

Based on the aforementioned facts, it can conclude that the problem of the blending elements has been solved by the PU-XFEM. And therefore, the proposed formulation of the PU-XFEM can be a new foundation of the XFEM instead of the standard XFEM.

Chapter 5

Application to Fatigue Crack Simulations in Real Structures

5.1 INTRODUCTION

Recently, several fatigue cracks are reported on aged steel bridges. Therefore, the clarification of the cause of the fatigue damage and the rationalization of the maintenance are urgently required (Japanese Society of Steel Construction 1995, Japan Road Association 2002).

The numerical simulation of the fatigue crack behavior in the aged steel bridge is effective to clarify the mechanism of fatigue crack propagation and to study the rationalization of maintenance for the bridge, due to the following reasons:

- (a) In order to clarify the causes of fatigue crack propagation, the detailed evaluations are possible.
- (b) The quantitatively evaluation of the critical aspect of fatigue crack is possible. Thus, the interval of inspections and the priority of repairs and reinforcement can be effectively determined.
- (c) A larger number of simulations of fatigue crack propagation are possible than fatigue tests using specimens.

- (d) The fatigue crack in the large and complex structures can be evaluated.

The FEM is widely used for the numerical analyses of structures. However, this method has been developed as the numerical method intended for the analyses of continuum. Due to this reason, there are serious problems related to mesh generation for crack propagation according to the following reasons:

- (a) It is required to model the boundaries of elements and double nodes along crack surfaces, taking into account of the discontinuities of displacement.
- (b) Either refinement of meshes or the use of singular elements (Barsoum 1976) is required in order to evaluate the fracture mechanics parameters with high accuracy.
- (c) The complicated remeshing procedures are required for the modeling of crack propagation.

According to above reasons, the simulation of the fatigue crack propagation using the FEM is not a simple task to evaluate the behavior of fatigue crack propagation in local parts of the complex and large-scaled civil engineering structures, such as steel deck systems of bridges, without advanced knowledge and experience in numerical modeling.

On the other hand, the XFEM is a numerical method that employs the local enrichment application to reproduce discontinuity or singularity in the local parts of the domain, though it is formulated in the framework of the FEM. As the feature of the XFEM, the crack can be modeled independently from the finite element meshes. Therefore, the XFEM has a potential ability to solve the problem of the complicated modeling for crack propagation using the FEM.

However, the standard XFEM has an essential problem of blending elements. In the crack analysis, the reproducibility of the *priori* knowledge near the crack tip by the enrichment is seriously decreased. It means that the approximation cannot be appropriately constructed and an inevitable large error is caused in the standard XFEM approximation.

In order to solve the problem of the blending elements, we proposed the PU-XFEM based on the PUFEM approximation in Chapter 4. In the PU-XFEM, the insufficient

convergence rate in the standard XFEM was overcome in the evaluation of the approximation accuracies. In addition, the effectiveness of the PU-XFEM was verified in the actual application to the crack analyses. Therefore, the problem of the blending elements has been solved by the PU-XFEM.

Another important fact is related to the functions of general-purpose FEM analysis softwares that have recently been enhanced. This enables the numerical investigation of the behavior of complex and large-scale structures (Alex *et al.* 2002, Miki *et al.* 2005).

The implementation of the PU-XFEM, which enables accurate and simple modeling of crack analyses, into the general-purpose FEM analysis softwares, which have the advantage in numerical analyses for large-scale structures, is effective for evaluation of the behavior of actual fatigue cracks in localized regions of complex and large-scale structures.

The nodal degrees of freedom are required to be increased when the XFEM is implemented into the general-purpose FEM analysis software. Giner *et al.* (2009) recently presented the implementation of the standard XFEM into ABAQUS (Dassault Systems Simulia Corp. 2007), which is one of the most well known general purpose FEM analysis software. Giner *et al.* developed an application to numerical analyses of two dimensional problem. In this method, the nodal degrees of freedom, which are originally defined as rotations around three dimensional axes, temperatures, etc. in the “Conventions” section of the ABAQUS User’s Manual, were redefined as those in the displacement fields for two dimensional problems. However, this redefinition of the nodal degrees of freedom requires the use of the coupled thermo-mechanical analysis procedure. It is therefore completely different from the original convention rules employed in ABAQUS. The consistency with the analytical algorithms formulated in ABAQUS is therefore required to be confirmed for the implementation of the XFEM. In addition, there is a serious limitation related to the application to three dimensional analyses, which require significant increase of nodal degrees of freedom. This is caused due to the limitation of the nodal degrees of freedom defined by the ABAQUS User’s Manual. Therefore, this proposal of the implementation of the XFEM into ABAQUS is not adequate from the view points of the analytical logic and the extension to the three dimensional analysis.

According to the above-mentioned facts, in this chapter it is developed a fatigue

crack simulation code based on ABAQUS through the implementation of the PU-XFEM approximation. In particular, it is proposed a method to increase the number of the nodal degrees of freedom, which is consistent with the original definition of the nodal degrees of freedom in the convention rules provided by the ABAQUS User's Manual and applicable to the extension to three dimensional analyses.

Using this simulation code, the behavior of a fatigue crack through thickness of a plate in a three dimensional structure can be quantitatively evaluated. We validate the developed PU-XFEM analysis code through its applications to the numerical simulations of the fatigue crack propagation in the intermediate transverse beam of the actual I-girder bridge and the orthotropic steel deck specimen using bulb rib simulating the real bridge structure.

An outline of this chapter is as follows: In Section 5.2, the PU-XFEM approximation in the application to the linear fracture mechanics is briefly reviewed. In Section 5.3, it is carried out the implementation into ABAQUS of the general-purpose FEM code and is presented the flow of entire analysis and the evaluation of direction and rate of fatigue crack propagation. In Section 5.4, the numerical examples are shown by using the developed PU-XFEM code based on ABAQUS. In Section 5.5, the conclusions of the study results achieved in this chapter are provided.

5.2 MODELING OF CRACK USING PU-XFEM

Arbitrary deformation near the crack tip is identified as a superposition of the three types of deformation modes, i.e., the opening mode (mode I), the sliding mode (mode II) and the tearing mode (mode III) (Irwin 1958, Gdoutos 2005).

In the most cases of the fatigue damage in the real bridges, it can be assumed that the contribution of the mode III is negligible, compared to those of the mode I and the mode II.

In this study, we therefore concentrate on the investigation of the three dimensional structures modeled using shell elements under the in-plane deformation composed of

only mode I and mode II, which is dominant in the crack behavior.

The crack modeling using the PU-XFEM approximation, which applies to the formulation of the in-plane deformation of the shell elements, is briefly described below. The details of this formulation are found in Chapter 4. The coordinate transformation required in the three dimensional analyses using shell elements are found in the references (for example, Zienkiewicz & Taylor 2005).

The PU-XFEM approximation of the in-plane displacement field $\mathbf{u}_{\text{ap}}(\mathbf{x})$ is defined as

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \varphi_0(\mathbf{x})\mathbf{v}_0^{\text{ap}}(\mathbf{x}) + \varphi_C(\mathbf{x})\mathbf{v}_C^{\text{ap}}(\mathbf{x}) \quad (5.1)$$

where $\varphi_0(\mathbf{x})$ and $\varphi_C(\mathbf{x})$ are the following partition of unity (PU) $\varphi_0(\mathbf{x}) + \varphi_C(\mathbf{x}) = 1$, defined as

$$\varphi_C(\mathbf{x}) = \sum_{I \in C} \phi_I(\mathbf{x}) \quad (5.2)$$

$$\varphi_0(\mathbf{x}) = 1 - \varphi_C(\mathbf{x}) \quad (5.3)$$

satisfying

$$\varphi_0(\mathbf{x}) + \varphi_C(\mathbf{x}) = 1 \quad (5.4)$$

$\mathbf{v}_0^{\text{ap}}(\mathbf{x})$ and $\mathbf{v}_C^{\text{ap}}(\mathbf{x})$ are the approximation functions defined on the supports Ω_0 of the PU $\varphi_0(\mathbf{x})$ and Ω_C of the PU $\varphi_C(\mathbf{x})$, respectively as

$$\mathbf{v}_0^{\text{ap}}(\mathbf{x}) = \sum_I \phi_I(\mathbf{x})\mathbf{u}_I + \sum_{I \in J} \phi_I(\mathbf{x})H(\mathbf{x})\mathbf{b}_I \quad (5.5)$$

$$\mathbf{v}_C^{\text{ap}}(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) \left(\mathbf{w}_I + \sum_{k=1}^4 \gamma_k(\mathbf{x})\mathbf{c}_I^k \right) \quad (5.6)$$

In Eqs. (5.2) – (5.6), $\mathbf{u}_I$, $\mathbf{b}_I$, $\mathbf{w}_I$ and $\mathbf{c}_I^k$ ($k = 1, \dots, 4$) are the nodal degrees of freedom.

J and C are the sets of nodes, respectively. $\phi_I(\mathbf{x})$ is the interpolation function used in the finite element approximation. $H(\mathbf{x})$ and $\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$) are the enrichment functions, which are the same functions used in crack modeling of standard XFEM, and are defined as

$$H(\mathbf{x}) = \begin{cases} 1 & \text{on } \Omega_+ \\ -1 & \text{on } \Omega_- \end{cases} \quad (5.7)$$

$$\gamma_k(\mathbf{x}) = \left\{ \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2} \sin \theta, \sqrt{r} \sin \frac{\theta}{2} \sin \theta \right\} \quad (5.8)$$

where $H(\mathbf{x})$ is the Heaviside function used for the presence of strong discontinuities. Ω_+ and Ω_- are the upper and lower sides of the domain defined on the supports Ω_I of node I containing the Heaviside enrichment. It is noted that this Heaviside enrichment does not lead to the problem caused by blending elements even in the standard XFEM. This is because the Heaviside enrichment function is defined as a constant in the partially enriched elements called “blending elements”. Therefore, the form of the Heaviside enrichment in the standard XFEM is useful in reducing the degrees of freedom as the approximation function $\mathbf{v}_0^{\text{ap}}(\mathbf{x})$ in Eq. (5.5).

$\gamma_k(\mathbf{x})$ ($k = 1, \dots, 4$) are the set of singular enrichment functions based on the near-field asymptotic solution. (r, θ) are the local polar co-ordinates at the crack tip.

In the modeling of the crack based on the PU-XFEM approximation, the enrichment near the crack tip is redefined as the product of the PU $\phi_C(\mathbf{x})$ and the approximation function $\mathbf{v}_C^{\text{ap}}(\mathbf{x})$. It means that it is possible to define this enrichment based on the domain on the support Ω_C of the PU $\phi_C(\mathbf{x})$. This definition is different from that of the standard XFEM, which is defined based on the respective nodes.

