

ON WELL-BEHAVED C*-ALGEBRAS RELATED TO ORDERS

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One of the outstanding problems in the theory of AW*-algebras is the monotone completeness of any AW*-algebras. For AW*-algebras of Type I, the answer is known to be yes (see Kaplansky [3]) but, for general AW*-algebras, this question is still open, although an impressive attack on this problem was made by Christensen and Pedersen [1].

In this note, we should like to make a survey of the development of the problem of monotonicity of AW*-algebras, with an outline of their proofs. This is a joint work with John D.M. Wright [6].

Let us recall that a C*-algebra A is an AW*-algebra if (1) each maximal abelian *-subalgebra of A is generated by its projections and (2) each orthogonal family of projections $\{e_\alpha\}$ in A has a supremum $\sum_A e_\alpha$ in $\text{Proj}(A)$ (the complete lattice of all projections in A).

A natural line of attack on this problem would be to use the second dual A'' or the weak closures of representatives of A on some Hilbert spaces.

Unfortunately, this is too naive. In general, the structure of the complete lattice $\text{Proj}(A)$ is not consistent with that of $\text{Proj}(A'')$ or that of the weak closures, because of the lack of the weak or strong topologies. In fact, if so, A would be a von Neumann algebra.

Let B be an AW^* -algebra and let C be a unital C^* -subalgebra of B . We say that C is normal in B if for every orthogonal family $\{e_\alpha\}$ in $\text{Proj}(C)$ with the supremum $\sum_C e_\alpha$ in $\text{Proj}(C)$, $\sum_C e_\alpha = \sum_B e_\alpha$.

Let A be an AW^* -algebra. Then A sits inside its regular completion $\hat{A}$ [2]. $\hat{A}$ is a monotone complete C^* -algebra (and so an AW^* -algebra) which is, in general, not a von Neumann algebra. We say that A is normal if A is normal in $\hat{A}$. So our first question is this:

Are all AW^* -algebras normal ?

It has been known for ten years that finite AW^* -algebras are normal [7], [2] and [4]. So, when establishing normality, we may confine our attention to properly infinite AW^* -algebras.

Quite recently, we showed that mild restrictions on the centre of an AW^* -algebra are sufficient to force it to be normal. In particular, all AW^* -factors are normal.

Let A be an AW^* -algebra whose centre is locally countably decomposable. Then A must be normal.

Detailed proof will appear in the Journal of the London Mathematical Society (see [6]).

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Let B be a C^* -algebra. A net of increasing projections in $\text{Proj}(B)$, $\{e_j\}_{j \in J}$ with the supremum $\text{LUB}_{\text{Proj}(B)} e_j$ in $\text{Proj}(B)$ is said to be well-behaved if $\text{LUB}_{\text{Proj}(B)} e_j$ is the supremum of $\{e_j\}_{j \in J}$ in the partially ordered set B_h . B is said to be well-behaved if, every such net $\{e_j\}_{j \in J}$ is well-behaved.

Let us begin with the following lemma which plays an important role in proving the theorem.

Lemma 1. ([4]) Let A be an AW^* -algebra. Then the following three conditions are equivalent.

- (1) A is normal;
- (2) A is well-behaved;
- (3) for every increasing net $\{e_j\}_{j \in J}$ in $\text{Proj}(A)$ with the supremum e in $\text{Proj}(A)$, whenever x , in A_h , satisfies $e_j x e_j \geq 0$ for all j , then $ex e \geq 0$.

Remark. (2) and (3) are equivalent even when A is a general unital C^* -algebra.

Outline of the proof (see [4]).

(1) $\nrightarrow$ (2). Suppose that A is normal, then, by a result of Pedersen and Saitô, for every increasing net $\{e_j\}_{j \in J}$ in $\text{Proj}(A)$,

$$\text{LUB}_{\text{Proj}(A)} e_j = \text{LUB}_{\text{Proj}(\hat{A})} e_j.$$

Since $\hat{A}$ is monotone complete and $\{e_j\}_{j \in J}$ is well-behaved in $\hat{A}$, and so it is well-behaved in A as well.

