

## On Numerical Semigroups of Genus 9

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### §1. Introduction.

Let  $\mathbb{N}_0$  be the additive semigroup of non-negative integers. A subsemigroup  $H$  of  $\mathbb{N}_0$  is called a *numerical semigroup* if the complement  $\mathbb{N}_0 \setminus H$  of  $H$  in  $\mathbb{N}_0$  is a finite set. The cardinality  $g(H)$  of the set  $\mathbb{N}_0 \setminus H$  is called the *genus* of  $H$ . In this paper we are interested in numerical semigroups of genus 9. For a non-singular complete irreducible curve  $C$  over an algebraically closed field  $k$  of characteristic 0 (which is called a *curve* in this paper) and its point  $P$  we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists \text{ a rational function } f \text{ on } C \text{ with } (f)_\infty = nP\}.$$

A numerical semigroup is *Weierstrass* if there exists a curve  $C$  with its point  $P$  such that  $H(P) = H$ . We have the following results:

**Fact 1.** *Every numerical semigroup of genus  $g \leq 8$  is Weierstrass.* (See Lax [10], Komeda [4] and Komeda-Ohbuchi [8] for the case  $g = 4$ ,  $5 \leq g \leq 7$  and  $g = 8$  respectively.)

We note that for any  $g \geq 16$  there exists a non-Weierstrass numerical semigroup of genus  $g$  (see Buchweitz [1].) A numerical semigroup  $H$  is *primitive* if the largest positive integer not in  $H$  is less than twice the least positive integer in  $H$ . Then we know the following fact:

**Fact 2.** *Every primitive numerical semigroup of genus 9 is Weierstrass.* (See Komeda [6].)

We want to study non-primitive numerical semigroups of genus 9.

### §2. Non-primitive numerical semigroups of genus 9.

An *n-semigroup*, i.e., a numerical semigroup in which the least positive integer is  $n$ . When  $n$  is lower, we have the following result:

**Fact 3.** *For  $1 \leq n \leq 5$  every  $n$ -semigroup is Weierstrass.* (See Maclachlan [11], Komeda [2] and [3] for the case  $n = 3$ ,  $n = 4$  and  $n = 5$  respectively.)

Moreover, we have the following facts for two kinds of numerical semigroups:

**Fact 4.** *Every  $g$ -semigroup of genus  $g$  is Weierstrass.* (See Pinkham [12].)

**Fact 5.** *There is a unique non-primitive  $(g - 1)$ -semigroup of genus  $g$ , which is Weierstrass. (See Komeda [5].)*

Therefore, we are interested in non-primitive  $n$ -semigroups of genus 9 for  $n = 6, 7$ .

### §3. Non-primitive 6-semigroups of genus 9.

**Definition 1.** A numerical semigroup  $H$  with  $\#M(H) = m$  is said to be of *toric type* if there are a positive integer  $l$ , monomials  $g_j$ 's ( $j = 1, \dots, l + m - 1$ ) in  $k[X_1, \dots, X_m]$  and a saturated subsemigroup  $S$  of  $\mathbb{Z}^l$  generated by  $b_1, \dots, b_{l+m-1}$  which generates  $\mathbb{Z}^l$  as a group such that

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[X_1, \dots, X_m] \\ \downarrow & \square & \downarrow \\ \text{Spec } k[S] & \hookrightarrow & \text{Spec } k[Y_1, \dots, Y_{l+m-1}] \end{array}$$

where the horizontal maps are the embeddings through the generators and the right vertical map is induced by the  $k$ -algebra morphism from  $k[Y_1, \dots, Y_{l+m-1}]$  to  $k[X_1, \dots, X_m]$  sending  $Y_j$  to  $g_j$ .

**Definition 2.** A  $2m$ -semigroup  $H$  is of *double covering type* if there is a double covering  $\pi : C \longrightarrow C_0$  of curves with ramification point  $P$  such that  $H(P) = H$ .