The set of nodes C in Eq. (5.2) is defined as

$$C = \left\{ I \in N \mid \left| \mathbf{x}_I - \mathbf{x}_{\text{tip}} \right| \leq R_C \right\} \quad (5.9)$$

where R_C is the radius using the definition of the nodal set C . The set of nodes C defined

by the radius R_C provides the ranges of the two supports Ω_0 and Ω_C , and their common part Ω^{blind} at the same time.

In the PU-XFEM approximation, the nodal degrees of freedom $\mathbf{u}_I$ and $\mathbf{b}_I$ are defined at all the nodes on the support Ω_0 . On the other hand, the nodal degrees of freedom $\mathbf{w}_I$ and $\mathbf{c}_I^k$ ($k = 1, \dots, 4$) are defined at all the nodes on the support Ω_C . Thus, it is useful to define the following sets of nodes B_0 and B_C in the blending domain Ω^{blind} , which is defined as a common part of the supports Ω_0 and Ω_C , as

$$B_0 = \left\{ I \in N \mid \Omega_I \cap \Omega^{\text{blind}} \neq \emptyset, I \notin C \right\} \quad (5.10)$$

$$B_C = \left\{ I \in N \mid \Omega_I \cap \Omega^{\text{blind}} \neq \emptyset, I \in C \right\} \quad (5.11)$$

Then, an set of nodes C' is defined as a complementary set of B_C in C , i.e.,

$$C' = C \setminus B_C \quad (5.12)$$

Finally, the set of nodes J in Eq. (5.5) is defined by using the set C' , as

$$J = \left\{ I \in N \mid \Omega_I \cap (D \cap \Omega_0) \neq \emptyset, I \notin C' \right\} \quad (5.13)$$

where D is the geometry of the crack.

In the above modeling of the set of nodes in the PUXFEM shown in Eqs. (5.9) – (5.12), the radius R_C and the initial geometry of the crack D and are treated as input data in the analysis of the crack propagation, as well as those in the standard XFEM. An example of the sets of nodes in Eqs. (5.9) – (5.13) near the crack is shown in Fig. 5.1.

The numerical integration procedure in each element is carried out by using the standard Gauss quadrature in the FEM except for the elements that are cut by the crack discontinuity. In the elements cut by the crack, the sub-division of the domain is required. In particular, in the element contains a crack tip, the almost polar integration is employed (Laborde *et al.* 2005, Béchet *et al.* 2005). Details of the above integration procedure are same in the standard XFEM and found in Chapter 2.3.4.

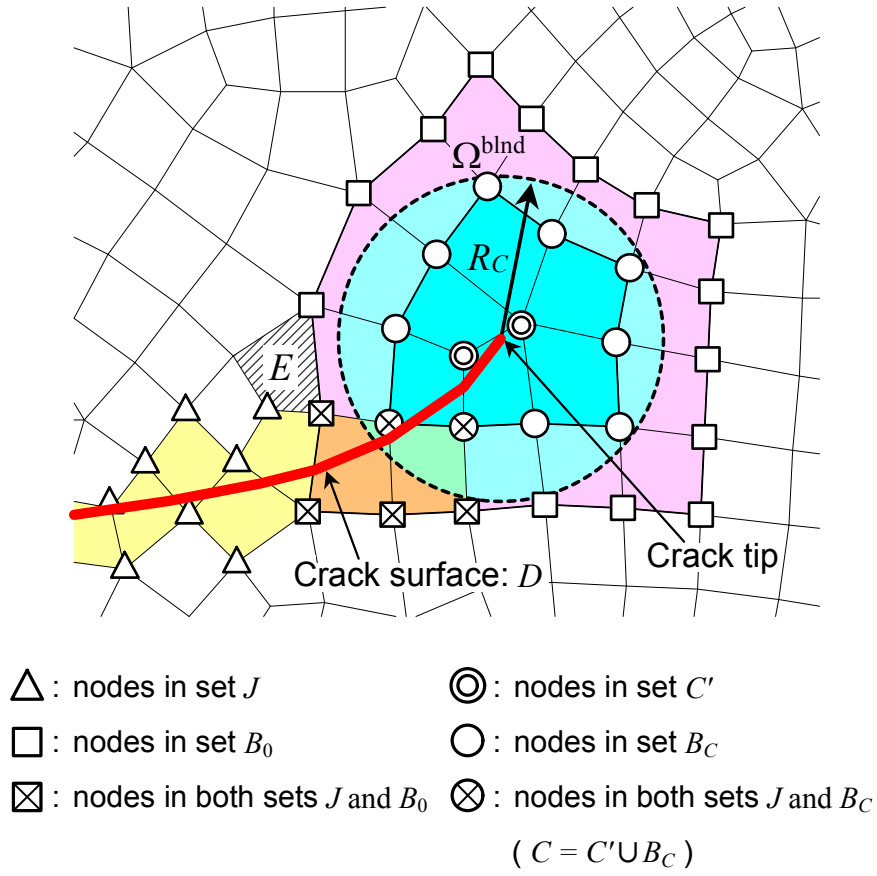


Figure 5.1 Modeling of the sets of nodes in the PU-XFEM

5.3 IMPLEMENTATION INTO GENERAL-PURPOSE FEM CODE

In this study, it is developed a fatigue crack simulation code applicable to a through thickness crack of a plate in a three dimensional structure modeled by shell elements. This code is composed of the general-purpose FEM analysis software ABAQUS (Dassault Systems Simulia Corp. 2007) and the following additional programs.

- (a) Program PRE-PRO for the implementation of the PU-XFEM approximation into ABAQUS
- (b) Program POST-PRO for evaluation of stress intensity factors, the calculation of the direction and rate of the fatigue crack propagation, and the redefinition of the geometries of the cracks
- (c) Program XFEM3DS for the control of the entire analysis

This section contains as follows: In Section 5.3.1, the outline of the program XFEM3DS is described, including the flow of the entire analysis. In Sections 5.3.2 and 5.3.3, the outlines of the programs PRE-PRO and POST-PRO are described, respectively. In particular, a new method to increase the nodal degrees of freedom, which is required in the implementation of the XFEM into the general-purpose FEM analysis software, is given in Section 5.3.2. The proposed method is consistent with the original definition of the nodal degrees of freedom given by the convention rules employed in ABAQUS and applicable to the extension to three dimensional analysis.

The details of the options available in the ABAQUS used in the following sections 5.3.1 and 5.3.2 are found in the ABAQUS User's Manual.

5.3.1 *Flow of Analysis*

The developed fatigue crack simulation code is composed of the software ABAQUS and the additional programs PRE-PRO, POST-PRO and XFEM3DS. In particular,

the program **XFEM3DS** controls the entire flow of the analysis as shown in Fig. 5.2. The procedures of the analysis using the developed code are given as follows.

- (a) Using the ABAQUS CAE, it is assembled the finite element model of the objective structure for analysis and write the model data into the input file **INPUT.inp**. The model data in the input file **INPUT.inp** is able to be submitted as the job into the ABAQUS solver, which assembles the global stiffness matrix and computes the system of equations.
- (b) The data of the initial crack geometry **CRACK.dat** is defined, corresponding to the coordinates of the finite element model made in (a)
- (c) To implement the PU-XFEM approximation, the program **PRE-PRO** is run to make a new input file **X_n.inp** (n is the step number of the analysis) by rewriting the part of the input file **INPUT.inp** defined in (a), based on the initial or new crack data **CRACK.dat** defined in (b) or (f).
- (d) The job defined by the input file **X_n.inp** is submitted into the ABAQUS solver. This procedure makes the output files **X_n.dat** through the assembly of the global stiffness matrix and the calculation of the system of the equations. The output files **X_n.dat** contains the numerical results of the nodal degrees of freedom related to the PU-XFEM approximation.
- (e) The program **POST-PRO** is run to calculate the stress intensity factors and to evaluate the direction and rate of the fatigue crack propagation, using the numerical results obtained in the output file **X_n.dat** in (e).
- (f) The data of the crack geometry **CRACK.dat** is redefined, based on the numerical result of the direction of the fatigue crack propagation in (e).
- (g) The procedures of (c) ~ (f) are repeated until the given condition of the analysis, based on the results of the stress intensity factors, length of crack, etc.
- (h) If necessary, the output files of **DS_n.dat** for the displacement field and **ST_n.dat** for the stress field in each analytical step n are produced, using the software **TECPLOT** (2008) for data visualization.

According to the above description, the developed PU-XFEM code easily enables a numerical simulation of the propagation of the fatigue cracks through thickness in a three dimensional structure modeled by shell elements.

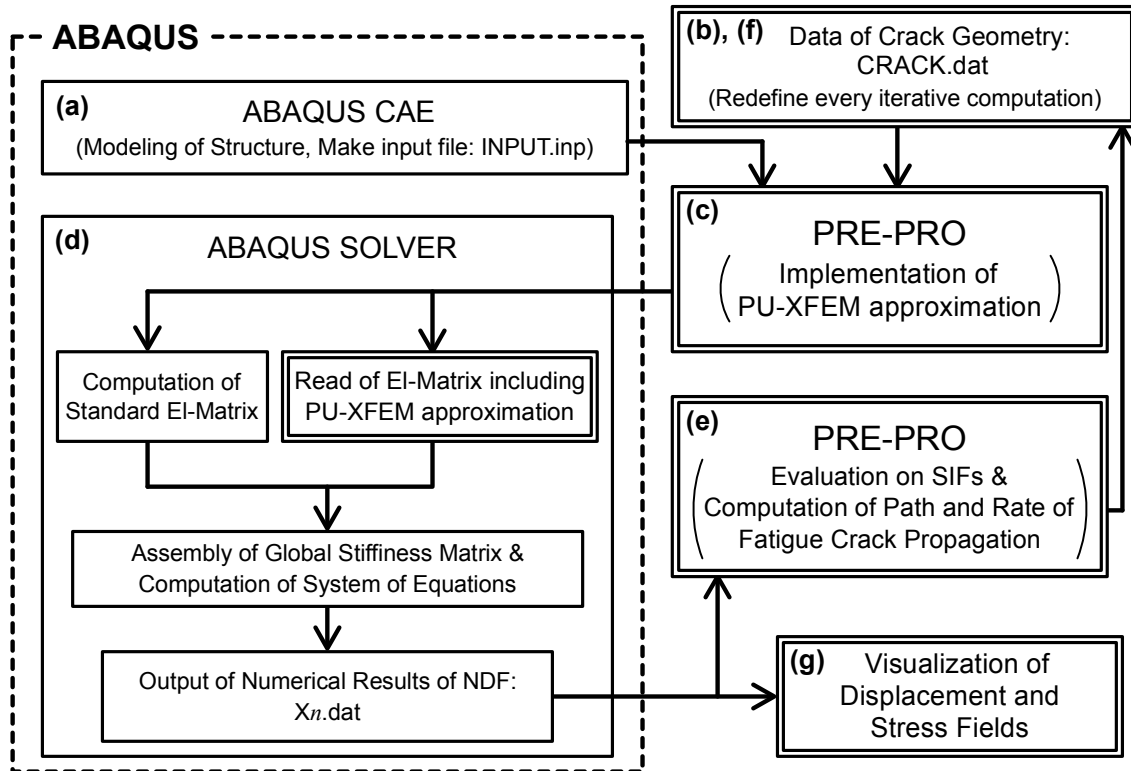


Figure 5.2 Flow chart of the developed fatigue crack simulation code based on implementation of the PU-XFEM into the ABAQUS
(Entire flow of analysis is controlled by the program XFEM3DS)

5.3.2 *Implementation of PU-XFEM Approximation into ABAQUS*

In this section, the method of the implementation of the PU-XFEM approximation into the ABAQUS is described.