Conversely, suppose that A is well-behaved. Then, for every orthogonal family $\{e_j\}_{j \in J}$ in $\text{Proj}(A)$, the net $\{\sum_{j \in F} e_j \mid F \text{ a non-empty finite subset of } J\}$ is well-behaved. So it follows that

$$\begin{aligned} \sum_A e_j &= \text{LUB}_{\text{Proj}(A)} \{ \sum_{j \in F} e_j \mid F \} \\ &= \text{LUB}_{\text{Proj}(\hat{A})} \{ \sum_{j \in F} e_j \mid F \} \\ &= \sum_{\hat{A}} e_j. \end{aligned}$$

(2) $\nrightarrow$ (3). It is given that $\{e_j\}_{j \in J}$ is an increasing net in $\text{Proj}(A)$ with the supremum $\text{LUB}_{\text{Proj}(A)} e_j$ ($= e$ say). Suppose that $\{e_j\}_{j \in J}$ satisfies (3).

The claim is that $\text{LUB}_{\text{Proj}(A)} e_j = \text{LUB}_{A_h} e_j$. We have only to check that $e_j \leq a$ for all j for some a in A_h implies

$e \leq a$. Suppose that such an a is given as above, then, because $a \geq 0$, it follows that

$$(a + 1/n)^{-1/2} e_j (a + 1/n)^{-1/2} \leq 1$$

for each j and n . Thus we get that

$$\|(a + 1/n)^{-1/2} e_j\| \leq 1$$

for each j and n . This implies that

$$e_j (e - e(a + 1/n)^{-1} e) e_j \geq 0$$

for all j and n . Since $\{e_j\}_{j \in J}$ satisfies (3), it follows that

$$e(e - e(a + 1/n)^{-1} e) e \geq 0$$

and $\|e(a + 1/n)^{-1} e\| \leq 1$ for all n . Thus we conclude that

$$(a + 1/n)^{-1/2} e (a + 1/n)^{-1/2} \leq 1$$

for all n . This implies that $e \leq a + 1/n$ for all n and so $e \leq a$ follows.

Conversely suppose that $\{e_j\}_{j \in J}$ is well-behaved. It is given $x \in A_h$ such that $e_j x e_j \geq 0$ for all j . To prove the claim, we may assume that $e = 1$ (consider it in eAe) and $\|x\| \leq 1$. Since

$$\begin{aligned} (1 + x)(1 - e_j)(1 + x) - (1 - x)(1 - e_j)(1 - x) \\ = 2x(1 - e_j) + 2(1 - e_j)x, \end{aligned}$$

we see that

$$\begin{aligned}
 e_j x e_j - x &= (1 - e_j)x(1 - e_j) - (1 - e_j)x - x(1 - e_j) \\
 &= (1/2)((1 - x)(1 - e_j)(1 - x) \\
 &\quad - (1 + x)(1 - e_j)(1 + x)) + (1 - e_j)x(1 - e_j) \\
 &\leq (1/2)(1 - x)(1 - e_j)(1 - x) + 1 - e_j
 \end{aligned}$$

because $\|x\| \leq 1$ and $x = x^*$. Take $y = (1/2)(|x| + x)$ and $z = (1/2)(|x| - x)$. We see that $x = y - z$, y, z in A_h , $yz = 0$ and z and y are non-negative. Moreover, y and z commute with x . Hence, it follows that

$$ze_j x e_j z - zxz \leq (1/2)z(1 - x)(1 - e_j)(1 - x)z + z(1 - e_j)z.$$

Since $ze_j x e_j z$ is non-negative for all j and $zxz = -z^3$,

we see that

$$z^3 \leq (1/2)z(1 - x)(1 - e_j)(1 - x)z + z(1 - e_j)z$$

for all j . Since $\{e_j\}_{j \in J}$ is well-behaved, this implies that

$$(1/2)z(1 - x)(1 - e_j)(1 - x)z + z(1 - e_j)z \downarrow 0 \text{ in } A_h,$$

and so $z^3 = 0$, that is, $z = 0$. This completes the proof.

Theorem 1. Finite AW*-algebras are normal.

Let $\{e_j\}_{j \in J}$ be any increasing net in $\text{Proj}(A)$ with the supremum e in $\text{Proj}(A)$. We shall show that $\{e_j\}_{j \in J}$ satisfies (3). To do this, we may assume that $e = 1$.

Suppose that x in A_h satisfies that $e_j x e_j \geq 0$ for all j .

