

A relation between instanton-type solutions of P_J ($J = \text{I, II, 34, IV}$)-hierarchies with a large parameter

By

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Abstract

We report a relation between instanton-type solutions for equations of P_J ($J = \text{I, II, 34, IV}$)-hierarchies with a large parameter. The content of these notes is a short summary of our forthcoming papers [2], [16] and [17].

§ 1. Definitions of P_J ($J = \text{I, II, 34, IV}$)-hierarchies with a large parameter η

We recall the definitions of equations of P_J ($J = \text{I, II, 34, IV}$)-hierarchies with a large parameter η given in [15], [8] and [9].

(i) The m -th members $(P_1)_m$ and $(P_{34})_m$ of P_1 , P_{34} -hierarchies with η

Let u_k and v_k ($k = 1, 2, \dots$) be unknown functions of t and c_k 's are constants. In what follows, δ_{jm} stands for Kronecker's delta.

• For $m = 1, 2, \dots$, $(P_1)_m$ has the following form (see [15]):

$$(1.1) \quad \begin{cases} \eta^{-1} \frac{du_j}{dt} = 2v_j, & j = 1, 2, \dots, m, \\ \eta^{-1} \frac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j), & j = 1, 2, \dots, m, \end{cases}$$

with the assumption $u_{m+1} = 0$. Here w_j is recursively defined by

$$(1.2) \quad w_j = \frac{1}{2} \sum_{k=1}^j u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \sum_{k=1}^{j-1} v_k v_{j-k} + c_j + \delta_{jm} t.$$

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- For $m = 1, 2, \dots$, $(P_{34})_m$ has the following form (see [9]):

$$(1.3) \quad \begin{cases} \eta^{-1} \frac{du_j}{dt} = 2v_j, & j = 1, 2, \dots, m, \\ \eta^{-1} \frac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j), & j = 1, 2, \dots, m, \end{cases}$$

with

$$(1.4) \quad u_{m+1} = -w_m + c_0 u_m + \frac{v_m^2 - \kappa^2}{2u_m}.$$

Here $\gamma (\neq 0)$, κ and $\{c_j\}_{j=0}^m$ are constants, and w_j is recursively defined by

$$(1.5) \quad \begin{aligned} w_j = & \frac{1}{2} \sum_{k=1}^j u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2} \sum_{k=1}^{j-1} v_k v_{j-k} + c_j \\ & + c_0 \left(2u_j - \sum_{k=1}^{j-1} u_k u_{j-k} \right) + \delta_{j,m-1} \gamma t + 2\delta_{jm} \gamma t c_0. \end{aligned}$$

Note that the form above has been slightly modified from the original form given by [9]. If we replace u_m (resp. v_m) in (1.3) and (1.4) with $u_m - \gamma t$ (resp. $v_m - \eta^{-1} \frac{\gamma}{2}$), then we have the original form of $(P_{34})_m$.

- (ii) The m -th members $(P_{\text{II}})_m$ and $(P_{\text{IV}})_m$ of P_{II} , P_{IV} -hierarchies with η

Let u_k and v_k ($k = 1, 2, \dots$) be unknown functions of t and c_k 's are constants.

- For $m = 1, 2, \dots$, $(P_{\text{II}})_m$ has the following form (see [8]):

$$(1.6) \quad \begin{cases} \eta^{-1} \frac{du_j}{dt} = -2(u_1 u_j + v_j + u_{j+1}) + 2c_j u_1, & j = 1, 2, \dots, m, \\ \eta^{-1} \frac{dv_j}{dt} = 2(v_1 u_j + v_{j+1} + w_j) - 2c_j v_1, & j = 1, 2, \dots, m, \end{cases}$$

with $u_{m+1} = \gamma t$ and $v_{m+1} = \kappa$. Here $\gamma (\neq 0)$, κ and c_j 's are constants, and w_j is recursively defined by

$$(1.7) \quad w_j = \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^j u_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k.$$