We can show the following:

**Theorem 1.** *Every non-primitive 6-semigroup of genus 9 is either of toric type or double covering type, hence Weierstrass. (See Komeda [9].)*

### §4. Non-primitive 7-semigroups of genus 9.

We know that every non-primitive 7-semigroup of genus 9 is generated by 5 or 6 elements. We list up all non-primitive 7-semigroups of genus 9.

**Remark 2.** A non-primitive 7-semigroup of genus 9 generated by 5 elements is one of the following:

$$\langle 7, 9, 10, 11, 13 \rangle, \quad \langle 7, 9, 10, 11, 12 \rangle, \quad \langle 7, 9, 10, 12, 13 \rangle, \quad \langle 7, 8, 11, 12, 13 \rangle.$$

**Theorem 3.** *Every non-primitive 7-semigroup of genus 9 generated by 5 elements is of toric type, hence Weierstrass. (See Komeda [7].)*

**Remark 4.** A non-primitive 7-semigroup of genus 9 generated by 6 elements is one of the following:

$$\langle 7, 9, 11, 12, 13, 17 \rangle, \quad \langle 7, 9, 11, 12, 13, 15 \rangle, \quad \langle 7, 10, 11, 12, 13, 16 \rangle, \quad \langle 7, 10, 11, 12, 13, 15 \rangle.$$

First, we shall show that  $\langle 7, 9, 11, 12, 13, 17 \rangle$  is of toric type. We set  $a_1 = 7$ ,  $a_2 = 9$ ,  $a_3 = 11$ ,  $a_4 = 12$ ,  $a_5 = 13$ ,  $a_6 = 17$ . Then we have a generating system of

relations among  $a_1, a_2, a_3, a_4, a_5$  and  $a_6$  as follows:

$$\begin{aligned} 3a_1 &= a_2 + a_4, 2a_2 = a_1 + a_3, 2a_3 = a_2 + a_5, 2a_4 = a_1 + a_6, 2a_5 = a_2 + a_6, \\ 2a_6 &= a_2 + a_4 + a_5, a_1 + a_5 = a_2 + a_3, a_1 + a_6 = a_3 + a_5, 2a_1 + a_3 = a_4 + a_5, \\ a_3 + a_6 &= a_1 + a_2 + a_4, a_5 + a_6 = a_1 + a_3 + a_4, 2a_1 + a_2 = a_3 + a_4, \\ 2a_1 + a_4 &= a_2 + a_6, a_4 + a_6 = a_1 + a_2 + a_5. \end{aligned}$$

We set

$$\begin{aligned} \mathbf{b}_i &= \mathbf{e}_i \in \mathbb{Z}^6, i = 1, \dots, 6, \mathbf{b}_7 = (1, 1, -1, 0, 0, 0), \mathbf{b}_8 = (1, 0, -1, 1, 0, 0), \\ \mathbf{b}_9 &= (1, 0, 1, -1, 1, 0), \mathbf{b}_{10} = (-1, 0, -1, 1, 0, 1), \mathbf{b}_{11} = (0, 0, 2, -1, 0, 0). \end{aligned}$$

Let  $S$  be the subsemigroup of  $\mathbb{Z}^6$  generated by  $\mathbf{b}_1, \dots, \mathbf{b}_{11}$ . Then  $\text{Spec } k[S]$  is a 6-dimensional affine toric variety. We have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[X_1, \dots, X_6] = \mathbb{A}^6 \\ \downarrow & \square & \downarrow^{\eta} \\ \text{Spec } k[S] & \hookrightarrow & \text{Spec } k[Y_1, \dots, Y_{11}] = \mathbb{A}^{11} \end{array}$$

where  $\eta : k[Y_1, \dots, Y_{11}] \longrightarrow k[X_1, \dots, X_6]$  is the  $k$ -algebra homomorphism sending  $Y_i$  to  $\xi_i$  for  $1 \leq i \leq 11$  where

$$\begin{aligned} \xi_1 &= X_1, \xi_2 = X_6, \xi_3 = X_3, \xi_4 = X_5, \xi_5 = X_1, \xi_6 = X_6, \\ \xi_7 &= X_5, \xi_8 = X_2, \xi_9 = X_4, \xi_{10} = X_4, \xi_{11} = X_2. \end{aligned}$$

Hence, the numerical semigroup  $\langle 7, 9, 11, 12, 13, 17 \rangle$  is Weierstrass.