If $x = x^+ - x^-$ ($x^+ x^- = 0$, $x^+ \geq 0$ and $x^- \geq 0$) and $x^- \neq 0$,

then there is a non-zero projection q in A and a positive number ϵ such that $x^- \geq \epsilon q$ and $(1 - q)x^+ = x^+$. Set $f_j = e_j \wedge q$ and we have

$$0 \leq f_j e_j x e_j f_j = f_j x f_j = f_j q x q f_j = -f_j x^- q f_j$$

$$\leq -\epsilon f_j q f_j \leq -\epsilon f_j$$

for all j and so $f_j = 0$ for all j , that is, $e_j \wedge q = 0$ for all j .

Note that

$$q = q - e_j \wedge q \sim e_j \vee q - e_j \leq 1 - e_j$$

for all j and A is finite, this implies that $q = 0$, because

$1 - e_j \neq 0$ in $\text{Proj}(A)$. This is a contradiction. Thus

$x^- = 0$, that is $x \geq 0$. This completes the proof.

Now we are in the position to discuss about the properly infinite case. Since, as you see, above proof depends on the finiteness assumption on A , we need to seek another way to establish the normality for properly infinite case.

Before going into the discussions, we need some definitions.

For a given index set β , a family $\{x_j \in A_h \mid j \in \beta\}$ is said to be order summable if

$$\{ \sum_{j \in F} x_j \mid F \text{ a non-empty finite subset of } \beta \}$$

is bounded above in A_h . When such a family is order summable its order sum is defined to be the supremum of the set

$$\{ \|\sum_{j \in F} x_j\| \mid F \text{ a non-empty finite subset of } \beta \}.$$

For a given indexed set β , a family $\{x_j \in A_h \mid j \in \beta\}$ is said to be well-behaved if the set

$$\{ \sum_{j \in F} x_j \mid F \text{ a non-empty finite subset of } \beta \}$$

has a supremum in A_h . When $\{x_j \mid j \in \beta\}$ is a family of orthogonal projections, this definition of well-behaved is consistent with our earlier one.

Let β be a given index set. The algebra A is said to be β -complete if each order summable, β -indexed family of positive elements in A is well-behaved.

It is clear that if A is β -complete and if γ is a set where $\#\gamma \leq \#\beta$, then A is γ -complete. We note further that A is α -complete for a sufficiently large ordinal α , then A is monotone complete, but we omit the details.

The rest of the discussions, we shall suppose that A is a properly infinite AW*-algebra.

We suppose for the moment that, for some infinite ordinal Ω , A has a system of matrix units $\{e_{ij}\}_{0 \leq i, j < \Omega}$ where

$e_{00} \sim 1$ in A .

A transfinite sequence $\{a_j\}_{j<\alpha}$ in $e_{00}Ae_{00}$ is said to be dilatable in A if there exists an orthogonal family of projections $\{p_j\}_{j<\alpha}$ in $(\sum_{j<\alpha} e_{jj})A(\sum_{j<\alpha} e_{jj})$ such that $e_{00}p_j e_{00} = x_j$ for each $j < \alpha$. Let α be an ordinal number. We call $e_{00}Ae_{00}$ α -dilatable if, whenever $\{x_j\}_{j<\alpha}$ is an order summable transfinite sequence of positive elements of $e_{00}Ae_{00}$, with order sum less than 1, then the transfinite sequence is dilatable in A .

The following lemma is a modification of an ingenious argument by Christensen and Pedersen [1].

Lemma 3. Let α be an ordinal with $\alpha \leq \Omega$. Let $e_{00}Ae_{00}$ be α -dilatable in A . Then A is α -complete.

Since $e_{00}Ae_{00}$ is $*$ -isomorphic to A , it suffices to show that $e_{00}Ae_{00}$ is α -complete. The proof is rather long. We shall omit the details. See [6].

We shall need the following lemma which is proved in [1, Lemma 3].