- For $m = 1, 2, \dots$, $(P_{\text{IV}})_m$ has the following form (see [8]):

$$(1.8) \quad \begin{cases} \eta^{-1} \frac{du_j}{dt} = -2(u_1 u_j + v_j + u_{j+1}) + 2c_j u_1 - 2\delta_{j,m-1} \gamma t, \\ \eta^{-1} \frac{dv_j}{dt} = 2(v_1 u_j + v_{j+1} + w_j) - 2c_j v_1, & j = 1, 2, \dots, m, \end{cases}$$

with

$$(1.9) \quad u_{m+1} = -\alpha_1, \quad v_{m+1} = -w_m - \frac{(v_m - \alpha_1)^2 - \alpha_2^2}{2(u_m - c_m)}.$$

Here $\gamma (\neq 0)$, α_1 , α_2 and c_j 's are constants, and w_j is defined by

$$(1.10) \quad w_j = \sum_{k=1}^{j-1} u_{j-k} w_k + \sum_{k=1}^j u_{j-k+1} v_k + \frac{1}{2} \sum_{k=1}^{j-1} v_{j-k} v_k - \sum_{k=1}^{j-1} c_{j-k} w_k + \delta_{jm} \gamma t v_1.$$

Note that the form above has been slightly modified from the original form given by [8]. If we replace u_m in (1.8) and (1.9) with $u_m - \gamma t$, then we have the original form of $(P_{\text{IV}})_m$.

§ 2. P_J ($J = \text{I, II, 34, IV}$)-hierarchical with η in terms of generating functions

In this note, we consider the represented forms with generating functions of unknown functions. Let θ denotes an independent variable.

(i) The m -th members $(P_{\text{I}})_m$ and $(P_{34})_m$ of P_{I} , P_{34} -hierarchies with η

We define generating functions U , V and C of $(P_{\text{I}})_m$ (resp. $(P_{34})_m$) by

$$(2.1) \quad \begin{aligned} U(\theta) &= \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \\ C(\theta) &= \sum_{k=1}^{\infty} (c_k + \delta_{km} t) \theta^{k+1} \quad (\text{resp. } C(\theta) = \sum_{k=1}^{\infty} c_k \theta^{k+1}), \end{aligned}$$

respectively. Here u_k , v_k , c_k ($k = 1, 2, \dots$) denote unknown functions and constants of $(P_{\text{I}})_m$ (resp. $(P_{34})_m$). In what follows, by $A \equiv B$ we mean that $A - B$ is zero modulo θ^{m+2} .

- $(P_{\text{I}})_m$ is rewritten in the following form

$$(2.2) \quad \eta^{-1} \frac{d}{dt} \begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} 2V\theta \\ -(1 + 2u_1\theta)(1 - U) + \frac{1 + 2C - \theta V^2}{1 - U} \end{pmatrix}$$

with the condition that the coefficients of θ^{m+1} of U and V are zero.

- $(P_{34})_m$ is rewritten in the following form

$$(2.3) \quad \begin{aligned} \eta^{-1} \frac{d}{dt} \begin{pmatrix} U\theta \\ V\theta \end{pmatrix} &\equiv \begin{pmatrix} 2V\theta \\ -(1 + 2(u_1 + c_0)\theta)(1 - U) + \frac{1 + 2C - \theta(V^2 - 2c_0)}{1 - U} \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 2\gamma t \theta^m (1 + (u_1 + 2c_0)\theta) \end{pmatrix} \end{aligned}$$

with the condition that the coefficient of θ^{m+1} of U (resp. V) is equal to the right hand side of (1.4) (resp. zero).

Remark that, if we compare the coefficients with respect to θ^j ($2 \leq j \leq m+1$) on the both sides of (2.2) (resp. (2.3)), we obtain (1.1) (resp. (1.3)).

