

Variants of subtlety in $P_\kappa\lambda$

— comparison of ideals defined by combinatorial principles —

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概要

A type of subtlety for $P_\kappa\lambda$ called “ A -subtle” is presented and compared with Menas’ notion “ M -subtle”. Using it we prove almost ineffability is consistencywise stronger than Shelah property although $P_\kappa\lambda$ is A -subtle for every $\lambda \geq \kappa$ if κ is subtle. The following are also shown (i) “ $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\}$ is A -subtle” has rather strong consequences. (ii) The subtle ideals are not λ -saturated, and completely ineffable ideal is not precipitous. (iii) $NAIn_{\kappa,\lambda} = NIn_{\kappa,\lambda}$ and $I_{\kappa,\lambda}$ does not have the partition property if $\lambda^{<\kappa} = 2^\lambda$. (iv) We can not prove “ κ is $\lambda^{<\kappa}$ -ineffable whenever κ is λ -ineffable”.

1 Notations and basic facts

Throughout this paper κ denotes a regular uncountable cardinal and λ a cardinal $\geq \kappa$. Let A be a set and a a set of ordinals with $|a| \leq |A|$. For any such pair (a, A) , $P_a A$ denotes the set $\{x \subset A : |x| < |a|\}$. Thus $P_\kappa\lambda$ denotes the set $\{x \subset \lambda : |x| < \kappa\}$ and, for $x \in P_\kappa\lambda$, $P_{x \cap \kappa} x = \{s \subset x : |s| < |x \cap \kappa|\}$.

Combinatorial properties originated for regular uncountable cardinals have been translated into $P_\kappa\lambda$ in [10], [6] and [7]. For $x \in P_\kappa\lambda$, $\hat{x}$ denotes the set $\{y \in$

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$P_\kappa\lambda : x \subset y\}$. We say $X \subset P_\kappa\lambda$ is *unbounded* if $X \cap \hat{x} \neq \emptyset$ for all $x \in P_\kappa\lambda$. Let $I_{\kappa\lambda} = \{X \subset P_\kappa\lambda : X \text{ is not unbounded}\}$.

We say I is an ideal on $P_\kappa\lambda$ if the following hold:

1. $I \subset PP_\kappa\lambda$,
2. $\emptyset \in I$ and $P_\kappa\lambda \notin I$,
3. $I_{\kappa,\lambda} \subset I$,
4. I is κ -complete; $\cup X \in I$ for any $X \subset I$ with $|X| < \kappa$.

Thus $I_{\kappa,\lambda}$ is the minimal ideal on $P_\kappa\lambda$.

We say $X \subset P_\kappa\lambda$ is *closed* if $\cup D \in X$ for any $\subset$ -chain $D \subset X$ with $|D| < \kappa$. X is called *club* if it is closed and unbounded.

Fact 1.1. *Let $X \subset P_\kappa\lambda$. Then, X is club if and only if there exists $f : \lambda \times \lambda \rightarrow \lambda$ such that $C_f := \{x \in P_\kappa\lambda : f''(x \times x) \subset x \text{ and } x \cap \kappa \in \kappa\} \subset X$.*

We say X is *stationary* if $X \cap C \neq \emptyset$ for any club C . Let $NS_{\kappa,\lambda} = \{X \subset P_\kappa\lambda : X \text{ is nonstationary}\}$. Let $I^+ = PP_\kappa\lambda \setminus I$. For $X \in I^+$ $I \restriction X$ denotes the set $\{Y \subset P_\kappa\lambda : Y \cap X \in I\}$, which is an ideal on $P_\kappa\lambda$ extending I . We say an ideal I is *normal* if I is closed under diagonal unions; $\nabla X := \{x \in P_\kappa\lambda : x \in \bigcup\{X_\alpha : \alpha \in x\}\} \in I$ for any $X = \{X_\alpha : \alpha < \lambda\} \subset I$. Note that I is normal if and only if every regressive function on $X \in I^+$ is constant on some $Y \in P(X) \cap I^+$, where a function f is said to be *regressive* if $f(x) \in x$ for any x in $\text{dom}(f) \setminus \{\emptyset\}$.

The relation $\prec$ is defined by $y \prec z$ if $y \in P_{x \cap \kappa} z$. An ideal I is *strongly normal* if for any $X \in I^+$ and $f : X \rightarrow P_\kappa\lambda$ such that $f(x) \prec x$ for all $x \in X$ there exists $Y \in P(X) \cap I^+$ such that $f \restriction Y$ is constant. This is equivalent to the following: for any $\{X_s : s \in P_\kappa\lambda\} \subset I$, $\nabla_{\prec} X_s := \{x : x \in \bigcup\{X_s : s \prec x\}\} \in I$. Clearly every strongly normal ideal is normal. A filter F on $P_\kappa\lambda$ and an ideal I on $P_\kappa\lambda$ are *dual* to each other if the following holds:

$$X \in F \text{ if and only if } P_\kappa\lambda \setminus X \in I \text{ for every } X \subset P_\kappa\lambda.$$

The dual filter of I will be denoted by I^* .

For $f : P_\kappa\lambda \rightarrow P_\kappa\lambda$, let $C_f = \{x \in P_\kappa\lambda : f''P_{x \cap \kappa} x \subset P_{x \cap \kappa} x\}$. We define an ideal $WNS_{\kappa,\lambda}$ by:

$$X \in WNS_{\kappa,\lambda} \text{ if and only if } X \subset P_\kappa\lambda \text{ and } X \cap C_f = \emptyset \text{ for some } f : P_\kappa\lambda \rightarrow P_\kappa\lambda.$$

The following are well-known [5], [8] and [1].

- Fact 1.2.** (1) $NS_{\kappa,\lambda}$ is the minimal normal ideal on $P_\kappa\lambda$.
 (2) $WNS_{\kappa,\lambda}$ is the minimal strongly normal ideal on $P_\kappa\lambda$ and $NS_{\kappa,\lambda} \subsetneq WNS_{\kappa,\lambda}$.
 (3) $WNS_{\kappa,\lambda}$ is proper if and only if κ is Mahlo or $\kappa = \nu^+$ with $\nu^{<\nu} = \nu$.
 (4) If κ is Mahlo, then $\{x \in P_\kappa\lambda : x \cap \kappa = |x \cap \kappa| \text{ is inaccessible}\} \in WNS_{\kappa,\lambda}^*$.
 (5) If $h : P_\kappa\lambda \rightarrow \lambda$ is a bijection, then $WNS_{\kappa,\lambda} = NS_{\kappa,\lambda} \upharpoonright \{x : h''P_{x \cap \kappa}x = x\}$.
 (6) If $\{s_\alpha : \alpha < \lambda^{<\kappa}\}$ is an enumeration of $P_\kappa\lambda$ and $f : P_\kappa\lambda \rightarrow P_\kappa\lambda^{<\kappa}$ is defined by $f(x) = \{\alpha : s_\alpha \prec x\}$, then $f_*(WNS_{\kappa,\lambda}) := \{X \subset P_\kappa\lambda^{<\kappa} : f^{-1}(X) \in WNS_{\kappa,\lambda}\} = WNS_{\kappa,\lambda^{<\kappa}}$ and $\{x \in P_\kappa\lambda : f(x) \cap \lambda = x\} \in WNS_{\kappa,\lambda}^*$.

All the notions defined above for $P_\kappa\lambda$ can be naturally translated into $P_{x \cap \kappa}x$ if $x \cap \kappa$ is regular uncountable. For instance, $X \subset P_{x \cap \kappa}x$ is unbounded if for any $y \in P_{x \cap \kappa}x$ there is $z \in X$ such that $y \subset z$, and $I_{x \cap \kappa, x}$ denotes the set $\{X \subset P_{x \cap \kappa}x : X \text{ is not unbounded in } P_{x \cap \kappa}x\}$ which is a $x \cap \kappa$ -complete ideal on $P_{x \cap \kappa}x$.