Lemma 4. Let e and p be projections in a unital C^* -algebra B such that $\|epe\| < 1$ and let x be a positive element of B such that $x + epe \leq e$. Let $\{f_{ij}\}_{1 \leq i, j \leq 2}$ be matrix units

for $M_2(C)$. Then there exists a projection q in $B\otimes M_2(C)$ such that q is orthogonal to $p\otimes f_{11}$ and

$$(e\otimes f_{11})q(e\otimes f_{11}) = x\otimes f_{11}.$$

Let $z = (1 - p)(1 - epe)^{-1}x(1 - epe)^{-1}(1 - p)$. Then $z \in A_h$ such that $(1 - p)z(1 - p) = z$, $eze = x$ and $0 \leq z \leq 1$.
Let

$$q = \begin{pmatrix} z & (z-z^2)^{1/2} \\ (z-z^2)^{1/2} & 1 - z \end{pmatrix}$$

via $\{f_{ij}\}_{1 \leq i, j \leq 2}$. Then q satisfies all the requirements.

Lemma 5. Let $\alpha < \Omega$ and let $\alpha + 1$ be the successor ordinal of α . Let $e_{00}Ae_{00}$ be α -dilatable in A . Then $e_{00}Ae_{00}$ is also $(\alpha + 1)$ -dilatable.

In fact, let $\{x_\xi\}_{\xi < \alpha+1}$ be an order summable transfinite sequence of positive elements of $e_{00}Ae_{00}$. Let its order sum be c , where $c < 1$. By hypothesis, there exists a family of orthogonal projections $\{p_\xi\}_{\xi < \alpha}$ in $(\sum_{i < \alpha} e_{ii})A(\sum_{i < \alpha} e_{ii})$ such that $e_{00}p_\xi e_{00} = x_\xi$ for each $\xi < \alpha$. By Lemma 3, A is α -complete and so

$$\sum_{i < \alpha} p_i = \text{LUB}_{A_h} \{ \sum_{i \in F} p_i \mid F \text{ a non-empty finite subset of } \alpha \}$$

Thus

$$e_{00}(\sum_{i<\alpha} p_i)e_{00} = \text{LUB}_{A_h} \{ \sum_{i \in F} x_i \mid F \text{ a finite subset of } \alpha \}$$

and so $\|e_{00}(\sum_{i<\alpha} p_i)e_{00}\| \leq c < 1$. Also

$$e_{00}(\sum_{i<\alpha} p_i)e_{00} + x_\alpha \leq ce_{00} < e_{00}.$$

We observe that $\sum_{i<\alpha} e_{ii} \sim e_{\alpha\alpha} \sim \sum_{i \leq \alpha} e_{ii}$. Let $f_{11} = \sum_{i<\alpha} e_{ii}$, let $f_{22} = e_{\alpha\alpha}$ and let f_{12} be any partial isometry in $(\sum_{i \leq \alpha} e_{ii})A(\sum_{i \leq \alpha} e_{ii})$ such that $f_{12}f_{12}^* = f_{11}$ and $f_{12}^*f_{12} = f_{22}$. Let $f_{21} = f_{12}^*$. Then, by the above lemma, there is a projection p_α in $(\sum_{i \leq \alpha} e_{ii})A(\sum_{i \leq \alpha} e_{ii})$, such that p_α is orthogonal to $\sum_{i<\alpha} p_i$ and $x_\alpha = e_{00}p_\alpha e_{00}$. Hence $\{x_\xi \mid \xi \leq \alpha\} = \{x_\xi \mid \xi < \alpha + 1\}$ is dilatable.

Lemma 6. Let α be an infinite ordinal such that $\alpha \leq \Omega$. Let A be ξ -complete for each $\xi < \alpha$. Then $e_{00}Ae_{00}$ is α -dilatable in A .

In fact, let $\{x_i\}_{i<\alpha}$ be an order summable transfinite sequence of positive elements of $e_{00}Ae_{00}$ with order sum c , where $c < 1$. To obtain a contradiction, let us assume that this transfinite sequence is not dilatable. Then, there exists a smallest ordinal β , such that $\{x_i\}_{i<\beta}$ is not dilatable, and $\beta \leq \alpha$. (Note that $\omega < \beta$ by the results of Christensen and Pedersen [1].)