(ii) The m -th members $(P_{\text{II}})_m$ and $(P_{\text{IV}})_m$ of P_{II} , P_{IV} -hierarchies with η

We define generating functions U , V and C of $(P_{\text{II}})_m$ (resp. $(P_{\text{IV}})_m$) by

$$(2.4) \quad U(\theta) = \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) = \sum_{k=1}^{\infty} v_k \theta^k \quad \text{and} \quad C(\theta) = \sum_{k=1}^{\infty} c_k \theta^k,$$

respectively. Here u_k , v_k , c_k ($k = 1, 2, \dots$) denote unknown functions and constants of $(P_{\text{II}})_m$ (resp. $(P_{\text{IV}})_m$).

• $(P_{\text{II}})_m$ is rewritten in the following form

$$(2.5) \quad \eta^{-1} \frac{d}{dt} \begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv 2 \begin{pmatrix} u_1(1-U+C)\theta - U - V\theta \\ -v_1(1-U+C)\theta + \frac{2UV + V^2\theta}{2(1-U+C)} + V \end{pmatrix}$$

with the condition that the coefficients of θ^{m+1} of U and V are γt and κ , respectively.

• $(P_{\text{IV}})_m$ is rewritten in the following form

$$(2.6) \quad \eta^{-1} \frac{d}{dt} \begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv 2 \begin{pmatrix} u_1(1-U+C)\theta - U - V\theta - \gamma t \theta^m \\ -v_1(1-U+C)\theta + \frac{2UV + V^2\theta}{2(1-U+C)} + V + \gamma t v_1 \theta^{m+1} \end{pmatrix}$$

with the condition that the coefficients of θ^{m+1} of U and V are equal to the right hand sides of (1.9), respectively.

Remark that, if we compare the coefficients with respect to θ^j ($2 \leq j \leq m+1$) on the both sides of (2.5) (resp. (2.6)), we obtain (1.6) (resp. (1.8)).

§ 3. The generating functions of the leading terms of 0-parameter solutions

As is shown in [4], each P_J -hierarchy has a formal power series of η^{-1} in the form

$$(3.1) \quad u_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{u}_{k,j}(t), \quad v_k(t) = \sum_{j=0}^{\infty} \eta^{-j} \hat{v}_{k,j}(t), \quad j = 1, \dots, m.$$

The solution taking the form of (3.1) is often called a 0-parameter solution, as the form does not have any free parameters. Let us define the generating functions of the leading terms $\hat{u}_{i,0}$ and $\hat{v}_{i,0}$ of their 0-parameter solutions of $(P_J)_m$ ($J = \text{I, II, 34, IV}$) by

$$(3.2) \quad \hat{u}_0(\theta) = \sum_{i=1}^{\infty} \hat{u}_{i,0} \theta^i, \quad \hat{v}_0(\theta) = \sum_{i=1}^{\infty} \hat{v}_{i,0} \theta^i.$$

Each explicit form of (3.2) for $(P_J)_m$ ($J = \text{I, II, 34, IV}$) is given as follows.

