

# WORD PROBLEMS FOR SEMIGROUPS \*

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Duncan and Gilman introduced the concept “word problems” for finitely generated semigroups in terms of formal languages. In this paper, we show that finitely generated semigroups with a regular [ resp. context-free] word problem has a presentation with regular [ resp. context-free] congruence classes.

## 1. Word problems and semigroups with a presentation with special congruence classes

We recall regular, context-free and one counter languages (see [4]).

**Definition 1** . Let  $X$  be finite alphabets.  $X^*$  is the set of all words over  $X$ .  $X^+$  is the set of all non-empty words over  $X$ . A subset of  $X^*$  is called a **language** over  $X$

**Definition 2** . An automaton is defined as a 5-tuple:

$M = (Q, X, \delta, q_0, F)$  where  $Q$  is a finite set of states,  $q_0$  is the start state,  $X$  is a finite set of the input alphabets,  $\delta : Q \times X_\epsilon \rightarrow Q$  is the transition function,  $X_\epsilon = X \cup \{\epsilon\}$ ,  $F \subset Q$  is the set of accepting states.

For  $q_0$  and  $w = x_1x_2 \cdots x_{n-1}x_n (\in X^*)$ , let  $q = \delta(\cdots \delta(\delta(q_0, x_1), x_2) \cdots), x_n)$ . Then we define  $\delta(q_0, w) = q$ .

The accepted language by the  $M$

$$L(M) = \{w \in \Sigma \mid \delta(q_0, w) = q (q \in F)\}$$

is called a regular language.

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\*This is an abstract and the paper will appear elsewhere.

**Definition 3 .** A Pushdown automaton is defined as a 7-tuple:

$M = (Q, X, \Gamma, \delta, q_0, Z_0, F)$  where  $Q$  is a finite set of states,  $q_0$  is the start state,  $X$  is a finite set of the input alphabets,  $\Gamma$  is a finite set of the stack alphabets,  $Z_0$  is the initial stack symbol,  $\delta : Q \times X_\epsilon \times \Gamma_\epsilon \rightarrow P(Q \times X_\epsilon)$  is the transition function, where  $P(S)$  denotes the power set of  $S$ ,  $X_\epsilon = X \cup \{\epsilon\}$ ,  $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$ .  $F \subset Q$  is the set of accepting states.

For  $q \in Q$ ,  $x \in X_\epsilon, w \in X^*$  and  $Z \in \Gamma_\epsilon, \alpha \in \Gamma^*$ , we define a relation  $\Rightarrow_M$  by  $(q, xw, Z\alpha) \Rightarrow_M (p, w, \beta\alpha')$  if  $(p, \beta) \in \delta((q, x, Z))$ . The transitive closure of the relation  $\Rightarrow_M$  is written as a relation  $\Rightarrow_M^*$ . The accepted language by  $M$

$$L(M) = \{w \in \Sigma \mid (q_0, w, Z_0) \Rightarrow_M^* (q, \epsilon, \epsilon) \ (q \in F)\}$$

is called a context-free language.

**Definition 4 .** A one counter machine is a pushdown automaton with a single stack alphabet. The accepted language by the one counter machine is called a one counter language.

**Definition 5 .** ([7]) A semigroup  $S$  has a presentation with regular [ resp. context-free ] congruence classes if there exists a finite set  $X$  and there exists a surjective homomorphism  $\phi$  of  $X^+$  to  $M$  such that for each word  $w \in X^+$ ,  $\phi^{-1}(\phi(w))$  is a regular [ resp. context-free ] language.

Let  $X$  be finite alphabets. For a word  $w = x_1x_2 \cdots x_{n-1}x_n \in X^*$ ,  $x_nx_{n-1} \cdots x_2x_1$  is called the reverse of  $w$  denoted by  $w^R$ .

**Definition 6 .** ([2]) Let  $S$  be a finitely generated semigroup, there exist finite alphabets  $X$  and there exists a surjective homomorphism  $\phi$  of  $X^+$  to  $S$ .