First we observe a type of subtlety for $P_\kappa\lambda$: $X \subset P_\kappa\lambda$ is *A-subtle* if for any $\{S_x \subset P_{x \cap \kappa}x : x \in P_\kappa\lambda\}$ and $C \in WNS_{\kappa,\lambda}^*$, there exist $y \prec z$ both in $C \cap X$ such that $S_y = S_z \cap P_{y \cap \kappa}y$.

The following are shown in §2:

- Theorem 1.3.** (1) κ is subtle if and only if $P_\kappa\lambda$ is A-subtle for every $\lambda \geq \kappa$.
 (2) If $X \subset P_\kappa\lambda$ is A-subtle, there exists $\{S_x \subset P_{x \cap \kappa}x : x \in X\}$ such that $\{x \in X : S_x = S \cap x\} \in WNS_{\kappa,\lambda}^+$ for every $S \subset P_\kappa\lambda$.
 (3) If κ is weakly Mahlo and $2^{\alpha^{<\kappa}} \leq \lambda$ for every $\alpha < \lambda$, there exists $\{S_x \subset P_{x \cap \kappa}x : x \in P_\kappa\lambda \text{ and } x \cap \kappa \text{ is regular}\}$ such that S_x is a club in $P_{x \cap \kappa}x$ and $\{x : S_x \not\subset C\} \in NS_{\kappa,\lambda}$ for any club $C \subset P_\kappa\lambda$.

The last statement is false for $\lambda = \kappa$.

In the next section we study large cardinal aspects of A-subtlety to prove an analogue of Baumgartner's theorem [4] for regular uncountable cardinals:

Theorem 1.4. If $X \subset P_\kappa\lambda$ is A-subtle, then $\{x \in X : X \cap P_{x \cap \kappa}x \text{ is } \Pi_n^m\text{-indescribable}\}$ is A-subtle for every $m, n < \omega$.

Thus our subtlety takes an appropriate place in $P_\kappa\lambda$ combinatorics. The next follows immediately.

Corollary 1.5. *If $cf(\lambda) \geq \kappa$ and $X \subset P_\kappa \lambda$ is almost ineffable, then $\{x \in X : X \cap P_{x \cap \kappa} x \text{ is Shelah}\}$ is almost ineffable.*

Hence “ κ is almost λ -ineffable” is an essentially stronger hypothesis than “ κ is λ -Shelah”. Kamo [12] already proved:

Fact 1.6. (Kamo) *If κ is λ -ineffable, then $\{x \in P_\kappa \lambda : P_{x \cap \kappa} x \text{ is not almost ineffable}\}$ is not ineffable.*

Thus we have the same hierarchy of combinatorial properties for $P_\kappa \lambda$ as for regular uncountable cardinals. Our proof is applicable for Kamo’s theorem and more simple than his.

Another corollary is:

Corollary 1.7. *If $\{x \in P_\kappa \lambda : x \cap \kappa < |x|\}$ is A -subtle, then $V \neq L[U]$*

Note that $L \models$ “ κ is subtle” if κ is subtle.

In §4 we turn to saturation of subtle ideals and show:

Theorem 1.8. (1) *The ideals of non-subtle sets are not λ -saturated.*

(2) *The ideal of non-completely ineffable sets is not precipitous*

The last section is devoted to almost ineffability and ineffability. Our results might be surprising comparing with Kamo’s theorem:

Theorem 1.9. *Suppose that $\lambda^{<\kappa} = 2^\lambda$. Then, X is almost ineffable if and only if X is ineffable.*

As a corollary we get:

Corollary 1.10. (1) *We can not prove in ZFC that κ is $\lambda^{<\kappa}$ -ineffable whenever κ is λ -ineffable.*

(2) *If $\lambda^{<\kappa} = 2^\lambda$, $I_{\kappa, \lambda}$ does not have the partition property*

2 The Subtle ideals on $P_\kappa \lambda$.

Menas [16] tried to introduce subtlety into $P_\kappa \lambda$ as follows (we call M -subtle in this paper):

Definition 2.1. Let $X \subset P_\kappa \lambda$. X is M -subtle if for any $\{S_x \subset x : x \in P_\kappa \lambda\}$ and a club $C \subset P_\kappa \lambda$ there exist $y \subsetneq z$ both in $C \cap X$ such that $S_y = S_z \cap y$.

This is not an essential generalization as proved in the same paper.

Fact 2.2. For every $\lambda \geq \kappa$, $P_\kappa \lambda$ is M -subtle if and only if κ is subtle.

We present a new definition of subtlety for $P_\kappa \lambda$ and use the word “ A -subtle” for it. In place of the filter of club sets we use $WNS_{\kappa, \lambda}$.

Definition 2.3. For $X \subset P_\kappa \lambda$, X is A -subtle if for any $\{S_x \subset P_{x \cap \kappa} x : x \in P_\kappa \lambda\}$ and $C \in WNS_{\kappa, \lambda}^*$, there are $y \prec z$ both in $C \cap X$ such that $S_y = S_z \cap P_{y \cap \kappa} y$. Set $I_M = \{X \subset P_\kappa \lambda : X \text{ is not } M\text{-subtle}\}$ and $I_A = \{X \subset P_\kappa \lambda : X \text{ is not } A\text{-subtle}\}$.

Remark 2.4. If $\lambda^{<\kappa} = \lambda$, then $\{x \in P_\kappa \lambda : h''P_{x \cap \kappa} x \subset x\} \in WNS_{\kappa, \lambda}^*$ for any bijection $h : P_\kappa \lambda \rightarrow \lambda$. Thus, in this case, $X \subset P_\kappa \lambda$ is A -subtle if and only if for any $\{S_x \subset x : x \in P_\kappa \lambda\}$ and $C \in WNS_{\kappa, \lambda}^*$ there are $y \prec z$ both in $C \cap X$ such that $S_y = S_z \cap y$.

We say κ is λ -subtle if $P_\kappa \lambda$ is A -subtle (in $P_\kappa \lambda$). If $P_\kappa \lambda$ is A -subtle, then it is M -subtle. So κ is assumed to be subtle in the rest of this section.

We collect several facts for subtle ideals:

Proposition 2.5. (1) $I_M \subset I_A$.

(2) I_M is a normal ideal on $P_\kappa \lambda$.

(3) I_A is a strongly normal ideal on $P_\kappa \lambda$.

(4) $\{x \in P_\kappa \lambda : x \cap \kappa \text{ is Mahlo}\} \in I_M^*$.

(5) If $\kappa \leq \delta < \lambda$ and $X \subset P_\kappa \lambda$ is A -subtle, then $X \restriction \delta := \{x \cap \delta : x \in X\} \subset P_\kappa \delta$ is A -subtle.

(6) If κ is Mahlo and $\{x \in P_\kappa \lambda : X \cap P_{x \cap \kappa} x \text{ is } A\text{-subtle}\} \in WNS_{\kappa, \lambda}^+$, then X is A -subtle.

(7) Let $\{s_\alpha : \alpha < \delta\}$ be an enumeration of $P_\kappa \lambda$ and $f(x) = \{\alpha : s_\alpha \prec x\}$ for $x \in P_\kappa \lambda$. Then, $f''X \subset P_\kappa \delta$ is A -subtle if and only if $X \subset P_\kappa \lambda$ is A -subtle.

(8) If λ is regular, $X \subset P_\kappa \lambda$ and $\{\alpha < \lambda : X \cap P_\kappa \alpha \text{ is } M\text{-subtle}\} \in NS_\lambda^+$, then X is M -subtle.

Proof. We only show (3). Let $X \subset P_\kappa \lambda$ be subtle and $f : X \rightarrow P_\kappa \lambda$ such that $f(x) \prec x$ for every $x \in X$. Suppose to the contradiction that $f^{-1}(\{a\})$ is not subtle for any $a \in P_\kappa \lambda$. For $a \in P_\kappa \lambda$, we fix $\{S_x^a \subset P_{x \cap \kappa} x : x \in P_\kappa \lambda\}$ and $D_a \in WNS_{\kappa, \lambda}^*$ such that $S_y^a \neq S_z^a \cap P_{y \cap \kappa} y$ for any $y \prec z$ both in $D_a \cap f^{-1}(\{a\})$.