Second, we shall show that  $\langle 7, 9, 11, 12, 13, 15 \rangle$  is of toric type. We set  $a_1 = 7, a_2 = 9, a_3 = 11, a_4 = 12, a_5 = 13, a_6 = 15$ . Then we have a generating system of relations among  $a_1, a_2, a_3, a_4, a_5$  and  $a_6$  as follows:

$$\begin{aligned} 3a_1 &= a_2 + a_4, 2a_2 = a_1 + a_3, 2a_3 = a_1 + a_6, 2a_4 = a_3 + a_5, 2a_5 = a_3 + a_6, \\ 2a_6 &= a_1 + a_3 + a_4, a_1 + a_5 = a_2 + a_3, a_1 + a_6 = a_2 + a_5, \\ 2a_1 + a_3 &= a_4 + a_5, a_3 + a_6 = 2a_1 + a_4, a_5 + a_6 = a_1 + a_2 + a_4, \\ 2a_1 + a_2 &= a_3 + a_4, 2a_1 + a_5 = a_4 + a_6, a_2 + a_6 = a_3 + a_5. \end{aligned}$$

We set

$$\begin{aligned} \mathbf{b}_i &= \mathbf{e}_i \in \mathbb{Z}^4, i = 1, \dots, 4, \mathbf{b}_5 = (1, 1, -1, 0), \mathbf{b}_6 = (-1, 1, 1, 0), \\ \mathbf{b}_7 &= (-1, 0, 2, 0), \mathbf{b}_8 = (2, 0, -1, 1), \mathbf{b}_9 = (-1, 2, 0, -1). \end{aligned}$$

Let  $S$  be the subsemigroup of  $\mathbb{Z}^4$  generated by  $\mathbf{b}_1, \dots, \mathbf{b}_9$ . Then  $\text{Spec } k[S]$  is a 4-dimensional affine toric variety. We have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[X_1, \dots, X_6] = \mathbb{A}^6 \\ \downarrow & \square & \downarrow^{a_\eta} \\ \text{Spec } k[S] & \hookrightarrow & \text{Spec } k[Y_1, \dots, Y_9] = \mathbb{A}^9 \end{array}$$

where  $\eta : k[Y_1, \dots, Y_9] \longrightarrow k[X_1, \dots, X_6]$  is the  $k$ -algebra homomorphism sending  $Y_i$  to  $\xi_i$  for  $1 \leq i \leq 9$  where

$$\xi_1 = X_1, \xi_2 = X_5, \xi_3 = X_2, \xi_4 = X_1, \xi_5 = X_3, \xi_6 = X_6, \xi_7 = X_3, \xi_8 = X_4, \xi_9 = X_4.$$

Hence, the numerical semigroup  $\langle 7, 9, 11, 12, 13, 15 \rangle$  is Weierstrass.

Third, we consider the semigroup  $\langle 7, 10, 11, 12, 13, 16 \rangle$ . We set

$$a_1 = 7, a_2 = 10, a_3 = 11, a_4 = 12, a_5 = 13, a_6 = 16.$$

Then we have a generating system of relations among  $a_1, a_2, a_3, a_4, a_5$  and  $a_6$  as follows:

$$3a_1 = a_2 + a_3, 2a_2 = a_1 + a_5, 2a_3 = a_2 + a_4, 2a_4 = a_3 + a_5,$$

$$2a_5 = a_2 + a_6, 2a_6 = a_1 + a_4 + a_5, a_1 + a_6 = a_2 + a_5, a_1 + a_6 = a_3 + a_4,$$

$$2a_1 + a_2 = a_3 + a_5, 2a_1 + a_3 = a_4 + a_5, 2a_1 + a_4 = a_2 + a_6,$$

$$2a_1 + a_5 = a_3 + a_6, a_4 + a_6 = a_1 + a_2 + a_3, a_5 + a_6 = a_1 + a_2 + a_4.$$