Using the option *USER ELEMENT provided by ABAQUS, it is possible to define arbitrary elements in the framework defined in the convention of ABAQUS. This approach can maintain the original framework of the ABAQUS.

In order to implement the PU-XFEM approximation using the option *USER ELEMENT, it is required to increase the degrees of freedom at the respective nodes near the crack. However, the respective degrees of freedom are prescribed in the ABAQUS convention rules, as shown in Table 5.1. If we follow the above definition of the degrees of freedom, it is difficult to simply increase the nodal degrees of freedom in the displacement field for the implementation of the XFEM.

According to the above-mentioned facts, a new method to increase the nodal degrees of freedom by adopting multiple-nodes is presented in the subsection **a)**. And then, the outline of the program PRE-PRO to implement the PU-XFEM approximation into the ABAQUS is presented in the subsection **b)**.

Table 5.1 Definition of the nodal degree of freedom in the ABAQUS convention rules

NDF No.	Definition
1 - 3	x, y, z -displacements
4 - 6	Rotations about the x, y, z -axis
7	Warping amplitude
8	Pore pressure, hydrostatic fluid pressure, or acoustic pressure
9	Electric potential
10	Connector material flow
11	Temperature
12	Second temperature
13	Third temperature
14	Etc.

a) Method to increase nodal degrees of freedom by adopting multiple-nodes

In the developed PU-XFEM code, “multiple nodes” are introduced at the coordinates of the nodes which are required to increase the nodal degrees of freedom for the implementation of the PU-XFEM. The provision of “multiple nodes” at the same coordinates can be assumed equivalent to the increase of the nodal degrees of freedom in the PU-XFEM approximation. This method to increase the nodal degrees of freedom in the developed PU-XFEM code is presented below in detail.

Four node shell element definition is used as a standard element in the developed PU-XFEM code. In the shell element, 6 degrees of freedom (3 degrees of freedom for translations and 3 degrees of freedom for rotations) are defined at the respective nodes. The respective elements are therefore composed of 24 degrees of freedom in total.

In the present study, the discontinuity of the displacement along the crack surface is not considered in the formulation of the out-of-plane deformation as shown in Section 5.2. Therefore, only 3 degrees of freedom for translations are defined at the additional nodes to increase the degrees of freedom. The number of the additional nodes for the implementation of the PU-XFEM approximation are determined, corresponding to the respective set of nodes J , C' , B_0 and B_C defined in Eqs. (5.9) – (5.13) (see Fig. 5.1), as shown in Table 5.2.

For example, the definition of the element E found in Fig. 5.1 is presented below. The closeup of the element E including node number is shown in Fig. 5.3. In this element, the node 1 is modeled by the composition of 2 nodes because the node 1 belongs to the set of nodes J . The node 2 is modeled by the composition of 7 nodes because the node 2 belongs to both of the sets of nodes J and B_0 . The node 3 is modeled by the composition of 6 nodes because the node 3 belongs to the set of nodes B_0 . The node 4 is a standard node which is not any set of nodes. As a result, to define the element shown in Fig. 5.3 by using the option *USER ELEMENT, the respective total numbers of nodes and degrees of freedom are therefore required as follows:

$$(\text{Number of the nodes}) = 2 + 7 + 6 + 1 = 16$$

$$(\text{Number of the degrees of freedom}) = 9 + 24 + 21 + 6 = 60$$

The above-mentioned method to increase the nodal degrees of freedom used in the developed PU-XFEM code has the following advantages:

- (a) This method is consistent with the original definition of the nodal degrees of freedom in the ABAQUS User's Manual. Therefore, it is possible to easily implement the XFEM approximation into the ABAQUS.
- (b) The original options for analysis, such as a coupled thermo-mechanical analysis procedure, are maintained without any modifications, except that the ABAQUS CAE, which is not available for the visualization of the numerical results.
- (c) Arbitrary degrees of freedom can be increased at the arbitrary nodes. It is therefore easily applicable to the extensions such as the three dimensional analysis, requiring the significant increase of the nodal degrees of freedom.
- (d) If it is possible to define an element using the arbitrary number of the nodes in the other general-purpose FEM analysis software, the implementation of the XFEM approximation is easily achieved by using the same method presented above.

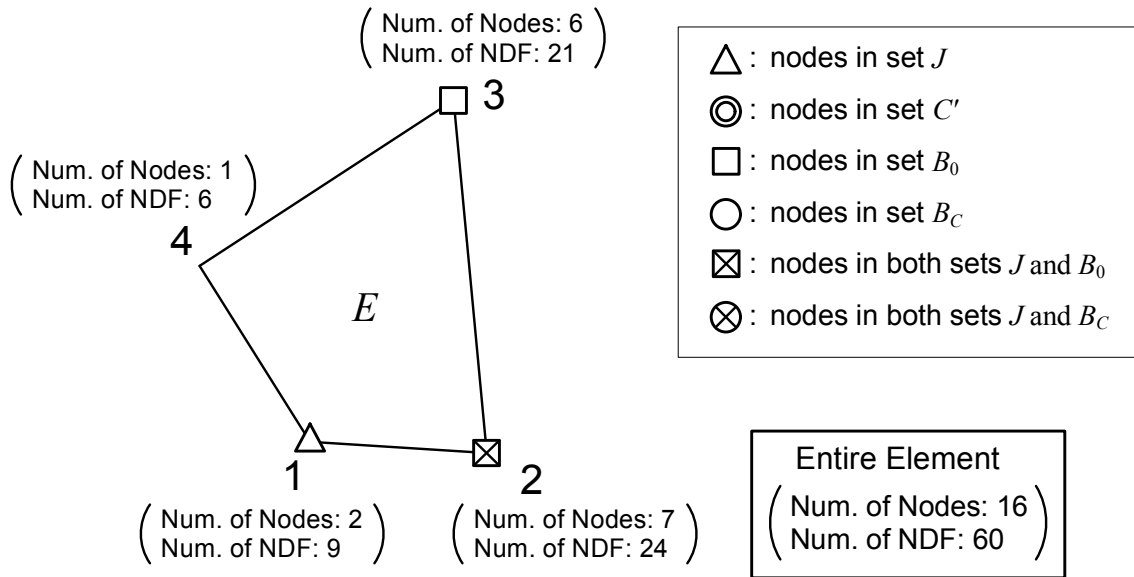


Figure 5.3 Example of the element implemented using *USER ELEMENT
 (the element E in Fig. 5.1)

Table 5.2 Number of the additional nodes and nodal degrees of freedom for the implementation of the PU-XFEM approximation

(a) Number of nodes and nodal degrees of freedom defined in the classical finite element approximation in four nodes shell element

	Num. of nodes	Num. of DOF
Classical FEM app.	4	24 (4×6)

(b) Number of additional nodes and nodal degrees of freedom corresponding to the respective set of nodes

Set of nodes	Num. of additional nodes	Num. of additional DOF
J	+ 1	+ 3 (1×3)
C'	+ 4	+ 12 (4×3)
B_0	+ 5	+ 15 (5×3)
B_C	+ 5	+ 15 (5×3)
J and B_0	+ 6	+ 18 (6×3)
J and B_C	+ 6	+ 18 (6×3)

b) Outline of the program PRE-PRO

The fatigue crack behavior can be evaluated in the theory of elasticity, assuming the infinitesimal deformation. In this case, the option *MATRIX, which directly introduces the arbitrary elemental stiffness matrices to the input file in the option *USER ELEMENT, enables to implement the PU-XFEM approximation into ABAQUS.

Based on the method to increase the nodal degrees of freedom by adopting multiple-nodes presented in the subsection a), the procedures in the program PRE-PRO are shown below.

- (i) All data of the coordinates of the nodes and the relationships between nodes and elements is read from the input file INPUT.inp of the structural model assembled in ABAQUS CAE.
- (ii) The respective set of nodes J , C' , B_0 and B_C are defined, based on the coordinates of the crack geometry in the file CRACK.dat and the respective nodes read from the input file INPUT.inp. And then, the additional nodes are introduced as multiple-nodes, as shown in Table 5.2.
- (iii) If some of the elemental nodes belong to the set of nodes J , C' , B_0 and B_C , the elemental stiffness matrix is calculated, considering the enrichment functions defined in Eqs. (5.7) and (5.8). In the case of the element cut by the crack, the sub-division of the domain along the crack surface is required in the numerical integration procedure.
- (iv) The new input file $X_n.inp$ (n is the step number of the analysis) is defined by rewriting the file of the input file INPUT.inp. In the definition of the new input file $X_n.inp$, the elemental stiffness matrices calculated in (iii) are introduced by using the options *USER ELEMENT and *MATRIX.

The input file $X_n.inp$, defined in the program PRE-PRO, can directly be submitted into the ABAQUS solver as the job.

5.3.3 Evaluation on Fracture Mechanics Parameters and Criteria of Fatigue Propagation

In the numerical simulation of the fatigue crack propagation, the calculation of the stress intensity factors and the modeling on the propagation of the fatigue crack are required. In the developed code, the original options for the fracture mechanics provided in the ABAQUS are not used because of the modeling of the cracks in the XFEM is defined independently from meshes. The program POST-PRO is therefore introduced to the modeling of the fatigue crack propagation. The outline of the program POST-PRO is described below.

In the developed code, the M -integral (interaction integral) (Yau *et al.* 1980) method is employed as a method to evaluate the stress intensity factors K_I and K_{II} . The M -integral method is a method based on the well-known J -integral method and generally employed in crack analyses by using the XFEM.