Let $h : P_\kappa \lambda \times P_\kappa \lambda \rightarrow P_\kappa \lambda$ be a bijection and set $T_x = h''(\{f(x)\} \times S_x^{f(x)}) \cap P_{x \cap \kappa} x$. Note that $C = \{x \in P_\kappa \lambda : h''(P_{x \cap \kappa} x \times P_{x \cap \kappa} x) \subset P_{x \cap \kappa} x\} \in WNS_{\kappa, \lambda}^*$. Since X is A -subtle and $E = C \cap \Delta_{\prec} D_a \in WNS_{\kappa, \lambda}^*$, there exist $y \prec z$ both in $E \cap X$ such that $T_y = T_z \cap P_{y \cap \kappa} y$. Then, $f(y) = f(z)$ and $S_y^{f(y)} = S_z^{f(z)} \cap P_{y \cap \kappa} y$. Set $a = f(y) = f(z)$. We have $y \prec z$ are both in $D_a \cap f^{-1}(\{a\})$ and $S_y^a = S_z^a$, which contradicts our assumption. $\square$

A natural question arises:

Question 2.6. *Can it be proved that $I_M \subsetneq I_A$?*

It turns out that “ A -subtle” is neither an essential generalization.

Theorem 2.7. *If κ is subtle, then $P_\kappa \lambda$ is A -subtle.*

Proof. Let $S_x \subset P_{x \cap \kappa} x$ for $x \in P_\kappa \lambda$ and $D \in WNS_{\kappa, \lambda}^*$. Since $\kappa^{<\kappa} = \kappa$, $WNS_{\kappa^+, \lambda}$ is proper.

We first show $\{x \in P_{\kappa^+} \lambda : D \cap P_\kappa x \in WNS_{\kappa, x}^*\} \in WNS_{\kappa^+, \lambda}^*$. Let $f : P_\kappa \lambda \rightarrow P_\kappa \lambda$ such that $C_f \subset D$. If $\{x \in P_{\kappa^+} \lambda : f'' P_\kappa x \subset P_\kappa x\} \notin WNS_{\kappa^+, \lambda}^*$, $X := \{x \in P_{\kappa^+} \lambda : \kappa \subset x \wedge \exists y_x \in P_\kappa x (f(y) \notin P_\kappa x)\} \in WNS_{\kappa^+, \lambda}^+$. Note that $\kappa = |x \cap \kappa|$ for every $x \in X$. By strong normality we have $y \in P_\kappa \lambda$ such that $Y := \{x \in X : y_x = y\} \in WNS_{\kappa^+, \lambda}^+$. $Y \cap \widehat{f(y)} = \emptyset$. Contradiction. Thus $Z := \{x \in P_{\kappa^+} \lambda : \kappa \subset x \wedge f \upharpoonright P_\kappa x : P_\kappa x \rightarrow P_\kappa x\} \in WNS_{\kappa^+, \lambda}^*$. For $x \in Z$ $D_x := \{s \in P_\kappa x : f'' P_{s \cap \kappa} s \subset P_{s \cap \kappa} s\} \in WNS_{\kappa, x}^*$. For every $x \in Z$ $D \cap P_\kappa x \in WNS_{\kappa, x}^*$ since $D_x \subset C_f \cap P_\kappa x \subset D$.

Note that κ is subtle if and only if $P_\kappa \kappa$ is A -subtle. Thus $P_\kappa y$ is A -subtle for every $y \in Z$. Now we consider $\{S_x : x \in P_\kappa y\}$ and $D \cap P_\kappa y$. There exist $x_1 \prec x_2$ both in $D \cap P_\kappa y$ such that $S_{x_1} = S_{x_2} \cap P_{x_1 \cap \kappa} x_1$. $\square$

The following observations suggest two ideals may be the same. The first appears in [16].

Fact 2.8. *If $X \subset P_\kappa \lambda$ is M -subtle and $S_x \subset x$ for each $x \in X$, then for any club $C \subset P_\kappa \lambda$ there exist x, y both in $C \cap X$ such that $x \subset y$, $x \cap \kappa < y \cap \kappa$, and $S_x = S_y \cap x$.*

Proposition 2.9. *If κ is subtle, then $X = \{x \in P_\kappa \lambda : x \cap \kappa = |x|\}$ is A -subtle.*

Proof. Note that $X \notin WNS_{\kappa, \lambda} ([1])$. Let $f : P_\kappa \lambda \rightarrow P_\kappa \lambda$ and $S_x \subset P_{x \cap \kappa} x$ for all $x \in P_\kappa \lambda$. We build a chain $\langle x_\alpha : \alpha < \kappa \rangle$ as follows:

Choose $x_0 \in X \cap C_f$ arbitrarily and $x_{\alpha+1} \in X \cap C_f$ so that $x_\alpha \prec x_{\alpha+1}$. For limit α let $x_\alpha = \bigcup \{x_\beta : \beta < \alpha\}$.

Set $x = \bigcup \{x_\alpha : \alpha < \kappa\}$. Then, $x \cap \kappa = |x| = \kappa$, $P_{x \cap \kappa} x = \bigcup \{P_{x_\alpha \cap \kappa} x_\alpha : \alpha < \kappa\}$, and there exists a club $D \subset \kappa$ such that for every $\alpha \in D$ $x_\alpha \cap \kappa = \alpha = |x_\alpha|$. Note that $x_\alpha \in C_f$ if α is regular. Let $g : P_{x \cap \kappa} x \rightarrow \kappa$ be any bijection. Then, we have a club $E \subset \{\alpha \in D : g''P_\alpha x_\alpha \subset \alpha\}$. Since E is subtle in κ , there exist regular $\beta < \gamma$ both in E such that $g''S_{x_\beta} = g''S_{x_\gamma} \cap \beta$. Since β and γ are regular, x_β and x_γ belong to C_f . $\square$

S. Baldwin[3] and others observed the consistency strength of the stationarity of $\{x \in P_\kappa \lambda : x \cap \kappa < |x|\}$, the complement of the set we mentioned now.

Fact 2.10. ([3],[14],[1])

- (1) *If κ is weakly inaccessible and λ Ramsey $> \kappa$, then $\{x \in P_\kappa \lambda : x \cap \kappa < |x|\}$ is stationary.*
- (2) *If $\{x \in P_\kappa \kappa^+ : x \cap \kappa < |x|\}$ is stationary, then $0^\dagger$ exists.*
- (3) *If $P_\kappa \lambda$ is Shelah, then $\{x \in P_\kappa \lambda : x \cap \kappa < |x|\} \in NSh_{\kappa, \lambda}^*$.*

Corollary 2.11. *Suppose that $0^\dagger$ does not exist and $S = \{x \in P_\kappa \lambda : h''P_{x \cap \kappa} x = x\}$ where $h : P_\kappa \lambda \rightarrow \lambda$ is a bijection. Then, $I_M \restriction S = I_A$. Thus $I_M = I_A$ if I_M is strongly normal.*

The same relation holds between subtlety and its another weakening, which relates to “ethereal” introduced by Kunen: $X \subset \kappa$ is *ethereal* if for any $\langle S_\alpha \in [\alpha]^\alpha : \alpha < \kappa \rangle$ and a club $C \subset \kappa$, there are $\beta < \gamma$ both in $C \cap X$ such that $|S_\beta \cap S_\gamma| = |\beta|$.

