

Standard Models of dP4 fibrations in Positive Characteristic

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Corti defined the notion of standard models of del Pezzo fibrations and studied the existence of them over $\mathbb{C}$ with a fixed generic fibre in [Cor]. We proved that existence of standard models of del Pezzo fibrations of degree 4 in characteristic > 2 . To show this, we use Kollár stability, which is introduced in [Kol] and [AFK].

We work over a field $k = \bar{k}$ with $\text{char}(k) \neq 2$.

0 Standard models of dP fibrations

We say that $\pi : X \rightarrow C$ is a *del Pezzo fibration* (dP fibration) of degree d if π is a flat, projective morphism from a threefold X to a curve C , and the generic fibre X_K of π is a smooth del Pezzo surface of degree d over the function field K of C .

Definition 1. A dP fibration $\pi : X \rightarrow C$ of degree $d \geq 3$ is called a *standard model* if the following conditions are satisfied:

- (1) X has only terminal singularities.
- (2) π has integral fibres.
- (3) $-K_X$ is a π -ample line bundle.

Question 2. Do there exist some standard models of del Pezzo fibrations as birational models for a given del Pezzo fibration $\pi : X \rightarrow C$?

Using techniques of descent, we only have to show that the existence of standard models $\pi : X \rightarrow C = \text{Spec } R$ with a fixed generic fibre X_K , where R is the local ring of a point on the curve C .

1 Settings

We work over a DVR R with the fraction field K .

- V : space of quadrics in n variables $x_1, \dots, x_n$ over R
 - $M := \text{Gr}(2, V)$: The moduli of pencils
 - $G := \text{GL}_n$: general linear group (group scheme over R)
- G naturally acts on M by coordinate changes.

A *weight system* (ρ, E) is a pair consisting of $\rho = (w_1, \dots, w_n) \in \mathbb{Z}^n$ and $E \in \text{GL}_n(R)$. Each weight system determines a K -point of G in the following form;

$$E^{-1} \text{diag}(t^{w_1}, \dots, t^{w_n}) E.$$

Thus a weight system induces an action by the corresponding K -point of G .

2 Semistability over curves

In order to construct the standard models, we apply *Kollár stability*, which is introduced in [AFK], to the moduli of pencils M .

Definition 3. A pencil $f \in M(R)$ is *semistable* if the following properties are satisfied:

- (1) The base locus X_K of the pencil f over K is smooth.
- (2) For every weight system (ρ, M) with $\rho = (w_1, \dots, w_n)$, the following inequality holds:

$$\text{mult}_\rho(f) \leq \frac{4}{n} \sum_{i=1}^n w_i.$$

We use the following inequality;

$$\text{mult}_\rho(f) \geq \text{val}_t(\rho \cdot g) + \text{val}_t(\rho \cdot h),$$

where $\{g, h\}$ is any R -basis of the pencil f and

$$\text{val}_t(\rho \cdot g) := \max\{N \mid \forall \text{coeff. of } \rho \cdot g \text{ is divisible by } t^N\}.$$

Theorem 4. (1) For any smooth $(2, 2)$ -complete intersection X_K over K , there exists a semistable model f s.t. the generic fibre of the base locus $X \rightarrow \text{Spec } R$ is X_K .

(2) Suppose $n = 5$. Let $f \in M(R)$ be a pencil which is semistable. Then the base locus $X \subset \mathbb{P}_R^4$ of the pencil f is a standard model of dP4 fibrations over $\text{Spec } R$.

3 Outline of the proof of Theorem 4

We show that the fibre X_0 of the closed point of R is integral.

Assume that X_0 is not integral. By the degree considerations, there are following two cases:

- (1) X_0 contains a plane.

- (2) X_0 is a union of two degree 2 surfaces Y_1, Y_2 (possibly $Y_1 = Y_2$ and X_0 is non-reduced).

Let $I \subset k[x_1, \dots, x_5]$ be the ideal which defines X_0 .

Case (1): In some coordinates, $I \subset (x_1, x_2)$. Then f is destabilized by $\rho = (1, 1, 0, 0, 0)$

Case (2): If $Y_1 \neq Y_2$ (resp. $Y_1 = Y_2$), since each degree 2 component is contained in some hyperplane ($\subset \mathbb{P}_k^4$), in some coordinates we may suppose $x_1 x_2 \in I$ (resp. $x_1^2 \in I$). In either case, f is destabilized by $\rho = (1, 0, 0, 0, 0)$.

References

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- [Cor] A. Corti, *Del Pezzo surfaces over Dedekind schemes*, Ann. of Math. (2) 144 (1996), no. 3, 641–683.
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