The strain and stress fields for the objective problem in the integrand of the M -integral is reproduced, based on the numerical results of the nodal degrees of freedom in the output file `Xn.dat` calculated through the ABAQUS solver.

In order to improve the numerical accuracy in the application to the kinked or curved crack, the mapping procedure considering the crack geometry is adopted for the strain and stress fields in the reference problems in the integrand of the M -integral. Detailed discussion of this topic is found in the reference by Shibamura & Utsunomiya (2008).

In the standard XFEM, a considerable error is observed in the stress field near the crack tip, especially in the area of the blending elements. On the other hand, in the PU-XFEM, the errors as shown in the standard XFEM can be successfully reduced. Therefore, two radiuses R_C and R_F , which are used in the respective definitions of the nodal set C and the M -integral area, can-therefore be modeled as the similarly minimal as

$$R_C = R_F = \max \left\{ 1.0h, \max_I \left\{ \left| \mathbf{x}_I - \mathbf{x}_{tip} \right| \right\} \right\} \quad (5.14)$$

where I is the set of nodes in the element including the crack tip, and h is average length of the sides in the element.

The behavior of the fatigue crack can be defined based on the stress intensity factors evaluated by the above M -integral method.

The maximum tangential stress criterion proposed by Erdogan & Sih (1963) is generally used as the criterion for the direction of the brittle fracture propagation θ_c . In this criterion, the direction of the propagation is calculated as

$$\theta_c = 2 \tan^{-1} \left\{ 0.25 \left(\frac{K_I}{K_{II}} \pm \sqrt{\left(\frac{K_I}{K_{II}} \right)^2 + 8} \right) \right\} \quad (5.15)$$

The application of the extension of the maximum tangential stress criterion to the fatigue crack propagation is proposed by Kitagawa *et al.* (1981). In this criterion, it is assumed that the direction θ_c corresponds to that of the maximum tangential stress range $\Delta\sigma_{\theta_{\max}}$ at the crack tip, as

$$\theta_c = 2 \tan^{-1} \left\{ 0.25 \left(\frac{\Delta K_I}{\Delta K_{II}} \pm \sqrt{\left(\frac{\Delta K_I}{\Delta K_{II}} \right)^2 + 8} \right) \right\} \quad (5.16)$$

The validation of the maximum tangential stress criterion for fatigue propagation including the case of the high-tension steel in the residual stress field is presented by Ohji *et al.* (1993).

The equivalent stress intensity factor range ΔK of the mixed-mode I and II crack (Tanaka *et al.* 2005) is assumed as

$$\Delta K = \Delta K_I \cos^3 \left(\frac{\theta_c}{2} \right) - 3 \Delta K_{II} \cos^2 \left(\frac{\theta_c}{2} \right) \sin \left(\frac{\theta_c}{2} \right) \quad (5.17)$$

The propagation rate of the fatigue crack is assumed by the Paris's law considering the threshold stress intensity factor ΔK_{th} , described using the equivalent stress intensity

factor range ΔK of the mixed-mode I and II crack, as

$$\frac{da}{dN} = C \left(\Delta K^m - \Delta K_{th}^m \right) \quad (5.18)$$

where C and m are the material constants. The constants and the threshold stress intensity factor range are determined as $C = 1.50 \times 10^{11}$, $m = 2.75$ and $\Delta K_{th} = 2.9 \text{ MPa}$ based on the average design curve in the fatigue design recommendations for steel structures published by the Japanese Society of Steel Construction (1995). It is noted that the above constants C , m and ΔK_{th} are identified in the conditions of the weld joints under the high tensile residual stress and the propagation rate da/dN is therefore evaluated on the safe side.

In the modeling of the crack propagation, the crack geometry is redefined by the extension of a prespecified length of crack segment in every iterative computation step. The cycle of the loads N is computed by the approximation of linear interpolation using the numerical results of the stress intensity factor range ΔK before and after of the propagation in each iterative computation step. The modeling of the fatigue crack propagation is illustrated in Fig. 5.4.

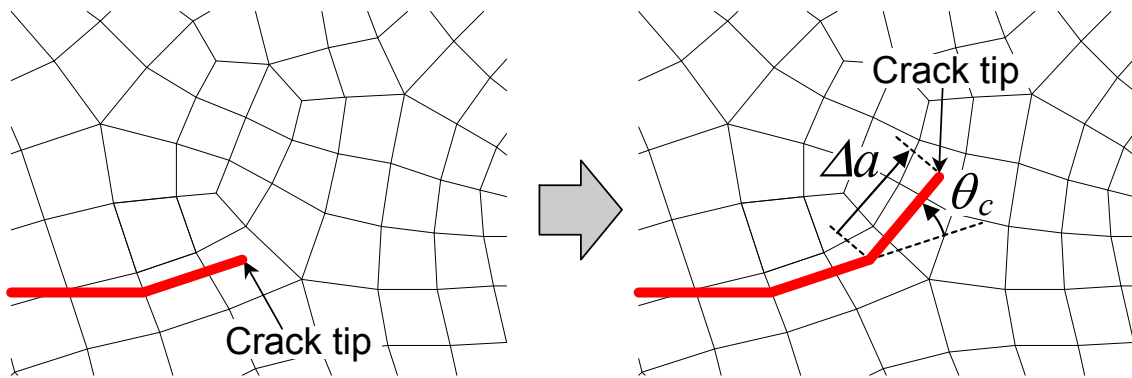


Figure 5.4 Modeling of crack propagation

5.4 NUMERICAL EXAMPLES

In this section, the performance of the developed PU-XFEM code based on the ABAQUS is evaluated for the case of application to numerical simulations of fatigue crack propagation in real structures. In Section 5.4.1, the developed code is applied to the fatigue crack detected in the intermediate transverse beam of I-girder bridge. In Section 5.4.2, the developed code is also applied to the fatigue test for the orthotropic steel deck specimen using bulb rib.

5.4.1 *Application to Fatigue Crack Detected in Intermediate Transverse Beam of I-girder Bridge*

There are numerous reports related to fatigue damage from the weld joints between flange and web in the intermediate transverse beams of the I-girder bridges. Most of these fatigue damages are observed to propagate into inside of the web of the beams as shown in Fig. 5.5. It is possible to evaluate the behaviors of this type of the fatigue damages without the contribution of deformation mode III.

In this section, it is evaluated the fatigue behavior in the intermediate transverse beam of I-girder bridge shown in Fig. 5.5. The observed crack has a length of about 55mm and the direction of the crack has been gradually changed according to the crack propagation, i.e., the crack intends to propagate vertically upward along the arrangement of the bolt connection after propagation into the inside of the web in the initial phase.

A typical member of the intermediate transverse beam is modeled as a numerical model used in this section (Model-1) shown in Fig. 5.6. The standard mesh size is employed as $h = 20.0\text{mm}$ as shown in Fig. 5.6(a). However, near the area of “A” in Fig. 5.5(a) where fatigue crack was detected, the minimum mesh size $h = 3.0\text{mm}$ is employed as shown in Fig. 5.6(b). The repeated relative displacement Δd is loaded at the locations of bolted joints. The value of the repeated relative displacement Δd is assumed as $\Delta d_{\min} = 0.0\text{mm}$ at minimum and $\Delta d_{\max} = 1.0\text{mm}$ at maximum in the

numerical model. Finer mesh size of $h = 5.0\text{mm}$ is employed near the bolted joints on the side of the detected crack in the web than the standard mesh size.

The initial condition of the crack defined in the file **CRACK.dat** is assumed at the location of the detected crack in the real bridge as $a_0 = 3.5\text{mm}$ as shown in Fig.5.6(b). This initial length of the crack is sufficiently small compared to the size of the entire beam.

The numerical results of the propagation path of the fatigue crack are shown in Fig. 5.7. The transition of the equivalent stress intensity factor range ΔK and the relationship between cycle of the loads N and crack length a are also shown in Figs. 5.8 and 5.9, respectively. The transition of the maximum principal stress according to fatigue crack propagation is shown in Fig. 5.10.

In Fig. 5.7, the crack propagates into the inner part of the web in the initial stage of the crack propagation. After that, the direction of the crack is gradually changed upward in the vertical direction up to the crack length a of 60mm. The results show the same behaviors as those of the fatigue crack detected in the real bridge shown in the Fig. 5.5. It is therefore found that the developed PU-XFEM code can successfully simulate the fatigue crack propagation in the real bridge.

In addition, the path of the crack propagation gradually changes in the outward direction of the web. Finally, it is estimated that the crack propagation is stopped near the bolt location and the crack does not reach the subsidiary fracture, because the equivalent stress intensity factor range ΔK is drastically decreased below the threshold stress intensity factor range ΔK_{th} as shown in Fig. 5.9.