Definition 2.12. *Let $X \subset P_\kappa \lambda$. We say X is weakly subtle if for any $\{S_x \in I_{x \cap \kappa, x}^+ : x \in P_\kappa \lambda\}$ and a club $C \subset P_\kappa \lambda$, there are $y \prec z$ both in $C \cap X$ such that $S_y \cap S_z \in I_{y \cap \kappa, y}^+$. Let $I_W = \{X \subset P_\kappa \lambda : X \text{ is not weakly subtle}\}$.*

We have the following:

- Proposition 2.13.** (1) I_W is a normal ideal.
- (2) *If $P_\kappa \lambda$ is weakly subtle, then κ is ethereal.*
- (3) $I_W \restriction S = I_A$ where $S = \{x \in P_\kappa \lambda : h''P_{x \cap \kappa} x = x\}$ for a bijection $h : P_\kappa \lambda \rightarrow \lambda$.

Three subtle ideals are interesting from the view of the diamond principles for $P_\kappa\lambda$. The following two cardinal version of diamond principle by Jech is wellknown.

Definition 2.14. Let $X \subset P_\kappa\lambda$. Then,

$\diamond_0(X)$: there exist $\{S_x \subset x : x \in X\}$ such that $\{x \in X : S_x = S \cap x\}$ is stationary for any $S \subset \lambda$. We simply write $\diamond_0$ for $\diamond_0(P_\kappa\lambda)$. Let $J_0 = \{X \subset P_\kappa\lambda : \diamond_0(X) \text{ does not hold}\}$.

The following are known (see [9], [10]):

- Fact 2.15.** (1) J_0 is a normal ideal on $P_\kappa\lambda$.
 (2) $L \models \text{"}\diamond_0(X) \text{ for any } X \subset P_{\omega_1}\lambda\text{"}$.
 (3) If $2^{<\kappa} < \lambda$, then J_0 is proper.
 (4) If $\diamond_0$ holds, then $NS_{\kappa,\lambda}$ is not 2^λ saturated.

In the context of I_A and I_W another version of diamond arises.

Definition 2.16. Let $X \subset P_\kappa\lambda$. Then,

$\diamond_1(X)$: there exists $\{S_x \subset P_{x \cap \kappa} : x \in X\}$ such that $\{x \in X : S_x = S \cap P_{x \cap \kappa}\}$ is stationary for any $S \subset P_\kappa\lambda$.

$\diamond_2(X)$: there exists $\{S_x \subset P_{x \cap \kappa} : x \in X\}$ such that $\{x \in X : S_x = S \cap P_{x \cap \kappa}\} \in WNS_{\kappa,\lambda}^+$ for any $S \subset P_\kappa\lambda$.

J_1 and J_2 are similarly defined as J_0 .

Of course $J_0 \subset J_1 \subset J_2$ and it is easily seen:

Lemma 2.17. (1) J_1 is a normal ideal on $P_\kappa\lambda$.

(2) J_2 is a strongly normal ideal on $P_\kappa\lambda$

(3) If $\lambda^{<\kappa} = \lambda$ and $h : P_\kappa\lambda \rightarrow \lambda$ is a bijection, then $J_2 = J_0 \restriction S$ with $S = \{x : h''P_{x \cap \kappa} = x\}$.

(4) $L \models \text{"}J_2 = WNS_{\omega_1,\lambda}\text{"}$.

(5) If J_2 is proper, then any ideal $\subset WNS_{\kappa,\lambda}$ is not $2^{\lambda^{<\kappa}}$ saturated.

Theorem 2.18. If κ is subtle, then J_2 is proper.

Proof. We show $\diamond_2(X)$ holds for every A -subtle $X \subset P_\kappa\lambda$.

By induction on $\prec$ we define $S_x \subset P_{x \cap \kappa}$ for $x \in X$ as well as $C_x \subset P_{x \cap \kappa}$.

If x is a $\prec$ minimal element of X , then $S_x = C_x = \emptyset$.

Suppose S_y and C_y is defined for every $y \in X \cap P_{x \cap \kappa} x$. If there exist $S \subset P_{x \cap \kappa} x$ and $C \in WNS_{x \cap \kappa, x}^*$ such that $S_y \neq S \cap P_{y \cap \kappa} y$ for any $y \in C$, then let S_x and C_x be any such S and C . We say this is the substantial case. Otherwise let $S_x = C_x = P_{x \cap \kappa} x$. To show $\{S_x : x \in X\}$ is a witness of $\diamond_2(X)$, let $S \subset P_\kappa \lambda$, $D \in WNS_{\kappa, \lambda}^*$, and $S_x \neq S \cap P_{x \cap \kappa} x$ for any $x \in X \cap D$. Since $X \cap D$ is A -subtle, we may assume that $D \cap P_{x \cap \kappa} x \in WNS_{x \cap \kappa, x}^*$ for every $x \in X \cap D$. Thus $S \cap P_{x \cap \kappa} x$ and $D \cap P_{x \cap \kappa} x$ witness the substantial case occurs for every $x \in X \cap D$. However $X \cap D$ is subtle hence there exist $y \prec z$ both in $X \cap D$ such that $S_y = S_z \cap P_{y \cap \kappa} y$. In particular $y \in D \cap P_{z \cap \kappa} z$. Contradiction. $\square$

Note that $\diamond_\kappa$ holds if κ is ethereal and $\kappa^{<\kappa} = \kappa$ [13].

Question 2.19. (1) Does $\diamond_1$ hold if $P_\kappa \lambda$ is weakly subtle?

(2) If κ is ethereal, then $P_\kappa \lambda$ is weakly subtle?

Let $\text{Reg} = \{x \in P_\kappa \lambda : x \cap \kappa \text{ is regular}\}$. We conclude this section by:

Proposition 2.20. Suppose κ is weakly Mahlo and $2^{\alpha^{<\kappa}} \leq \lambda$ for every $\alpha < \lambda$. Then, there exists $\{S_x : x \in P_\kappa \lambda, x \cap \kappa \text{ is regular}\}$ such that

1. $S_x \subset P_{x \cap \kappa} x$ club,
2. for every club $C \subset P_\kappa \lambda$ $\{x : S_x \not\subset C\} \in NS_{\kappa, \lambda} \upharpoonright \text{Reg}$.

Proof. Let $\{C_\alpha : \kappa^+ \leq \alpha < \lambda\} = \{X : \exists \beta < \lambda \text{ } X \text{ is a club of } P_\kappa \beta\}$ be an enumeration and C_α a club of $P_\kappa \beta(\alpha)$. Set $B = \{x : \forall \alpha \in x \beta(\alpha) \in x\}$. Then, $B \in NS_{\kappa, \lambda}^*$. Let $S_x = \{y \in P_{x \cap \kappa} x : \forall \alpha \in x y \cap \beta(\alpha) \in C_\alpha\}$.

For every α $\{x \in P_\kappa \lambda : C_\alpha \cap P_{x \cap \kappa}(x \cap \beta(\alpha)) \text{ is a club of } P_{x \cap \kappa}(x \cap \beta(\alpha))\} \in (NS_{\kappa, \lambda} \upharpoonright \text{Reg})^*$. Thus, $\{x \in P_\kappa \lambda : \{z \in P_{x \cap \kappa} x : z \cap \beta(\alpha) \in C_\alpha\} \text{ is a club of } P_{x \cap \kappa} x\} \in (NS_{\kappa, \lambda} \upharpoonright \text{Reg})^*$. Hence $A = \{x \in P_\kappa \lambda : S_x \text{ is a club of } P_{x \cap \kappa} x\} \in (NS_{\kappa, \lambda} \upharpoonright \text{Reg})^*$.

Pick any club $C \subset P_\kappa \lambda$. We have $f : \lambda \times \lambda \rightarrow \lambda$ with $C_f \subset C$. Define g by $C_f \upharpoonright \beta = C_{g(\beta)}$ for $\beta < \lambda$. Note that $\beta = \beta(g(\beta))$.