Let  $S$  be the subsemigroup of  $\mathbb{Z}^4$  generated by

$$\mathbf{b}_i = \mathbf{e}_i \in \mathbb{Z}^4, i = 1, \dots, 4, \mathbf{b}_5 = (2, -1, 0, 0), \mathbf{b}_6 = (3, -2, 0, 0),$$

$$\mathbf{b}_7 = (-1, 2, 1, 0), \mathbf{b}_8 = (-2, 2, 1, 1), \mathbf{b}_9 = (4, -3, -1, 0).$$

Then  $\text{Spec } k[S]$  is a 4-dimensional non-normal variety such that we have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[X_1, \dots, X_6] = \mathbb{A}^6 \\ \downarrow & \square & \downarrow^{a_\eta} \\ \text{Spec } k[S] & \hookrightarrow & \text{Spec } k[Y_1, \dots, Y_9] = \mathbb{A}^9 \end{array}$$

Here  $\eta : k[Y_1, \dots, Y_9] \longrightarrow k[X_1, \dots, X_6]$  is the  $k$ -algebra homomorphism sending  $Y_i$  to  $\xi_i$  for  $1 \leq i \leq 9$  where

$$\xi_1 = X_2, \xi_2 = X_1, \xi_3 = X_1, \xi_4 = X_3, \xi_5 = X_5, \xi_6 = X_6, \xi_7 = X_3, \xi_8 = X_4, \xi_9 = X_4.$$

Lastly we investigate the semigroup  $\langle 7, 10, 11, 12, 13, 15 \rangle$ . We set

$$a_1 = 7, a_2 = 10, a_3 = 11, a_4 = 12, a_5 = 13, a_6 = 15.$$

Then we have a generating system of relations among  $a_1, a_2, a_3, a_4, a_5$  and  $a_6$  as follows:

$$\begin{aligned} 3a_1 &= a_2 + a_3, 2a_2 = a_1 + a_5, 2a_3 = a_1 + a_6, 2a_4 = 2a_1 + a_2, \\ 2a_5 &= 2a_1 + a_4, 2a_6 = a_1 + a_3 + a_4, a_1 + a_6 = a_2 + a_4, a_2 + a_6 = a_4 + a_5, \\ 2a_1 + a_2 &= a_3 + a_5, 2a_1 + a_3 = a_4 + a_5, 2a_1 + a_4 = a_3 + a_6, \\ 2a_1 + a_5 &= a_4 + a_6, a_2 + a_5 = a_3 + a_4, a_5 + a_6 = a_1 + a_2 + a_3. \end{aligned}$$

Let  $S$  be the subsemigroup of  $\mathbb{Z}^3$  generated by

$$\begin{aligned} \mathbf{b}_i &= \mathbf{e}_i \in \mathbb{Z}^3, i = 1, \dots, 3, \mathbf{b}_4 = (1, 1, -1), \mathbf{b}_5 = (1, -1, 1), \\ \mathbf{b}_6 &= (2, -2, 1), \mathbf{b}_7 = (-2, 1, 1), \mathbf{b}_8 = (-1, 3, -1). \end{aligned}$$

Then  $\text{Spec } k[S]$  is a 3-dimensional non-normal variety where we have a fiber product

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[X_1, \dots, X_6] = \mathbb{A}^6 \\ \downarrow & \square & \downarrow \circ \eta \\ \text{Spec } k[S] & \hookrightarrow & \text{Spec } k[Y_1, \dots, Y_8] = \mathbb{A}^8 \end{array}$$

Here  $\eta : k[Y_1, \dots, Y_8] \longrightarrow k[X_1, \dots, X_6]$  is the  $k$ -algebra homomorphism sending  $Y_i$  to  $\xi_i$  for  $1 \leq i \leq 8$  where

$$\xi_1 = X_2, \xi_2 = X_4, \xi_3 = X_6, \xi_4 = X_1, \xi_5 = X_5, \xi_6 = X_3, \xi_7 = X_1, \xi_8 = X_3.$$

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