$(P_{\text{I}})_m$	$\hat{u}_0 = 1 - \sqrt{\frac{1+2C}{1+2\hat{u}_{1,0}\theta}}, \quad \hat{v}_0 = 0.$ Here $\hat{u}_{1,0}$ is taken so that the coefficient of θ^{m+1} in $\hat{u}_0$ is zero.
$(P_{34})_m$	$\hat{u}_0 \equiv 1 - \sqrt{\frac{1+2(C+c_0\theta)}{(1+2(\hat{u}_{1,0}+c_0)\theta)(1-2\gamma t\theta^m)}},$ $\hat{v}_0 = 0,$ where $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ are taken so that the coefficients of θ^{m+1} in $\hat{u}_0$ and $\hat{v}_0$ are equal to $-\hat{w}_{m,0} + c_0\hat{u}_{m,0} + \frac{(\hat{v}_{m,0}^2 - \kappa^2)}{2\hat{u}_{m,0}}$ and 0, respectively. Here $\hat{w}_{m,0}$ is defined by (1.5) with u_k, v_k and w_k being replaced by $\hat{u}_{k,0}$, $\hat{v}_{k,0}$ and $\hat{w}_{k,0}$.
$(P_{\text{II}})_m$	$\hat{u}_0 = (1+C)\left(1 - \sqrt{\frac{1}{(1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}}\right),$ $\hat{v}_0\theta = (1+C)\left(-1 + (1+\hat{u}_{1,0}\theta)\sqrt{\frac{1}{(1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2}}\right).$ Here $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ are taken so that the coefficients of θ^{m+1} in $\hat{u}_0$ and $\hat{v}_0$ are γt and κ, respectively.
$(P_{\text{IV}})_m$	$\hat{u}_0 \equiv (1+C)(1 - \sqrt{1/f(t, \theta)}),$ $\hat{v}_0\theta \equiv (1+C)(-1 + (1+\hat{u}_{1,0}\theta)\sqrt{1/f(t, \theta)}) - \gamma t\theta^m,$ $f(t, \theta) := (1+\hat{u}_{1,0}\theta)^2 - 2\hat{v}_{1,0}\theta^2 - 2\gamma t\theta^m(1+2\hat{u}_{1,0}\theta - c_1\theta),$ where $\hat{u}_{1,0}$ and $\hat{v}_{1,0}$ are taken so that the coefficients of θ^{m+1} in $\hat{u}_0$ and $\hat{v}_0$ are equal to $-\alpha_1$ and $-\hat{w}_{m,0} - \frac{(\hat{v}_{m,0} - \alpha_1)^2 - \alpha_2^2}{2(\hat{u}_{m,0} - c_m)}$, respectively. Here $\hat{w}_{m,0}$ is defined by (1.10) with u_k, v_k and w_k being replaced by $\hat{u}_{k,0}$, $\hat{v}_{k,0}$ and $\hat{w}_{k,0}$.

§ 4. Instanton-type solutions of $(P_J)_m$ ($J = \text{I, II, 34, IV}$)

We prepare some notation. Let $\nu_{\pm 1}(t), \dots, \nu_{\pm m}(t)$ of $(P_J)_m$ denote the roots of the following algebraic equation $\Lambda_J(\lambda, t) = 0$ of λ with $\nu_k = -\nu_{|k|}$ if $k < 0$:

- If $J = \text{I}$ (resp. 34), then $\Lambda_J(\lambda, t)$ is defined by

$$g_J(\lambda)^m - \sum_{k=1}^m \hat{u}_{k,0} g_J(\lambda)^{m-k}$$

with

$$g_J(\lambda) = \frac{\lambda^2 - 8\hat{u}_{1,0}}{4} \quad (\text{resp. } g_{34}(\lambda) = \frac{\lambda^2 - 8(\hat{u}_{1,0} + c_0)}{4}).$$

Here $\hat{u}_{k,0}$'s are defined by (3.2) of $(P_J)_m$ ($J = \text{I, 34}$) respectively.

- If $J = \text{II}$ or IV , then $\Lambda_J(\lambda, t)$ is defined by

$$g_J(\lambda)^m - \sum_{k=1}^m (\hat{u}_{k,0} - c_k) g_J(\lambda)^{m-k}$$

with

$$g_J(\lambda) = -\hat{u}_{1,0} - \sqrt{\frac{\lambda^2}{4} + 2\hat{v}_{1,0}}.$$

Here $\hat{u}_{k,0}, \hat{v}_{1,0}$ are defined by (3.2) of $(P_J)_m$ ($J = \text{II, IV}$) respectively and so are c_k .

Let Ω be an open subset in $\mathbb{C}_t$ and the two conditions are always assumed:

- (A1) The roots $\nu_i(t)$'s ($1 \leq |i| \leq m$) are mutually distinct for each $t \in \Omega$.
- (A2) The function $p_1\nu_1(t) + \dots + p_m\nu_m(t)$ does not vanish identically on Ω for any $(p_1, \dots, p_m) \in \mathbb{Z}^m \setminus \{0\}$.

Then we have the following theorem.

