

A Summary of “Sheaves of Structures, Heyting-Valued Structures, and a Generalization of Łoś’s Theorem”

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This is a summary of the author’s doctoral dissertation [Ara21].

Sheaf-theoretic constructions have been used in universal algebra and model theory. For example, we can obtain the product (resp. an ultraproduct) of a family of structures from some sheaf by taking the set of global sections (resp. a stalk). Such a sheaf is an example of **sheaves of structures**. A sheaf of structures naturally gives rise to a **Heyting-valued structure**. The latter is useful to describe the logical behavior of the former. Via this construction, sheaves of structures can be applied to universal algebra and model theory.

On the other hand, it is well-known that the category $\mathbf{Sh}(X)$ of sheaves on a space X and the category $\mathbf{Set}(\mathcal{O}(X))$ of $\mathcal{O}(X)$ -valued sets¹ are categorically equivalent. From the viewpoint of categorical logic, sheaves of structures and Heyting-valued structures are **structures in the toposes** $\mathbf{Sh}(X)$ and $\mathbf{Set}(\mathcal{O}(X))$, respectively. While the categorical semantics in $\mathbf{Sh}(X)$ is well-known as sheaf semantics by Kripke and Joyal, that in $\mathbf{Set}(\mathcal{O}(X))$ has appeared less explicitly in the literature.

In the first half of this dissertation (§2 and §3), we revisit the theory of sheaves of structures and Heyting-valued structures from the categorical point of view. After reviewing preliminaries on sheaves, Heyting-valued sets, and first-order categorical logic, we give a categorical description of basic notions for Heyting-valued structures. In particular, we point out that forcing values of formulas, which are usually defined by induction on the construction of formulas, can be obtained by considering the strict relations classifying the interpretations of the given formulas. We also describe the Heyting-valued structure derived from a sheaf of structures and the relationship between forcing values and Kripke–Joyal semantics.

The second half of this dissertation (§4) is devoted to a generalization of Łoś’s theorem for Heyting-valued structures. We first examine how the filter-

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¹Let $\mathcal{O}(X)$ denote the complete Heyting algebra of open sets of X .

quotient construction for sheaves of structures can be generalized to Heyting-valued structures. We then prove a Łoś-type theorem for filter-quotients by generic filters (**Theorem 4.11**), which extends an analogous theorem of Caicedo [Cai95] for sheaves of structures. We also give a characterization of Heyting-valued structures for which Łoś’s theorem holds with respect to any maximal filter by introducing a variant of the *maximum principle* (**Theorem 4.15**). This characterization generalizes a similar theorem due to Pierobon & Viale [PV20] for *Boolean-valued* structures. Combining our two theorems yields a simplified proof of Ellerman’s ultrastalk theorem [Ell74].

References

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