Then

(1) for each  $w \in X^+$ , the set  $\{w\#v^R \mid \phi(w) = \phi(v)\}$  is called a word problem for the semigroup  $S$ .

(2)  $S$  has a regular [ resp. regular context-free ] word problem if the set  $\{w\#v^R \mid \phi(w) = \phi(v)\}$  is a regular [ resp. context-free ] language over  $X \cup \{\#\}$ .

**Theorem 1 .** *A finitly generated semigroup with a regular [ resp. context-free, one counter] word problem has a presentation with regular [resp. context-free, one counter] congruence classes.*

## 2. Regular word problems and semigroups with a presentation with regular congruence classes

The property of that a semigroup has a regular word problem is very strong. Actually, the following is known.

**Result 1 .** ([3], Proposition 3.1). *A semigroup  $S$  has a regular word problem if only if  $S$  is finite.*

On the other hand, we have

**Definition 7 .** *A semigroup  $S$  is called residually finite if for each pair of elements  $s, t \in S$  there is a homomorphism  $\phi$  of  $S$  such that  $\phi(S)$  is a finite semigroup and  $\phi(s)$  and  $\phi(t)$  are not equal.*

**Result 2 .** ([7], Theorem 5) *If a finitely generated semigroup  $S$  has a presentation with regular congruence classes, then  $S$  is residually finite.*

**Result 3 .** ([3], Proposition 4.2) *If  $S$  is a finitely generated semigroup with a one counter word problem, then there exist finitely many elements  $a_i, b_i, c_i \in S \cup \{\epsilon\}$  such that every element of  $S$  is represented by a word of the form  $a_i b_i^n c_i$  for some  $i$  and some  $n \geq 0$ .*

## 3. Context-free word problems and semigroups with a presentation with context-free congruence classes

**Definition 8 .** *Let  $L$  be a language over  $A$ . The syntactic congruence  $\sigma_L$  on  $A^+$  is defined by  $w\sigma_L w'$  if and only if*

$$\{(x, y) \in A^+ \times A^+ \mid xwy \in L\} = \{(x, y) \in A^+ \times A^+ \mid xw'y \in L\}.$$

*Then the factor semigroup  $A^+/\sigma_L$  is called syntactic semigroup of  $L$  denoted by  $Syn(L)$ .*

**Lemma 1** . ([7]). Let  $S$  be a finitely generated semigroup and  $\phi$  a surjective homomorphism of  $A^+$  to  $S$ .

For an element  $s$  of  $S$ , let  $L = \phi^{-1}(s)$ .

Define a congruence  $\sigma_s = \{(a, b) \in S \times S \mid xay = s \text{ if and if } xby = s \ (x, y \in S^1)\}$  which is called a syntactic congruence on  $S$ .

Then the syntactic semigroup  $A^+/\sigma_L$  of  $L$  is isomorphic to the factor semigroup  $S/\sigma_s$  of  $S$ . ( $S/\sigma_s$  is called the syntactic semigroup of  $S$  at  $s$ .)

**Theorem 2** . A semigroup  $S$  has a presentation with context-free [resp. one counter] congruence classes if and only if for each  $s \in S$ , the syntactic semigroup  $S/\sigma_s$  has context-free [resp. one counter] congruence classes.

**Example 1** . Let  $A = \{a, b\}$  and a context-free language  $L = \{a^n b^n, b^n a^n \mid n \in \mathbb{N}\}$ . Then all of  $\sigma_L$ -classes are  $\overline{ab} = \{a^p b^p \mid p \in \mathbb{N}\}$ ,  $\overline{a^n} = \{a^n\}$ ,  $\overline{b^n} = \{b^n\}$ ,  $\overline{a^{n+1}b} = \{a^{p+n} b^p \mid p \in \mathbb{N}\}$ ,  $\overline{ab^{n+1}} = \{a^q b^{q+n} \mid q \in \mathbb{N}\}$ ,  $\overline{ba} = \{b^p a^p \mid p \in \mathbb{N}\}$ ,  $\overline{ba^{n+1}} = \{b^p a^{p+n} \mid p \in \mathbb{N}\}$ ,  $\overline{b^{n+1}a} = \{b^{q+n} a^q \mid q \in \mathbb{N}\}$ .  $0 = A^* b A^* a A^* b A^* \cup A^* a A^* b A^* a A^*$ . Then