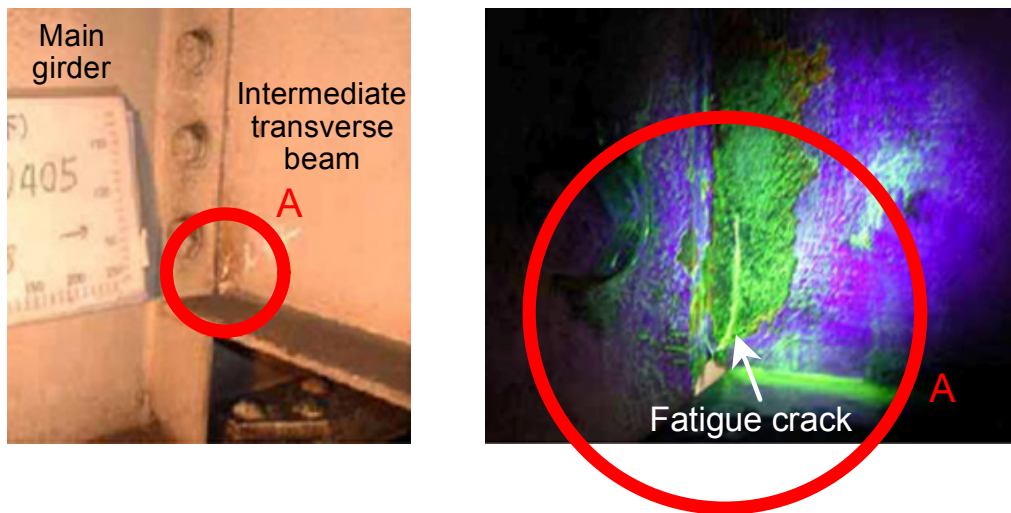
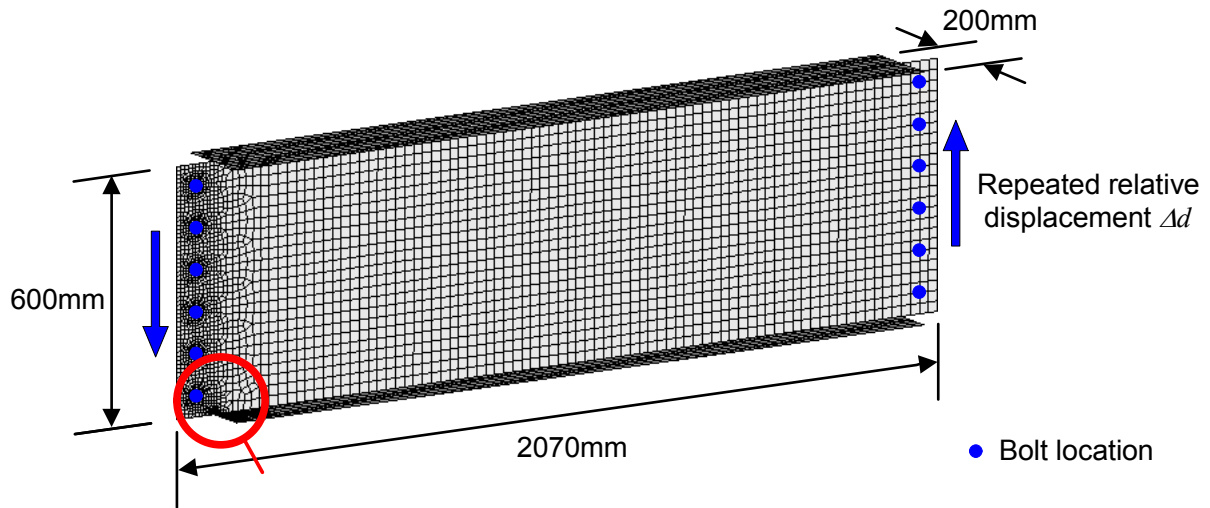
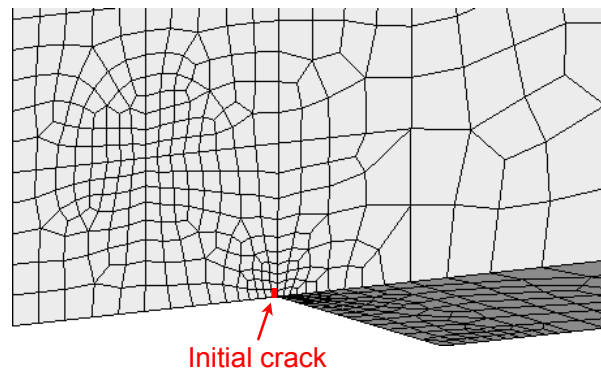


Figure 5.5 Fatigue crack detected in weld joint between flange and web in the intermediate transverse beam of I-girder bridges

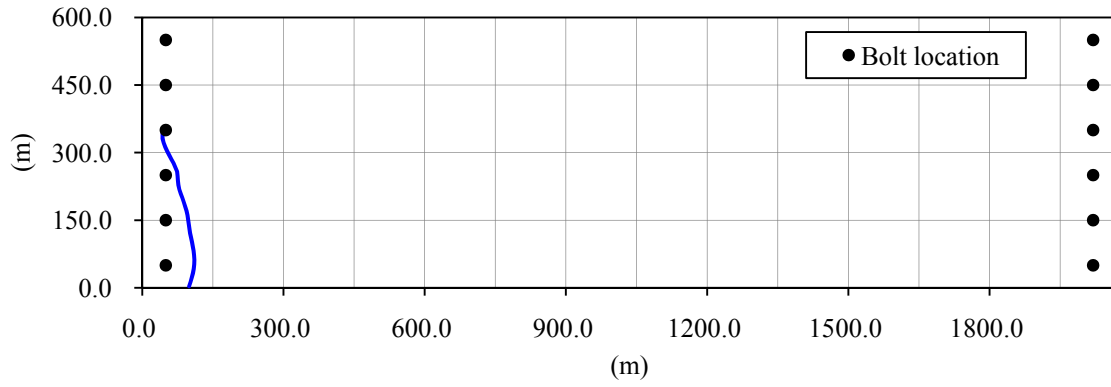


(a) Entire model and detected fatigue crack location “A”



(b) Detail around “A” and initial crack

Figure 5.6 Numerical model of the intermediate transverse beam of I-girder bridge (Model-1)



(a) Entire web

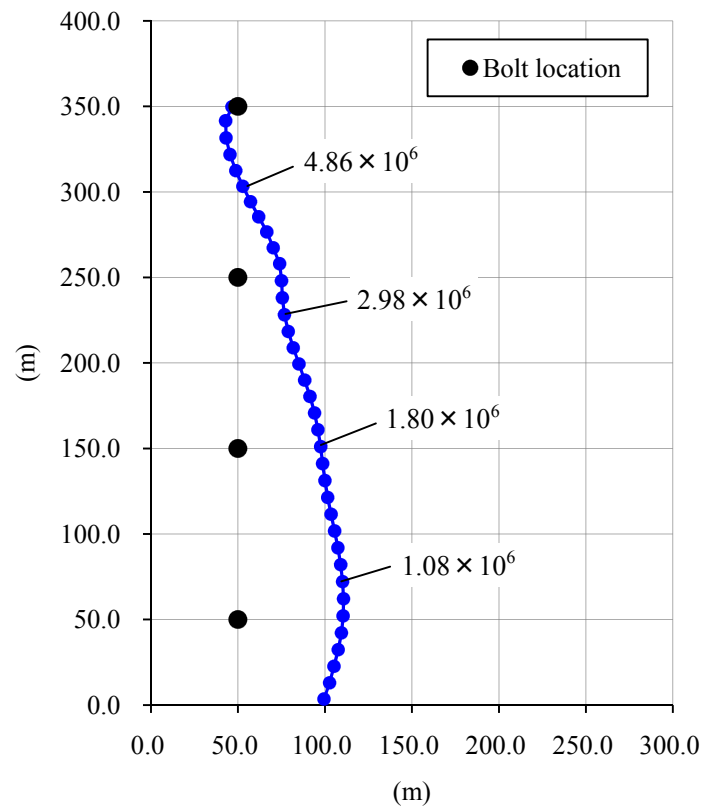

(b) Detail near crack propagation path
(figures in the graph show the cycle of loads N)

Figure 5.7 Numerical results of the crack propagation path (Model-1)

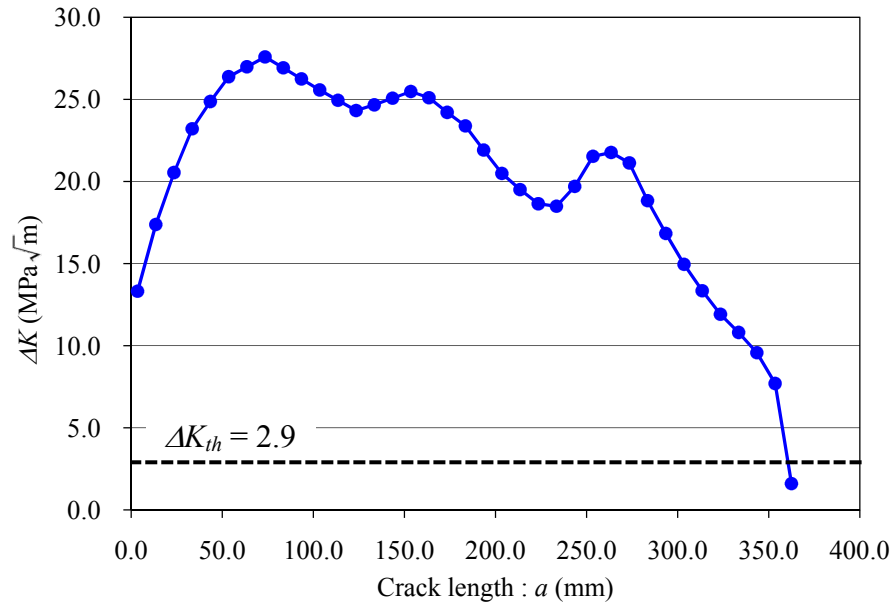


Figure 5.8 Transition of the equivalent stress intensity factor range ΔK (Model-1)

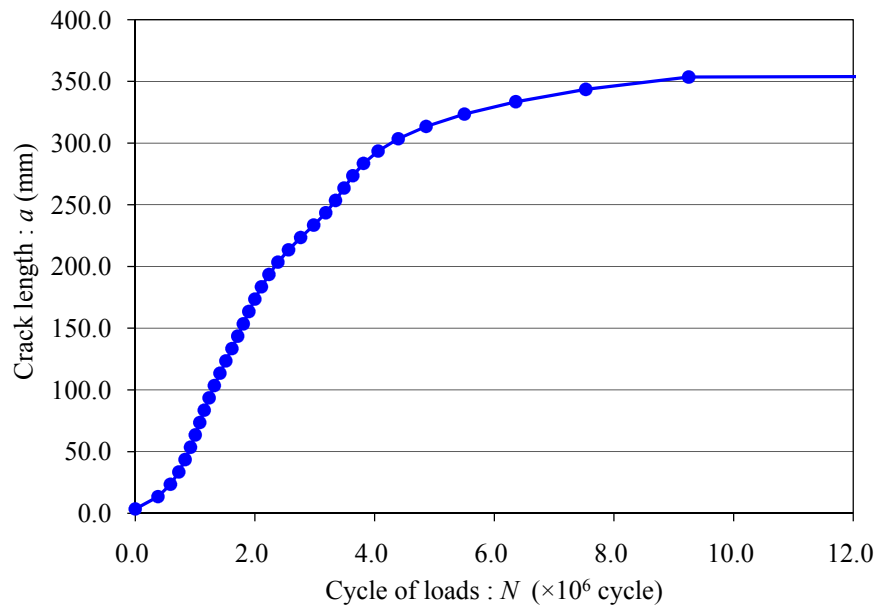


Figure 5.9 Relationship between cycle of loads N and crack length a (Model-1)