Let γ be any non-zero ordinal strictly less than β . From the definition of β , $\{x_i\}_{i<\gamma}$ is dilatable. So, there

is a family $\{p_i\}_{i<\gamma}$ of orthogonal projections in $(\sum_{i<\gamma} e_{ii})A(\sum_{i<\gamma} e_{ii})$ such that $x_i = e_{00}p_i e_{00}$ for each $i < \gamma$. Let M_γ be the set of all such families of orthogonal projections and let $M = \cup\{M_\gamma \mid 0 < \gamma < \beta\}$. Clearly $M \neq \emptyset$. For $\Gamma = \{p_i\}_{i<\gamma_1} \in M$ and $\Xi = \{q_j\}_{j<\gamma_2} \in M$, we define $\Gamma \leq \Xi$ to mean that $\gamma_1 \leq \gamma_2$ and, for all $i < \gamma_1$, $p_i = q_i$. This partially orders M inductively. So, by Zorn's lemma, M has a maximal element $\{p_i\}_{i<\zeta}$. Then, by applying the argument of Lemma 5 to $\{x_\xi\}_{\xi<\zeta+1}$, we find a projection p_ζ in $(\sum_{i \leq \zeta} e_{ii})A(\sum_{i \leq \zeta} e_{ii})$ such that p_ζ is orthogonal to $\sum_{i<\zeta} p_i$ and $e_{00}p_\zeta e_{00} = x_\zeta$. Thus $\{p_i\}_{i<\zeta+1}$ is in M . This contradicts maximality. Hence the assumption that $\{x_i\}_{i<\alpha}$ was not dilatable must be false. Hence $e_{00}Ae_{00}$ is α -dilatable in A .

By using these lemmas, we have the following:

Theorem 2. Let A be a properly infinite AW*-algebra. Let Ω be an infinite ordinal such that there exists an Ω -indexed system of matrix units in A , $\{e_{ij}\}_{0 \leq i,j < \Omega}$. Then A is Ω -complete.

In fact, assume that A is not Ω -complete. Then there is a first ordinal β , $\beta \leq \Omega$, such that A is not β -complete. So, for $\alpha < \beta$, A is α -complete. So, by the above lemma,

$e_{00}Ae_{00}$ is β -dilatable in A . Then, by Lemma 3, A is β -complete. This is a contradiction. So A must be Ω -complete.

Corollary 1([1]). Let A be a properly infinite AW*-algebra. Then A is monotone σ -complete.

Since A has a countable system of matrix units, A is ω -complete. So, A is monotone σ -complete.

Now we are in the position to discuss about normality in properly infinite AW*-algebras.

Theorem 3. Let A be a properly infinite AW*-algebra whose centre, Z , is locally countably decomposable. Then A is normal.

Outline of the proof. (See [6].)

Let A be an infinite AW*-factor. Let Π be an infinite set of orthogonal projections in A . Let $\aleph$ be the cardinality of Π . Then A is $\aleph$ -complete. Since A is monotone σ -complete, there is nothing further to prove if Π is countable. So let us suppose Π to be uncountably infinite.

We may decompose Π into a family of disjoint set $\{ \Pi_\lambda \mid \lambda \in \Lambda \}$, where each Π_λ is of the same cardinality as Π ,

and where $\#\Pi = \#\Lambda$. Let $p_\lambda = \sum \Pi_\lambda$. Then $\{p_\lambda \mid \lambda \in \Lambda\}$ is an orthogonal family of non-zero projections. We shall show that each p_i is infinite. Suppose that p_i is finite for some i . Then $p_i A p_i$ is a finite AW*-factor and so it is σ -finite. Since Π_i is uncountable. This is a contradiction. So all p_i are infinite. Since A is infinite, there exists a minimal infinite projection e_0 in A such that $e_0 \preceq p_i$ for all $i \in \Lambda$. So there is a set Π' of mutually orthogonal family of projections $\{e_\lambda \mid \lambda \in \Lambda\}$ such that $e_\lambda \sim e_0$ for all $\lambda \in \Lambda$. By Zorn's lemma, Π' can be extended to a maximal collection Γ of mutually orthogonal infinite projections, each of which is equivalent to e_0 . Clearly $\#\Gamma \geq \aleph$. By a general theory of AW*-algebras, one can find a $\#\Gamma$ -homogeneous partition of 1 in A . Hence we can construct $\#\Gamma$ -system of matrix units $\{e_{ij} \mid ij \in \Gamma\}$ in A such that $e_{ii} \sim 1$ for all i . Since $\#\Lambda \leq \#\Gamma$, A must be $\#\Lambda$ -complete, and so Π is well-behaved. For the general case, see [6].

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