Let $x \in A \cap B \cap C_f$. For every $\beta \in x$ $g(\beta) \in x$. Hence for every $y \in S_x$ and $\beta \in x$ we have $y \cap \beta \in C_{g(\beta)} = C_f \upharpoonright \beta$. If $\{\xi, \zeta\} \subset y$, then $\{\xi, \zeta, f(\xi, \zeta)\} \subset \beta$ for some $\beta \in x$. Choose $z \in C_f$ such that $y \cap \beta = z \cap \beta$. Then, $f(\xi, \zeta) \in z \cap \beta = y \cap \beta$. Hence $y \in C_f$. We have shown that $S_x \subset C_f \subset C$ for every $x \in A \cap B \cap C_f$. $\square$

Remark 2.21. This is false for $\lambda = \kappa$.

3 Subtlety and large cardinals

Recall that $X \subset \kappa$ is Π_n^m -indescribable if for any $R \subset V_\kappa$ and Π_n^m sentence φ such that $(V_\kappa, \in, R) \models \varphi$, there exists $\alpha \in X$ such that $(V_\alpha, \in, R \cap V_\alpha) \models \varphi$.

Fact 3.1. (1) Suppose that λ is weakly compact. Then, $X \subset P_\kappa \lambda$ is M -subtle if and only if $\{\alpha < \lambda : X \cap P_\kappa \alpha \text{ is not } M\text{-subtle}\}$ is not Π_1^1 -indescribable.

(2) I_M is a normal ideal on $P_\kappa \lambda$ such that $\{x \in P_\kappa \lambda : x \cap \kappa \text{ is not } \Pi_n^m\text{-indescribable}\} \in I_M$ for every $m, n < \omega$.

Carr[7] defined $P_\kappa \lambda$ -version of indescribability.

Definition 3.2. A sequence $\langle V_\alpha(\kappa, \lambda) : \alpha \leq \kappa \rangle$ is recursively defined as follows:

$$\begin{aligned} V_0(\kappa, \lambda) &= \lambda \\ V_{\alpha+1}(\kappa, \lambda) &= P_\kappa(V_\alpha(\kappa, \lambda)) \cup V_\alpha(\kappa, \lambda) \\ V_\alpha(\kappa, \lambda) &= \bigcup \{V_\beta(\kappa, \lambda) : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal} \end{aligned}$$

This definition can be carried out for $x \in P_\kappa \lambda$ if $x \cap \kappa$ is inaccessible. For such x we consider the structure $(V_{x \cap \kappa}(x \cap \kappa, x), \in)$ in the same way as $(V_\kappa(\kappa, \lambda), \in)$.

Definition 3.3. We say $X \subset P_\kappa \lambda$ is Π_n^m -indescribable if for any $R \subset V_\kappa(\kappa, \lambda)$ and Π_n^m sentence φ such that $(V_\kappa(\kappa, \lambda), \in, R) \models \varphi$, there exists $x \in X$ such that $x \cap \kappa = |x \cap \kappa|$ and $(V_{x \cap \kappa}(x \cap \kappa, x), \in, R \cap V_{x \cap \kappa}(x \cap \kappa, x)) \models \varphi$.

Lemma 3.4. If $X \subset P_\kappa \lambda$ is A -subtle and $S_x \subset P_{x \cap \kappa} x$ for $x \in P_\kappa \lambda$, then $\{x \in X : \{y \in X \cap P_{x \cap \kappa} x : S_y = S_x \cap P_{y \cap \kappa} y\} \text{ is not } \Pi_n^m\text{-indescribable for some } m, n\}$ is not A -subtle.

Proof. Otherwise, by κ -completeness of I_A , $Y := \{x \in X : \{y \in X \cap P_{x \cap \kappa} x : S_y = S_x \cap P_{y \cap \kappa} y\} \text{ is not } \Pi_n^m\text{-indescribable}\}$ is subtle for some $m, n < \omega$. We may assume that $x \cap \kappa$ is inaccessible for all $x \in Y$. For $x \in Y$ there exist $R_x \subset V_{x \cap \kappa}(x \cap \kappa, x)$ and a Π_n^m sentence φ_x such that $(V_{x \cap \kappa}(x \cap \kappa, x), \in, R_x) \models \varphi_x$ while $(V_{y \cap \kappa}(y \cap \kappa, y), \in, R_x \cap V_{y \cap \kappa}(y \cap \kappa, y)) \models \neg \varphi_x$ for any $y \in X \cap P_{x \cap \kappa} x$ with $S_y = S_x \cap P_{y \cap \kappa} y$. By κ -completeness again, we can assume for all $x \in Y$ $\varphi_x = \varphi$ for some φ .

Since Y is subtle, there are $y \prec z$ both in Y such that $R_y = R_z \cap V_{y \cap \kappa}(y \cap \kappa, y)$ and $S_y = S_z \cap P_{y \cap \kappa} y$. Then, $y \in X \cap P_{z \cap \kappa} z$, $S_y = S_z \cap P_{y \cap \kappa} y$ and $(V_{y \cap \kappa}(y \cap \kappa, y), \in, R_z \cap V_{y \cap \kappa}(y \cap \kappa, y)) \models \varphi_z$, which is a contradiction. $\square$

As a corollary we have:

Theorem 3.5. $\{x \in P_\kappa\lambda : P_{x \cap \kappa}x \text{ is not } \Pi_n^m\text{-indescribable}\} \in I_A \text{ for every } m, n < \omega$.

This theorem derives strong facts.

Lemma 3.6. *If $\{x \in P_\kappa\lambda : 2^{x \cap \kappa} \leq |x|\}$ is A -subtle, then $\{x \in P_\kappa\lambda : o(x \cap \kappa) \geq 1\}$ is A -subtle.*

Proof. Note that κ is measurable if $P_\kappa 2^\kappa$ is Π_1^1 -indescribable ([6],[7]). Let $X = \{x \in P_\kappa\lambda : P_{x \cap \kappa} 2^{x \cap \kappa} \text{ is } \Pi_1^1\text{-indescribable}\}$. Then, X is A -subtle. For $x \in X$ $x \cap \kappa$ is measurable and $\{y \in P_{x \cap \kappa} 2^{x \cap \kappa} : y \cap \kappa \text{ is measurable}\}$ is Π_1^1 -indescribable. Hence $o(x \cap \kappa) \geq 1$. $\square$

Note that $L[U] \models$ “there exist $\kappa < \lambda$ such that $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\} \in NS_{\kappa,\lambda}^+$ ”.

Corollary 3.7. (1) $L[U] \models$ “ $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\}$ is not A -subtle”.
(2) $L[U] \models$ “ $\neg \exists \kappa (\kappa \text{ is } \kappa^+\text{-Shelah})$ ”.

Thus the existence of a cardinal κ such that $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\}$ is A -subtle is rather strong in consistency strength. On the other hand subtlety is a Π_1^1 property of $P_\kappa\lambda$. The following proposition says the size of κ is not necessarily large.

Proposition 3.8. *The least cardinal κ such that κ is κ^+ -subtle is not κ^+ -Shelah.*

In the rest of this section we compare the almost ineffability and Shelah property using I_A , which reveals very useful in large cardinal hierarchy.

Carr [6] defined Shelah property as a $P_\kappa\lambda$ generalization of weak compactness. We show: *if $P_\kappa\lambda$ is subtle then there exist many $x \in P_\kappa\lambda$ such that $P_{x \cap \kappa}x$ is Shelah.*

Definition 3.9. *Let $X \subset P_\kappa\lambda$. We say X is Shelah if for any $\{f_x \in {}^x x : x \in P_\kappa\lambda\}$ there is $f : \lambda \rightarrow \lambda$ such that for every $y \in P_\kappa\lambda$ the set $\{x \in X \cap \hat{y} : f \restriction y = f_x \restriction y\} \in I_{\kappa,\lambda}^+$.*

We say X is almost ineffable (ineffable) if for any $\{f_x \in {}^x x : x \in P_\kappa\lambda\}$ there is $f : \lambda \rightarrow \lambda$ such that $\{x \in X : f \restriction x = f_x\} \in I_{\kappa,\lambda}^+ (NS_{\kappa,\lambda}^+)$.