- (1)  $L$  is a one counter language.
- (2)  $\text{Syn}(L)$  has a linear growth function. Actually,  $(|\overline{A^n}| \leq 6n)$ .
- (3)  $\text{Syn}(L)$  have a context-free word problem but does not have a one counter word problem.
- (4)  $\text{Syn}(L)$  has a presentation with one counter congruence classes.

**Example 2** . Let  $A = \{a, b\}$  and  $L = \{a^n b^m a^m b^n \mid m, n \in \mathbb{N}\}$ . Then all of  $\sigma_L$ -classes are  $\overline{a^i} = \{a^i\}$  ( $i \in \mathbb{N}$ ),  $\overline{b^i} = \{b^i\}$  ( $i \in \mathbb{N}$ ),  $\overline{a^i b^j} = \{a^i b^j\}$  ( $i, j \in \mathbb{N}$ ),  $\overline{b^i a^j} = \{b^i a^j\}$  ( $i, j \in \mathbb{N}$ ),  $\overline{a^i b^j a^k} = \{a^i b^j a^k\}$  ( $i, j, k \in \mathbb{N}$ ),  $\overline{b^i a^j b^k} = \{b^i a^j b^k\}$  ( $i, j, k \in \mathbb{N}$ ),  $\overline{a^i b^j a^j b} = \{a^i b^j a^j b\}$  ( $i, j \in \mathbb{N}$ ),  $\overline{ab^j a^j b^i} = \{ab^j a^j b^i\}$  ( $i, j \in \mathbb{N}$ ).  $0 = \cup_{i \neq j} A^* a b^i a^j b A^*$ .

Then

- (1)  $L$  is a context-free language.
- (2)  $\text{Syn}(L)$  does not have a presentation with context-free congruence classes.

**Example 3** . Let  $A = \{a, b, c\}$  and  $L = \{a^n b^n c^n | n \in \mathbb{N}\}$ . Then all of  $\sigma_L$ -classes are  $\{1\}$ ,  $\overline{a^q b^r} = \{a^q b^r\}$ ,  $(q, r \in \mathbb{N})$ ,  $\overline{b^q c^r} = \{b^q c^r\}$ ,  $(q, r \in \mathbb{N})$ ,  $\overline{a^i b^q c^r} = \{a^i b^q c^r | q, r, i \in \mathbb{N}, 2 \leq r \leq q\}$ ,  $\overline{a^p b^q c} = \{a^p b^q c^i | p, q, i \in \mathbb{N}, 2 \leq p \leq q\}$ ,  $\overline{abc} = \{a^n b^n c^n | n \in \mathbb{N}\}$ ,  $0 = X^* b a X^* \cup X^* c b X^* \cup X^* c a X^* \cup X^* a c X^* \cup \{a^p b^q c^r | p, q, r \in \mathbb{N}, \text{either } q \leq p \text{ or } q \leq r\}$ .  
Then

(1)  $\text{Syn}(L)$  has a polynomial growth function. Actually,  $\left(|\overline{A^n}| = \frac{n(n+1)}{2}\right)$ .

Therefore,

(2)  $\text{Syn}(L)$  does not have a one-counter word problem.

(3)  $\text{Syn}(L)$  has a presentation with one counter congruence classes.

Finally we pose the following two open problems.

**Problem 1** . Characterize a finitely generated semigroup  $S$  with a one counter word problem.

**Problem 2** . Characterize a finitely generated semigroup  $S$  which has a presentation with one counter congruence classes.

## References

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