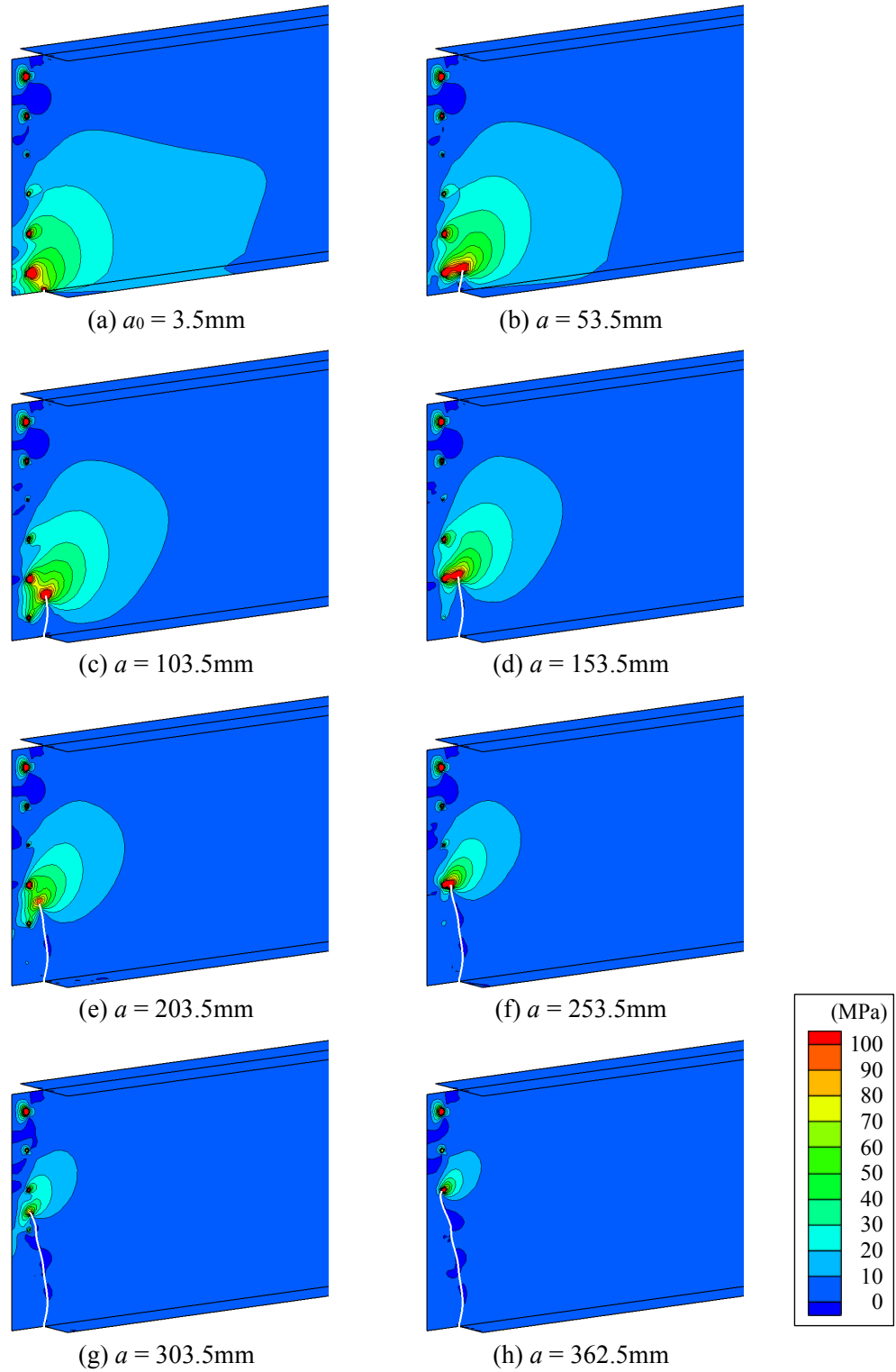


Figure 5.10 Transition of the maximum principal stress at $\Delta d_{\max} = 1.0\text{mm}$ according to fatigue crack propagation (Model-1)

5.4.2 Application to Fatigue Test for Orthotropic Steel Deck Specimen Using Bulb Rib

Recently, a lot of fatigue damage related cases are reported in the weld joints between bulb ribs and lateral ribs in the orthotropic steel deck bridge (Tabata *et al.* 2007).

Most of these cracks were detected at the lower edge of the weld joints in near the bottom of the slits of the lateral ribs as shown in Fig. 5.11. However, these cracks propagated in the various directions in the lateral ribs. Therefore, the causes and mechanisms of the behaviors of these cracks have not been clarified due to its complexity.

In this section, it is performed the numerical simulation to evaluate the fatigue crack behavior in the fatigue tests of the orthotropic steel deck specimen by Sakiya *et al.* (2009), which has the same structural detail as the real bridge, as shown in Fig. 5.12.

The test specimen is composed of three lateral ribs and five bulb ribs, etc. The fatigue crack was detected in the weld joint between the central lateral rib and the leftmost bulb rib (see Figs. 5.14(b) and (c)). The fatigue crack propagation path in the specimen is shown in Fig. 5.13. The propagated length of the crack is found to be $a = 33.1$ mm in the fatigue test. It is noted that Fig. 5.13 is a photograph of the cut piece of the test specimen including the stop hole made after fatigue test. It is also noted that the holes near the stop hole are those for bolt connection with the angle steel reinforcement after the stop hole was made for the further fatigue test. That is, two types of holes do not give an effect on the fatigue tests and numerical simulations because these holes were made for other objectives.

The numerical model used in this section (Model-2) and its meshes are shown in Fig. 5.14. The standard mesh size is employed as $h = 20.0$ mm. However, the minimum mesh size $h = 3.0$ mm is employed near the “B” of the weld joint between the central lateral rib and the leftmost bulb rib, where the fatigue crack was detected as shown in Figs. 5.14(b) and (c). The minimum mesh size of 3.0 mm is about one tenth of the eventual length of the crack observed in the fatigue test.

The locations of the loading are also shown in Fig. 5.14(a). Although the loading is changed stepwise (see Fig. 5.17), the loading in the numerical simulation is determined corresponding to the length of the crack in the fatigue test. As the boundary conditions,

the vertical and horizontal displacements of the parts of the sole plates just under the main girders are assumed to be fixed and free, respectively.

Initial condition of the crack defined in the file **CRACK.dat** is assumed as $a_0 = 9.3\text{mm}$, which was detected at the cycle of the loads $N = 1.0 \times 10^5$ in the initial stage of the fatigue test, as shown in Fig. 5.14(c).

The numerical results of the propagation path of the fatigue crack are shown in Fig. 5.15 with comparison to the results of the fatigue test. The numerical results show a good agreement with the results of fatigue test, though we evaluate the fatigue crack in the local part in the complex structure of the orthotropic steel deck specimen, which has same structural detail as the real bridge. Thus the developed PU-XFEM code could accurately simulate the propagation path of the fatigue crack in the case of a real structure.

The transition of the equivalent stress intensity factor range ΔK and the relationship between cycle of the loads N and crack length a are shown in Figs. 5.16 and 5.17, respectively. The propagation rate in the numerical results is evaluated on the safe side as twice fast as in the results of the fatigue test. This difference is caused that the design curve for the propagation rate is gained in the conditions of the weld joints under the high tensile residual stress. According to above discussion, it is concluded that the developed PU-XFEM analysis code has been sufficiently effective for the evaluation of the fatigue propagation behavior.

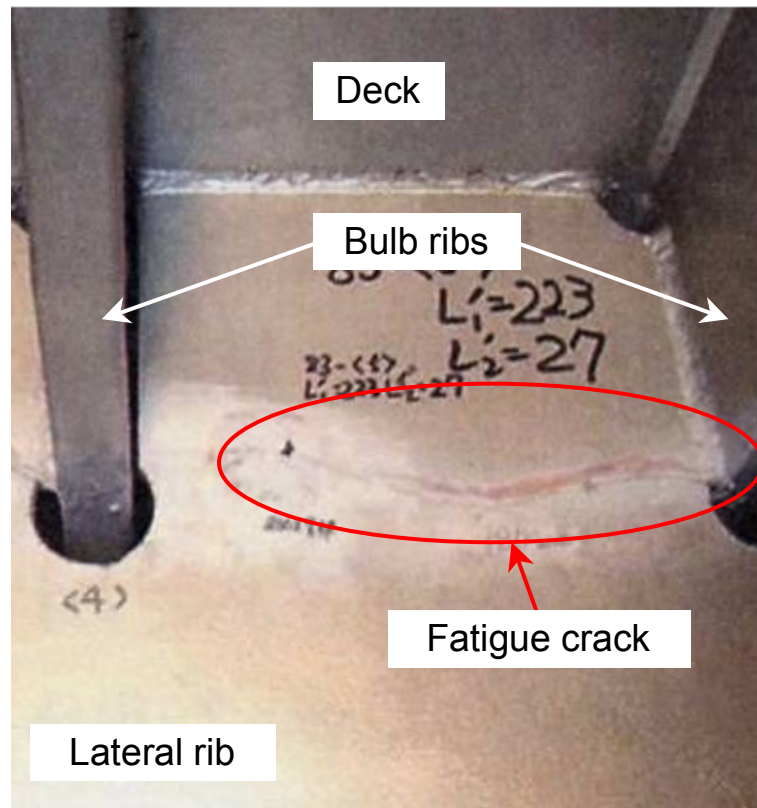


Figure 5.11 Fatigue crack through thickness at the weld joint between bulb rib and lateral rib in the orthotropic steel deck

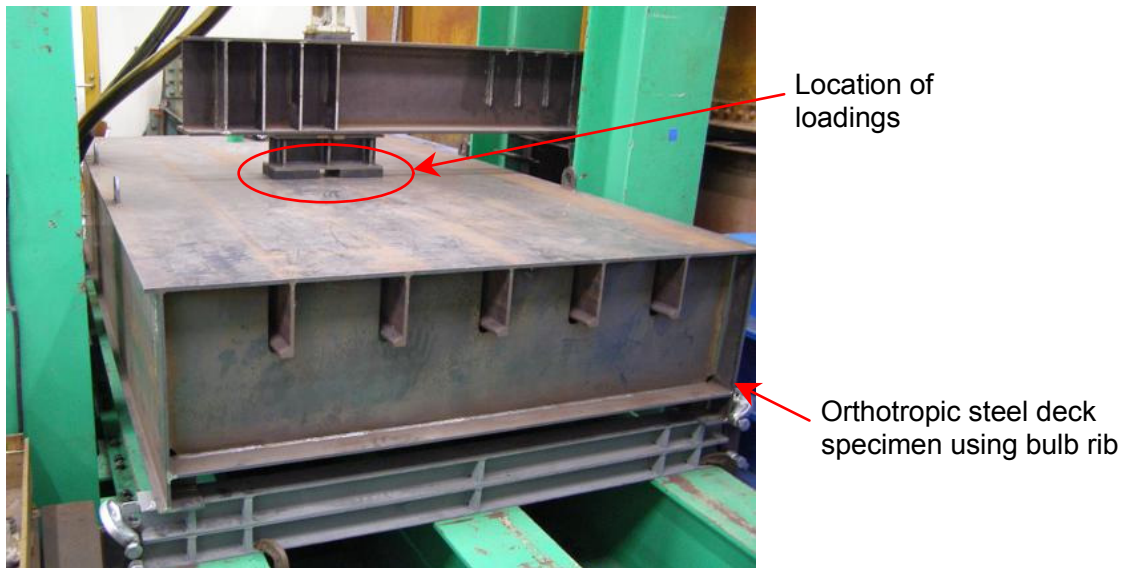


Figure 5.12 Orthotropic steel deck specimen which has the same structural detail as real bridge