Theorem 4.1 ([2], [17]). *We have instanton-type solutions of $(P_J)_m$ ($J = \text{I, 34}$) with free $2m$ -parameters $(\beta_{-m}, \dots, \beta_m) \in \mathbb{C}^{2m}[[\eta^{-1}]]$ of the form*

$$(4.1) \quad \begin{aligned} U &= \hat{u}_0 + (1 - \hat{u}_0)u, \quad V = \hat{v}_0 + (1 - \hat{v}_0)v, \\ \begin{pmatrix} u \\ v \end{pmatrix} &= \sum_{1 \leq |k| \leq m} f_k^J(\tau, t; \eta) A(\nu_k), \quad A(\nu_k) = \begin{pmatrix} a(\nu_k) \\ \frac{\nu_k}{2} a(\nu_k) \end{pmatrix}, \end{aligned}$$

$$(4.2) \quad f_k^J(\tau, t; \eta) = \sum_{j=1}^{\infty} \left(\sum_{\substack{\ell \geq 0, p \in \mathbb{Z}^m \\ 2\ell + |p| = j}} f_{k,p,\ell}^J(t) e^{p \cdot \tau} \right) \eta^{-j/2},$$

with $a(\nu_k) = \frac{\theta}{1 - \theta g_J(\nu_k)} = \sum_{j=0}^{\infty} g_J(\nu_k)^j \theta^{j+1}$. Here $(\hat{u}_0, \hat{v}_0)$, ν_k and g_J of $(P_J)_m$ ($J = \text{I, 34}$) have been defined in the previous section respectively. For the explicit forms of $f_{k,p,\ell}^J(t)$ ($J = \text{I, 34}$), see [2] and [17].

The following describes the relation between instanton-type solutions of $(P_1)_m$ and $(P_{34})_m$.

Theorem 4.2 ([17]). *An instanton-type solution of $(P_{34})_m$ is transformed algebraically to that of $(P_1)_m$ by the replacements of all terms which are depending on their 0-parameter solutions.*

Theorem 4.3 ([16]). *We have instanton-type solutions of $(P_J)_m$ ($J = \text{II, IV}$) with free $2m$ -parameters $(\beta_{-m}, \dots, \beta_m) \in \mathbb{C}^{2m}[[\eta^{-1}]]$ of the form*

$$(4.3) \quad \begin{aligned} U &= \hat{u}_0 + (1 - \hat{u}_0 + C)u, & V &= \hat{v}_0 + (1 - \hat{u}_0 + C)v, \\ \begin{pmatrix} u \\ v \end{pmatrix} &= \sum_{1 \leq |k| \leq m} f_k^J(\tau, t; \eta) A(\nu_k), & A(\nu_k) &:= \begin{pmatrix} a(\nu_k) \\ \varrho_J(\nu_k) a(\nu_k) \end{pmatrix} \end{aligned}$$

$$(4.4) \quad f_k^J(\tau, t; \eta) = \sum_{j=1}^{\infty} \left(\sum_{\substack{\ell \geq 0, p \in \mathbb{Z}^m \\ 2\ell + |p| = j}} f_{k,p,\ell}^J(t) e^{p \cdot \tau} \right) \eta^{-j/2},$$

where $a(\nu_k) = \frac{\theta}{1 - \theta g_J(\nu_k)} = \sum_{j=0}^{\infty} g_J(\nu_k)^j \theta^{j+1}$ and $\varrho_J(\nu_k) := -\frac{\nu_k}{2} + \sqrt{\frac{\nu_k^2}{4} + 2\hat{v}_{1,0}}$. Here $(\hat{u}_0, \hat{v}_0)$, ν_k and g_J of $(P_J)_m$ ($J = \text{II, IV}$) have been defined in the previous section respectively. For the explicit forms of $f_{k,p,\ell}^J(t)$ ($J = \text{II, IV}$), see [16].

The following describes the relations between instanton-type solutions of $(P_{\text{II}})_m$ and $(P_{\text{IV}})_m$.

Theorem 4.4 ([16]). *An instanton-type solution of $(P_{\text{II}})_m$ is transformed algebraically to that of $(P_{\text{IV}})_m$ by the replacements of all terms which are depending on their 0-parameter solutions.*

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