Let $NSh_{\kappa\lambda} = \{X \subset P_\kappa\lambda : X \text{ is not Shelah}\}$ and $NAIn_{\kappa\lambda} = \{X \subset P_\kappa\lambda : X \text{ is not almost ineffable}\}$.

We often say κ is λ -Shelah (almost λ -ineffable) if $P_\kappa\lambda$ is Shelah (almost ineffable).

Clearly X is Shelah if X is almost ineffable, and X is almost ineffable if X is ineffable. It is known that $NSh_{\kappa\lambda}$ and $NAIn_{\kappa\lambda}$ are strongly normal ideals if $cf(\lambda) \geq \kappa$. Moreover, Kamo [12] proved the following:

Fact 3.10. (Kamo) *If $X \subset P_\kappa\lambda$ is ineffable and $cf(\lambda) \geq \kappa$, then $\{x \in X : X \cap P_{x \cap \kappa}x \text{ is almost ineffable}\}$ is ineffable.*

The following follows immediately from definition and the remark after Definition 2.1 with strong normality of $NAIn_{\kappa\lambda}$.

Proposition 3.11. *If $cf(\lambda) \geq \kappa$ and $X \subset P_\kappa\lambda$ is almost ineffable, then X is A -subtle.*

Carr [7] proved the following:

Fact 3.12. *Let $X \subset P_\kappa\lambda$ is Π_1^1 -indescribable, then X is Shelah. The converse is also true if $cf(\lambda) \geq \kappa$.*

Corollary 3.13. *If $X \subset P_\kappa\lambda$ is subtle, then $Y = \{x \in X : X \cap P_{x \cap \kappa}x \text{ is not Shelah}\}$ is not A -subtle.*

Corollary 3.14. *Let $cf(\lambda) \geq \kappa$. If $X \subset P_\kappa\lambda$ is almost ineffable, then $\{x \in P_\kappa\lambda : X \cap P_{x \cap \kappa}x \text{ is Shelah}\}$ is almost ineffable. In particular, $\{x \in P_\kappa\lambda : x \cap \kappa \text{ is } x\text{-Shelah}\} \in NAIN_{\kappa,\lambda}^*$ if κ is almost λ -ineffable.*

This corollary tells that almost ineffability is much stronger hypothesis than Shelah property. For instance, suppose that κ is almost κ^+ -ineffable. Then $\{x \in P_\kappa\lambda : o.t.(x) = (x \cap \kappa)^+\} \in NAIN_{\kappa,\lambda}^*$ hence $\{x \in P_\kappa\lambda : x \cap \kappa \text{ is } (x \cap \kappa)^+\text{-Shelah}\} \in NAIN_{\kappa,\lambda}^*$. Thus, below the least α that is almost α^+ -ineffable, stationary many β which is β^+ -Shelah exist.

4 Saturation of subtle ideals on $P_\kappa\lambda$

Now we turn to saturation of subtle ideals.

Proposition 4.1. *Let I be a normal λ saturated ideal on $P_\kappa\lambda$ with λ regular. Then, $\{x \in P_\kappa\lambda : o.t.(x) \text{ is regular}\} \in I^*$ hence $\{x \in P_\kappa\lambda : x \cap \kappa < |x|\} \in I^*$.*

Proof. Let G be P_I generic for V . By λ saturation the generic ultrapower $Ult(V, G)$ is well-founded. Let $j : V \rightarrow M \cong Ult(V, G)$ be an generic embedding. Since λ is regular in $V[G]$, $M \models “\lambda \text{ is regular}”$ and $[\langle o.t.(x) \mid x \in P_\kappa \lambda \rangle]_G = o.t.(j“\lambda) = \lambda$. Clearly $\{x \in P_\kappa \lambda : x \cap \kappa \in \kappa \wedge x \setminus \kappa \neq \emptyset\} \in I^*$. $\square$

Theorem 4.2. *Neither I_A nor I_M is λ saturated.*

Proof. First suppose that λ is regular. We know $X = \{x \in P_\kappa \lambda : x \cap \kappa = |x| \text{ and } \kappa \in x\}$ is A -subtle. For $x \in X$ $o.t.(x)$ is singular.

Second assume λ is singular and the subtle ideal on $P_\kappa \lambda$, say I , is λ saturated. In fact I is δ saturated for some regular $\delta < \lambda$. Then $I \restriction \delta$ is δ saturated and $\{x \in P_\kappa \delta : x \cap \kappa = |x|\} \notin I \restriction \delta$. Contradiction. $\square$

Remark 4.3. *By Cummings' theorem I_A is not λ^+ -saturated if $cf(\lambda) < \kappa$.*

By definition ineffability can be seen as a strengthening of Shelah property. One of the strongest version is complete ineffability defined as follows:

Definition 4.4. *An ideal I on $P_\kappa \lambda$ is (λ, λ) -distributive if for any $X \in I^+$ and $\{W_\alpha : \alpha < \lambda\}$ which is an I -partition of X with $|W_\alpha| \leq \lambda$ there exist $Y \in P(X) \cap I^+$ and $\{X_\alpha : \alpha < \lambda\}$ such that $X_\alpha \in W_\alpha$ and $Y \setminus X_\alpha \in I$ for every $\alpha < \lambda$.*

We say κ is completely λ ineffable if there is a normal (λ, λ) -distributive ideal on $P_\kappa \lambda$. When κ is completely λ ineffable, the minimal normal (λ, λ) -distributive ideal on $P_\kappa \lambda$, say I , is called completely ineffable ideal and $X \in I^+$ is said to be completely ineffable.

If I is a normal (λ, λ) -distributive ideal on $P_\kappa \lambda$ and $X \in I^+$, the following hold [11]:

1. For any $\{f_x \in {}^x x : x \in X\}$ there is $f : \lambda \rightarrow \lambda$ such that $\{x \in X : f_x = f \restriction x\} \in I^+$
2. I is strongly normal.

In the rest of this section we assume $cf(\lambda) \geq \kappa$ and I is a normal (λ, λ) -distributive ideal on $P_\kappa \lambda$. Hence κ is Mahlo, $\lambda^{<\kappa} = \lambda$, and I is strongly normal. Let $\mathbb{P}_I = (I, \subset)$, $G \subset \mathbb{P}_I$ be V generic, and $j : V \prec M \cong Ult(V, G)$ a generic elementary embedding defined in $V[G]$.

By normality of I $P(\lambda)^V \subset M$. Moreover $[\langle P_{x \cap \kappa} x \mid x \in P_\kappa \lambda \rangle]_G = j'' P_\kappa \lambda \in M$, $j'' P_\kappa \lambda = P_\kappa j'' \lambda$, and $j(X) \cap j'' P_\kappa \lambda = j'' X \in M$ for every $X \in P(P_\kappa \lambda)^V$ by strong normality. Conversely we have:

Lemma 4.5. *For every $[f]_G \in j''^\lambda j'' V \cap M$ there exists $g \in V$ such that $[f]_G = j(g) \upharpoonright j'' \lambda = j'' g$.*

Proof. Suppose $f \in V$, $X \in I^+$ with $X \Vdash "[f]_G : j'' \lambda \rightarrow j'' V"$. For $Y := \{s \in X : \text{dom}(f(s)) = s\} \in G$, $\alpha < \lambda$, and $x \in V$, let $X_{\alpha, x} = \{s \in Y \cap \hat{\alpha} : f(s)(\alpha) = x\}$. Then $W_\alpha := \{X_{\alpha, x} : x \in V\} \cap I^+$ is a disjoint I -partition of Y and $|W_\alpha| \leq \lambda$. By (λ, λ) -distributivity there is $g : \lambda \rightarrow V$ such that for every $\alpha < \lambda$ $Y \setminus X_{\alpha, g(\alpha)} \in I$. Hence $Z := \Delta_\alpha X_{\alpha, g(\alpha)} \in I^+ \cap P(Y)$ and for every $s \in Z$ and $\alpha \in s$ $f(s)(\alpha) = g(\alpha)$, that is, $Z \Vdash "[f]_G = j(g) \upharpoonright j'' \lambda"$. $\square$

Remark 4.6. *By (λ, λ) -distributivity of I , λ remains a cardinal in $V[G]$ hence in M .*

Corollary 4.7. (1) *If $X \in M$, $X \subset j'' V$, and $|X|^M \leq \lambda$, then $X = j'' Y$ for some $Y \in V$.*

(2) $P_\kappa j'' \lambda = j''(P_\kappa \lambda)$ and $P(P_\kappa j'' \lambda)^M = \{j'' X : X \in P(P_\kappa \lambda)\}$.