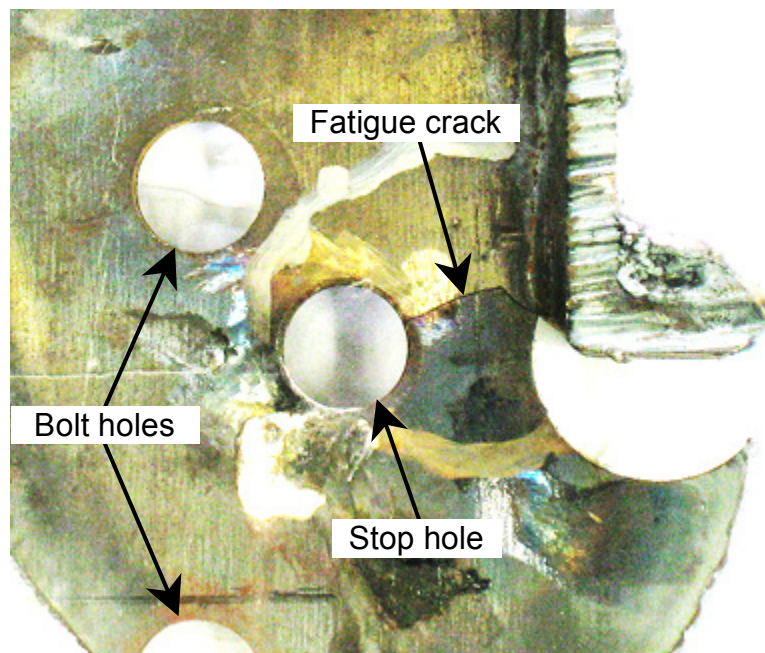
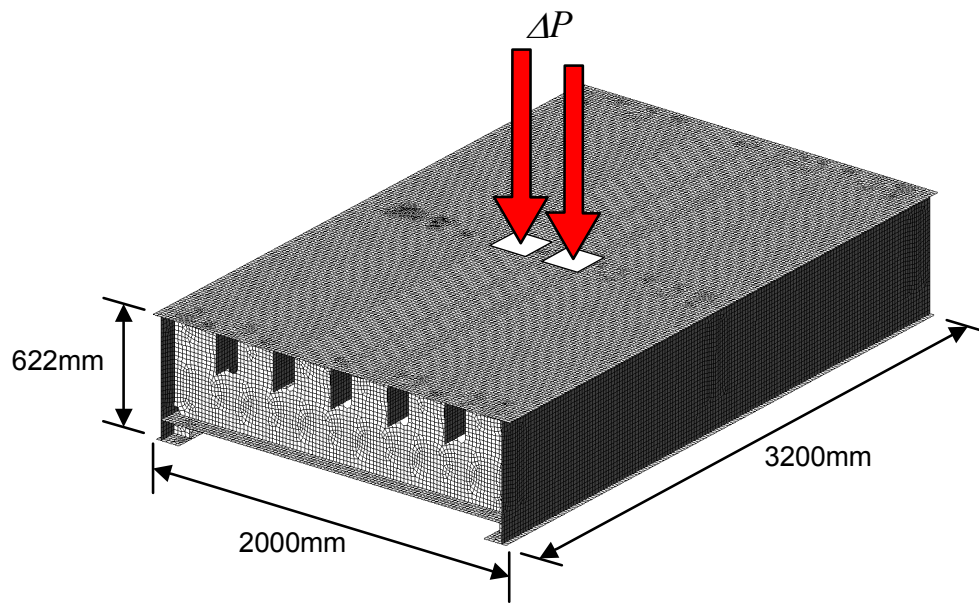
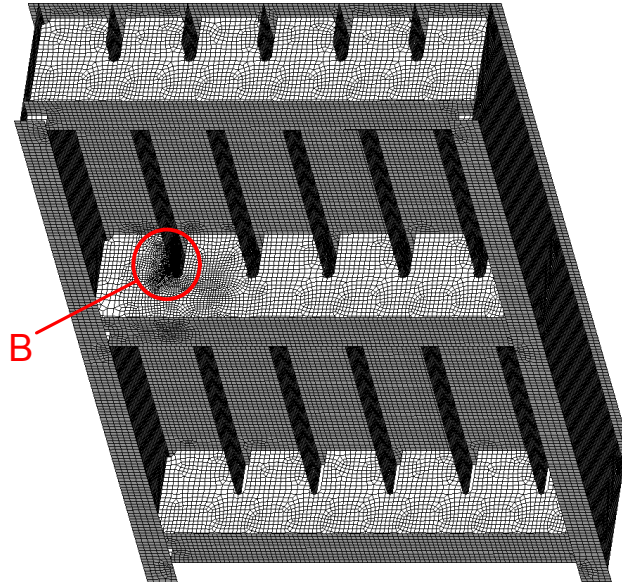


Figure 5.13 Fatigue crack propagation path detected in fatigue test

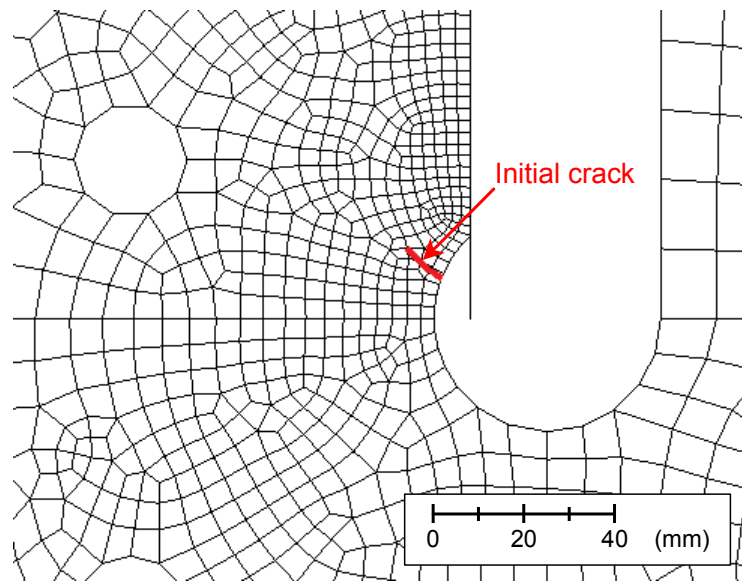


(a) Upper view of the entire numerical model and location of loading



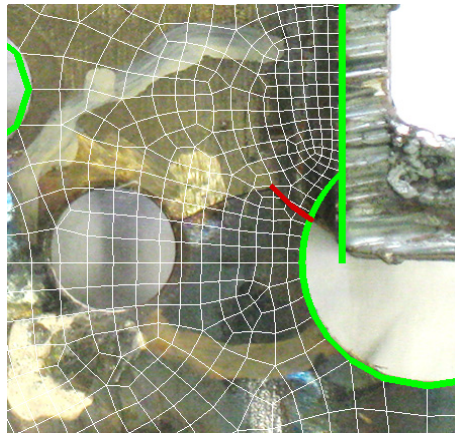
(b) Lower view of the entire numerical model and fatigue crack location “B”

Figure 5.14 Numerical model of the orthotropic steel deck specimen (Model-2)

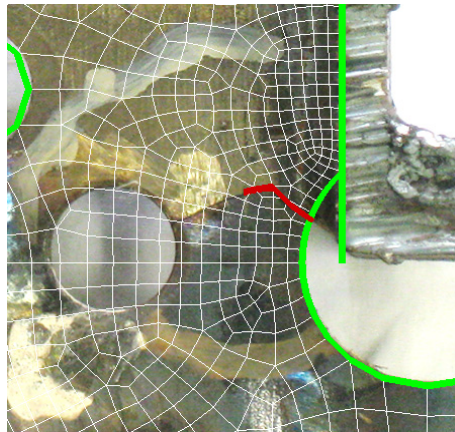


(c) Detail around “B” and initial crack

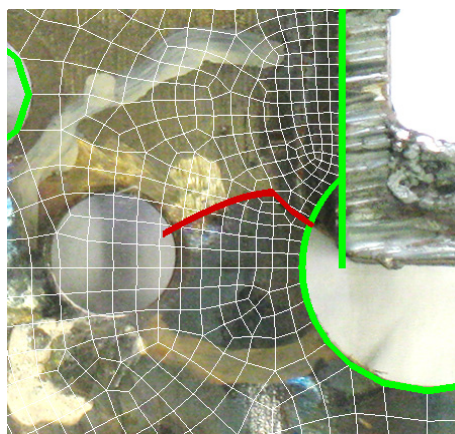
Figure 5.14 Numerical model of the orthotropic steel deck specimen (Model-2)



(a) $a_0 = 9.30\text{mm}$



(b) $a = 15.15\text{mm}$



(c) $a = 33.50\text{mm}$

Figure 5.15 Comparison of the results of fatigue propagation path in the numerical simulation and fatigue test (Model-2)

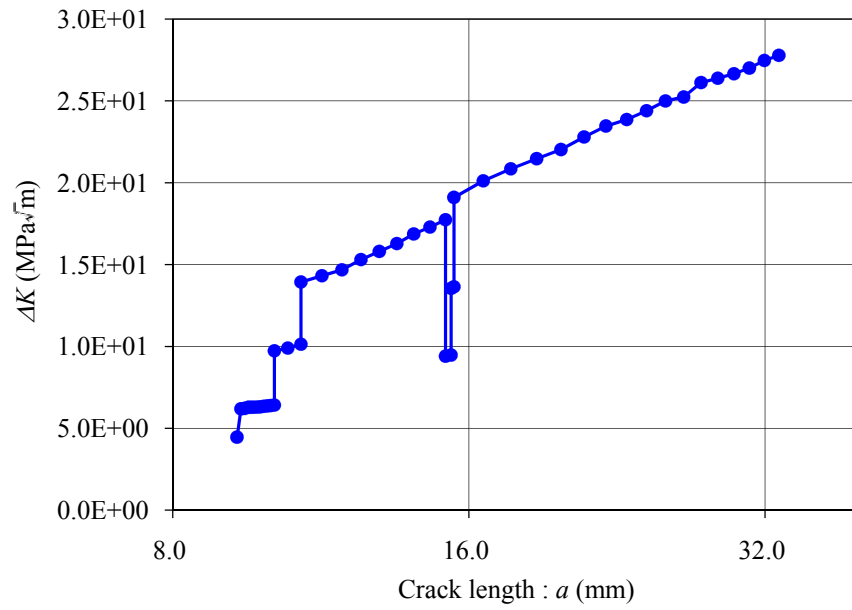


Figure 5.16 Transition of the equivalent stress intensity factor range ΔK (Model-2)