Proof. (1) Note that the collapsing map $\pi : j'' \lambda \rightarrow \lambda$ is a bijection in M . Choose any surjection $f \in M$ from $j'' \lambda$ to X . By the lemma $f = j(g) \upharpoonright j'' \lambda$ for some $g \in V$. Then $X = j(g)''(j'' \lambda) = \{j(g)(j(\alpha)) : \alpha < \lambda\} = \{j(g(\alpha)) : \alpha < \lambda\}$. Hence $X = j'' Y$ where $Y = g'' \lambda$. Now (2) is clear. $\square$

Next we present another characterization of completely ineffable subsets of $P_\kappa \lambda$.

Definition 4.8. *Let A be a set of ordinals. We inductively define $In_\alpha(\kappa, A) \subset P(P_\kappa A)$ as follows:*

- (1) $X \in In_0(\kappa, A)$ if $X \in NS_{\kappa, A}^+$, that is, for every $f : A \times A \rightarrow P_\kappa A$ there exists $x \in X$ such that $f''(x \times x) \subset P(x)$.
- (2) $X \in In_{\alpha+1}(\kappa, A)$ if for every $f : P_\kappa A \rightarrow P_\kappa A$ such that $f(x) \subset x$ for any $x \in P_\kappa A$ there is $S \subset A$ with $\{x \in X : f(x) = x \cap S\} \in In_\alpha(\kappa, A)$.
- (3) If α is a limit ordinal, $In_\alpha(\kappa, A) = \bigcap_{\beta < \alpha} In_\beta(\kappa, A)$.

Thus $X \in In_1(\kappa, \lambda)$ if and only if $X \subset P_\kappa \lambda$ is ineffable. Clearly $In_\alpha(\kappa, A) \subset In_\beta(\kappa, A)$ if $\beta < \alpha$. Hence there is α such that $In_\alpha(\kappa, A) = In_{\alpha+1}(\kappa, A)$. If

$In_\alpha(\kappa, A) = In_{\alpha+1}(\kappa, A) \neq \emptyset$, $P(P_\kappa A) \setminus In_\alpha(\kappa, A)$ is the minimal normal (λ, λ) -distributive ideal, that is, completely ineffable ideal on $P_\kappa A$.

Definition 4.9. We say I is precipitous if $\Vdash_{\mathbb{P}_I} \text{"Ult}(V, G) \text{ is well-founded"}$.

Lemma 4.10. Let $cf(\lambda) \geq \kappa$ and I be a normal (λ, λ) -distributive precipitous ideal on $P_\kappa \lambda$. For any $X \in P(P_\kappa \lambda)$ and α , $X \in In_\alpha(\kappa, \lambda)$ if and only if $M \models "j''X \in In_\alpha(\kappa, j''\lambda)"$.

Proof. By induction on α . Assume first X is stationary, $f : j''\lambda \times j''\lambda \rightarrow P_\kappa j''\lambda$, and $f \in M$. Since $P_\kappa j''\lambda = j''(P_\kappa \lambda)$, $f = j(g) \upharpoonright j''\lambda \times j''\lambda$ for some $g \in V$ such that $g : \lambda \times \lambda \rightarrow P_\kappa \lambda$. There is $x \in X$ such that $g''(x \times x) \subset P(x)$. Then $M \models "j(g)(a) \subset j(x) \text{ for all } a \in j(x) \times j(x)"$. Since $j(x) = j''x \subset j''\lambda$, $M \models "f(a) \subset j(x) \text{ for all } a \in j(x) \times j(x)"$. Hence $M \models "j''X \in NS_{\kappa, j''\lambda}^+"$.

Assume conversely $M \models "j''X \in NS_{\kappa, j''\lambda}^+"$, $f \in V$, and $f : \lambda \times \lambda \rightarrow P_\kappa \lambda$. Since $M \models "j(f) \upharpoonright (j''\lambda \times j''\lambda) : j''\lambda \times j''\lambda \rightarrow P_\kappa j''\lambda"$, there is $y \in j''X$ such that $j(f)''(y \times y) \subset P(y)$. For $x \in X$ with $j(x) = y$, $j(f)(j(a)) \subset j(x)$ for every $a \in x \times x$. Hence $f''(x \times x) \subset P(x)$, showing that X is stationary.

The case α is a limit ordinal is clear by our definition of $In_\alpha(\kappa, \lambda)$ and $In_\alpha(\kappa, j''\lambda)$. We prove for $\alpha = \beta + 1$. Suppose first $X \in In_\alpha(\kappa, \lambda)$, $M \models "f : P_\kappa j''\lambda \rightarrow P_\kappa j''\lambda$ and $f(j(x)) \subset j(x)$ for every $x \in P_\kappa \lambda$ ". We have $g \in V$ with $f = j(g) \upharpoonright j''P_\kappa \lambda$. Since $g(x) \subset x$ for every $x \in P_\kappa \lambda$, there is $S \subset \lambda$ such that $Y := \{x \in X : g(x) = x \cap S\} \in In_\beta(\kappa, \lambda)$. By inductive hypothesis $j''Y \in In_\beta(\kappa, j''\lambda)$. For every $x \in Y$ $j(g)(j(x)) = j(g(x)) = j(x \cap S) = j''(x \cap S) = j(x) \cap j''S$ and $j''S \in M$. Now $f(y) = y \cap j''S$ for every $y \in j''Y$ hence $j''X \in In_\alpha(\kappa, j''\lambda)$.

Suppose second $M \models "j''X \in In_\alpha(\kappa, j''\lambda)"$, $f \in V$, and $f(x) \subset x$ for every $x \in P_\kappa \lambda$. Set $j(f) \upharpoonright j''P_\kappa \lambda = g$. Since $M \models "g(x) \subset x \text{ for all } x \in j''P_\kappa \lambda"$, there is $T \subset j''\lambda$ such that $S := \{y \in j''X : g(y) = y \cap T\} \in In_\beta(\kappa, j''\lambda)$. By the fact $S \in P(P_\kappa j''\lambda)$ we have $Y \in P(P_\kappa \lambda)^V$ with $S = j''Y$. For every $x \in Y$ $g(j(x)) = j(x) \cap T$. By the same reason for S , $T = j''T_1$ for some $T_1 \in V$. For every $x \in Y$ $j(f(x)) = g(j(x)) = j(x) \cap j''T_1 = j''(x \cap T_1) = j(x \cap T_1)$. So $f(x) = x \cap T_1$ for all $x \in Y$. The inductive hypothesis shows $Y \in In_\beta(\kappa, \lambda)$. Hence $X \in In_\alpha(\kappa, \lambda)$. $\square$

Theorem 4.11. Suppose $cf(\lambda) \geq \kappa$ and I is a normal (λ, λ) -distributive precipitous ideal on $P_\kappa \lambda$. Then $\{x : x \cap \kappa \text{ is completely o.t.}(x)\text{-ineffable}\} \in I^*$.