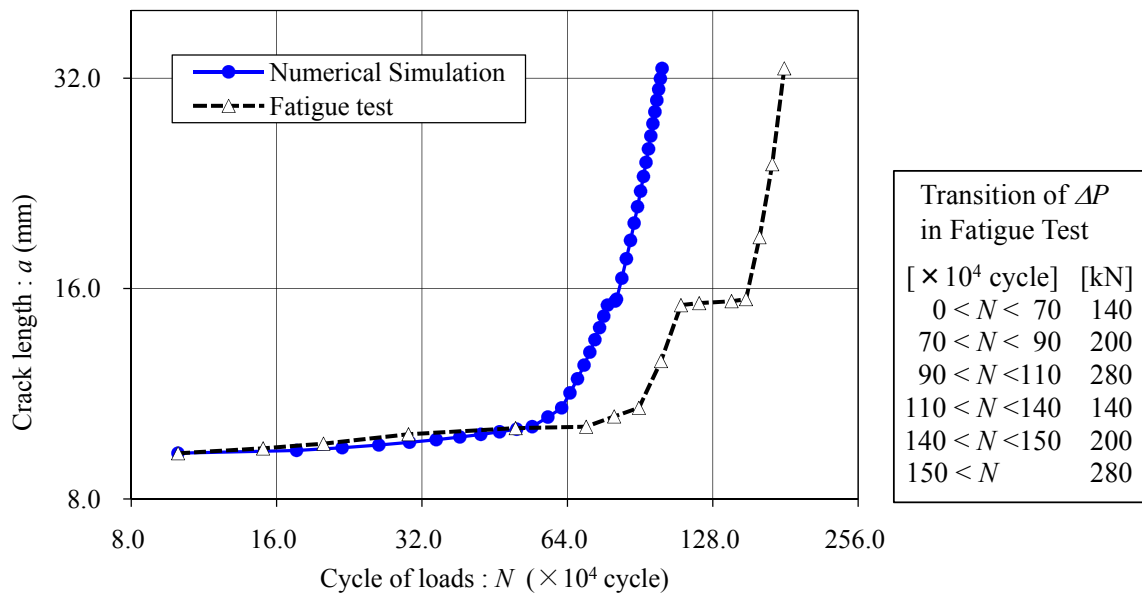


Figure 5.17 Relationship between cycle of loads N and crack length a , and transition of loading conditions (Model-2)

5.5 CONCLUDING REMARKS

In this chapter, it is developed a fatigue crack simulation code by implementing the PU-XFEM in order to evaluate the behavior of a fatigue crack in the local part of large-scale civil engineering structures. The developed code is based on the general-purpose FEM analysis software ABAQUS, through the introduction of the following additional programs;

- (a) Program for the implementation of the PU-XFEM approximation into the ABAQUS
- (b) Program for the evaluation of the stress intensity factors, the calculation of the direction and rate of the fatigue crack propagation, and the redefinition of the geometries of the cracks
- (c) Program for the control of the entire analysis

This developed code is applicable to a through thickness crack of a plate in a three dimensional structure modeled by shell elements.

In particular, it is proposed a method in order to increase the nodal degrees of freedom by adopting multiple-nodes. This method is consistent with the original definition of the nodal degrees of freedom in the convention rules provided by the ABAQUS User's Manual.

And then, the performance of the developed PU-XFEM code was evaluated by its application to the fatigue crack propagation in the following real structures.

The fatigue behavior in the intermediate transverse beam of the real I-girder bridge was evaluated. In the numerical result, the crack propagates into the inner part of the web in the initial stage of the crack propagation, and then the direction of the crack was changed upward in the vertical direction. These results show the same trend as those of the fatigue crack detected in the real bridge. In addition, the path of the crack propagation gradually changed in the outward direction of the web. It was estimated that the crack propagation was finally stopped near the bolt location and the crack does not reach the subsidiary fracture due to the decrement of the equivalent stress intensity factor range.

The fatigue behavior in the fatigue tests of the orthotropic steel deck specimen, which has the same structural detail as the real bridge, was evaluated. The numerical results showed good agreement with the results of fatigue test. In particular, the propagation path was possible to be accurately reproduced, though we evaluated the fatigue crack in the local part in the complex three dimensional structures. The propagation rate in the numerical results was evaluated on the safe side as twice fast as that in the results of the fatigue test. The numerical results are however found to be useful, because the adopted design curve for the propagation rate is based on the severe condition of the weld joints under the high residual stress.

It is concluded that the developed PU-XFEM analysis code is effective for the quantitative evaluation on the path and rate of the fatigue propagation including the estimation of termination of crack propagation. The achievement of the study in this chapter therefore can be used as a basis for developing numerical simulation tools for fatigue cracks in three dimensional large-scale civil engineering structures.

Chapter 6

Conclusions

In this thesis, the essential problem of the blending elements in the XFEM was solved by the reformulation of the XFEM approximation. And then, the fatigue crack simulation code was developed in order to apply to the actual use for real structures. The concluding remarks of the study in this thesis were summarized in this final chapter.

In Chapter 3, the influence of the blending elements relating the enrichment near crack tip in the application of the standard XFEM to the two dimensional linear fracture mechanics was evaluated by the verification using basic numerical models. In the evaluation on the reproducibility of the stress fields, large errors were found inside the domain of the blending elements. In the evaluation on the numerical accuracies of the fracture mechanics parameters, the lack of accuracies was caused when the integration domain of the J -integral is overlapped with the domain of the blending elements. It was also found that the numerical results can be evaluated with good accuracies under the condition that the integration domain of the J -integral is included in the domain of the fully enriched elements. Based on the condition for good accuracies in the evaluations on the fracture mechanics parameters, the effective modeling of the crack tip enrichment and J -integral as the method to avoid the influence of the blending elements in the framework of the standard XFEM were proposed. Although this proposal can be easily implemented, the numerical results showed stable and sufficient accuracies in the evaluation of the fracture mechanics parameters comparing with the results in the

existing method. Therefore, it was found out that the proposed modeling is effective in the framework of the standard XFEM approximation.

However, this proposed modeling is not a foundational solution of the problem of the blending elements. That is, any problems except the numerical accuracy of the fracture mechanics parameters cannot be improved by the proposal.

In Chapter 4, in order to foundationally solve the problem of the blending elements, we have reformulated an approximation of the XFEM based on the concept of the PUFEM approximation, which assures the approximation accuracy, as the ‘PU-XFEM’. The PU-XFEM approximation $\mathbf{u}_{\text{ap}}(\mathbf{x})$ has the form of

$$\mathbf{u}_{\text{ap}}(\mathbf{x}) = \varphi_0(\mathbf{x})\mathbf{v}_0^{\text{ap}}(\mathbf{x}) + \varphi_1(\mathbf{x})\mathbf{v}_1^{\text{ap}}(\mathbf{x})$$

where the sets of $\varphi_0(\mathbf{x})$ and $\mathbf{v}_0^{\text{ap}}(\mathbf{x})$, and $\varphi_1(\mathbf{x})$ and $\mathbf{v}_1^{\text{ap}}(\mathbf{x})$ are the PU and the approximate functions, relating to the classical finite element approximation and the enrichment approximation including the *priori* knowledge of the solution, respectively. The PU $\varphi_1(\mathbf{x})$ relating to enrichment is active only in the local area. The blending elements have been redefined as the elements in the common part of the supports of the PU $\varphi_0(\mathbf{x})$ and $\varphi_1(\mathbf{x})$ in the PU-XFEM approximation. In the evaluation of the approximation accuracies for a one dimensional problem, it is shown that the lack of convergence rate in the standard XFEM has been overcome in the proposed PU-XFEM. In the application to the linear fracture mechanics, the following features of the PU-XFEM have been obtained in comparison with the standard XFEM.

- (a) The significant improvement on the both of accuracy and convergence rate is found in the convergence study using the relative L_2 norm on numerical errors
- (b) The enrichment near the crack tip is dominated in the enriched domain. It means that the numerical results have a possibility of the exact solution.
- (c) The stress field around the crack tip is exactly reproduced in the enriched domain.
- (d) The numerical results of the linear fracture mechanics parameters have shown a significant improvement of the accuracy in the cases of coarse mesh and an assurance of the path independency of the J -integral.

According to the above-mentioned fact, it was found out that the problem of the blending elements has been solved by the PU-XFEM.

In Chapter 5, in order to evaluate a fatigue crack behavior in the local part of large scale structures, the fatigue crack simulation code is developed through implementation of the PU-XFEM approximation. The developed code is based on the well known general-purpose FEM analysis software ABAQUS and the following additional programs: (a) the program for the implementation of the PU-XFEM approximation, (b) the program for the calculation of the J -integral and the evaluation of fatigue crack behavior, and (c) the program for the control of the entire analysis. This developed code is applicable to a through thickness crack of a plate in a three dimensional structure modeled by shell elements. In particular, we proposed the new method to increase the nodal degrees of freedom by adopting multiple-nodes. This method is consistent with the original definition of the nodal degrees of freedom in the convention rules in the ABAQUS User's Manual. The performance of the developed PU-XFEM code was evaluated through its applications to the fatigue crack propagation in the following two real structures. In the application to the fatigue crack in the intermediate transverse beam of the real I-girder bridge, the numerical results show the same tendency of the fatigue crack detected in the real bridge. In addition, it was estimated that the crack propagation is finally terminated near the bolt location and the crack does not reach the subsidiary fracture. In the application to the fatigue crack in the fatigue tests of the orthotropic steel deck specimen using bulb rib simulating the real bridge structure, the numerical results showed good agreement with the results of fatigue test, though the behavior of the fatigue crack in the local part in the complex three dimensional structures was evaluated. It is concluded that the developed PU-XFEM analysis code is effective for the quantitative evaluation on the behavior of the fatigue crack propagation including the estimation of its termination.

Based on the aforementioned facts, we conclude that the proposed PU-XFEM is a correct formulation of the XFEM to obtain its essential features, by solving the problem of the blending elements. Therefore, the formulation of the PU-XFEM can be a new foundation of the XFEM instead of the standard XFEM. In addition, the implementation of the PU-XFEM in fracture mechanics is possible to be a basis for developing numerical simulation tool for actual fatigue crack behavior in three dimensional civil

engineering structures, in order to clarify the mechanism of fatigue crack propagation and to rationalize the maintenance for aged steel bridges.

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Associated Publications

Aword

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