Proof. Since $V \models \text{"}\kappa \text{ is completely } \lambda\text{-ineffable"}$, $In_\alpha(\kappa, \lambda) = In_{\alpha+1}(\kappa, \lambda) \neq \emptyset$ for some α . $In_\alpha(\kappa, \lambda) \neq \emptyset$ implies $In_\alpha(\kappa, j''\lambda) \neq \emptyset$. To show $In_\alpha(\kappa, j''\lambda) = In_{\alpha+1}(\kappa, j''\lambda)$, assume $X \in In_\alpha(\kappa, j''\lambda) - In_{\alpha+1}(\kappa, j''\lambda)$. Since $X \in P(P_\kappa j''\lambda)^M$, there is $Y \in P(P_\kappa \lambda)^V$ such that $X = j''Y$. Then $Y \in In_\alpha(\kappa, \lambda) = In_{\alpha+1}(\kappa, \lambda)$ hence $X = j''Y \in In_{\alpha+1}(\kappa, j''\lambda)$. Contradiction. $\square$

Lemma 4.12. *Let κ be Mahlo, $X \subset P_\kappa \lambda$, and α an ordinal.*

- (1) *If $\{x \in P_\kappa \lambda : X \cap P_{x \cap \kappa} x \in In_\alpha(x \cap \kappa, x)\} \in In_\alpha(\kappa, \lambda)$, then $X \in In_\alpha(\kappa, \lambda)$.*
- (2) *If $X \in In_\alpha(\kappa, \lambda)$, then $\{x \in P_\kappa \lambda : X \cap P_{x \cap \kappa} x \notin In_\alpha(x \cap \kappa, x)\} \in In_\alpha(\kappa, \lambda)$.*
- (3) *If $X \subset P_\kappa \lambda$ is completely λ ineffable, then $\{x \in X : X \cap P_{x \cap \kappa} x \text{ is not completely ineffable}\}$ is completely ineffable.*

Proof. (1) By induction on α . Suppose that $\{x \in P_\kappa \lambda : X \cap P_{x \cap \kappa} x \in NS_{x \cap \kappa, x}^+\} \in NS_{\kappa, \lambda}^+$ and $f : \lambda^2 \rightarrow P_\kappa \lambda$. There is x such that $X \cap P_{x \cap \kappa} x \in NS_{x \cap \kappa, x}^+$ and $f \upharpoonright x \times x : x \times x \rightarrow P(x)$. Then we can find $y \in X \cap P_{x \cap \kappa} x$ such that $f''(y \times y) \subset P(y)$. Hence X is stationary.

Let α be a limit ordinal and $Y := \{x \in P_\kappa \lambda : X \cap P_{x \cap \kappa} x \in In_\alpha(x \cap \kappa, x)\} \in In_\alpha(\kappa, \lambda)$. For any $\beta < \alpha$ $Y \subset \{x \in P_\kappa \lambda : X \cap P_{x \cap \kappa} x \in In_\beta(x \cap \kappa, x)\} \in In_\alpha(\kappa, \lambda) \subset In_\beta(\kappa, \lambda)$. By induction hypothesis $X \in In_\beta(\kappa, \lambda)$. Thus $X \in In_\alpha(\kappa, \lambda)$.

Let $\alpha = \beta + 1$ and $f(x) \subset x$ for all $x \in P_\kappa \lambda$. For each $x \in Y$ there is $S_x \subset x$ such that $\{y \in X \cap P_{x \cap \kappa} x : f(y) = y \cap S_x\} \in In_\beta(x \cap \kappa, x)$. Since $Y \in In_{\beta+1}(\kappa, \lambda)$ we have $S \subset \lambda$ such that $Z := \{x \in Y : S_x = x \cap S\} \in In_\beta(\kappa, \lambda)$. For $x \in Z$ and $y \in X \cap P_{x \cap \kappa} x$ $f(y) = y \cap S_x = y \cap x \cap S = y \cap S$. So $\{x \in P_\kappa \lambda : \{y \in X : f(y) = y \cap S\} \cap P_{x \cap \kappa} x \in In_\beta(x \cap \kappa, x)\} \in In_\beta(\kappa, \lambda)$. By induction hypothesis $\{y \in X : f(y) = y \cap S\} \in In_\beta(\kappa, \lambda)$ hence $X \in In_{\beta+1}(\kappa, \lambda)$.

(2) We prove this by induction on κ . We may assume $A := \{x \in X : X \cap P_{x \cap \kappa} x \in In_\alpha(x \cap \kappa, x)\} \in In_\alpha(\kappa, \lambda)$. For any $x \in A$, by inductive hypothesis, $B_x := \{y \in P_{x \cap \kappa} x : X \cap P_{x \cap \kappa} x \cap P_{y \cap \kappa} y = X \cap P_{y \cap \kappa} y \notin In_\alpha(y \cap \kappa, y)\} \in In_\alpha(x \cap \kappa, x)$. By (1) and the fact $B_x = \{y \in P_\kappa \lambda : X \cap P_{y \cap \kappa} y \notin In_\alpha(y \cap \kappa, y)\} \cap P_{x \cap \kappa} x$, the conclusion holds.

(3) Let β be the least ordinal such that $In_{\beta+1}(\kappa, \lambda) = In_\beta(\kappa, \lambda)$ and $X \in In_\beta(\kappa, \lambda)$. We may assume $W := \{x \in X : X \cap P_{x \cap \kappa} x \text{ is completely ineffable}\} \in In_\beta(\kappa, \lambda)$. For $x \in W$ let β_x be the least ordinal such that $In_{\beta_x+1}(x \cap \kappa, x) = In_{\beta_x}(x \cap \kappa, x) \neq \emptyset$. By (2), for each $x \in W$ $T_x := \{y \in P_{x \cap \kappa} x : X \cap P_{x \cap \kappa} x \cap P_{y \cap \kappa} y = X \cap P_{y \cap \kappa} y \notin In_{\beta_x}(y \cap \kappa, y)\} \in In_{\beta_x}(x \cap \kappa, x)$. Set $\gamma = \max(\beta, \bigcup_{x \in W} \beta_x)$ and $T = \{x \in P_\kappa \lambda :$

$X \cap P_{x \cap \kappa} x \notin \text{In}_\gamma(x \cap \kappa, x)\}$. Then $T_x \subset T \cap P_{x \cap \kappa} x$ for every $x \in W$. Hence $T \in \text{In}_\gamma(\kappa, \lambda) = \text{In}_\beta(\kappa, \lambda)$ $\square$

Now we conclude by the following corollary:

Corollary 4.13. *If $\text{cf}(\lambda) \geq \kappa$, then the completely ineffable ideal on $P_\kappa \lambda$ is not precipitous.*

5 Ineffability and almost ineffability

In this section we show $\text{NAIn}_{\kappa, \lambda} = \text{NIn}_{\kappa, \lambda}$ if $\text{cf}(\lambda) < \kappa$ and $\lambda^{<\kappa} = 2^\lambda$. Thus, two ideals are the same for “small” cofinality points if GCH holds.

Then we use it to prove that for λ with cofinality less than κ , “ κ is λ -ineffable” does not always imply “ κ is $\lambda^{<\kappa}$ -ineffable”. This contrasts to the fact “ κ is $\lambda^{<\kappa}$ -(super)compact if κ is λ -(super)compact”.

Fact 5.1. (1) $\text{WNS}_{\kappa, \lambda} \subsetneq \text{NAIn}_{\kappa, \lambda} \subset \text{NIn}_{\kappa, \lambda}$.

(2) If $P_\kappa \lambda \notin \text{NAIn}_{\kappa, \lambda}$ and $\text{cf}(\lambda) \geq \kappa$, then $\lambda^{<\kappa} = \lambda$.

(3) If $\text{cf}(\lambda) \geq \kappa$, then $\{x \in P_\kappa \lambda : P_{x \cap \kappa} x \in \text{NAIn}_{x \cap \kappa, x}\} \in \text{NIn}_{\kappa, \lambda}$ and $\text{NAIn}_{\kappa, \lambda} \subsetneq \text{NIn}_{\kappa, \lambda}$.

We have known quite little when $\text{cf}(\lambda) < \kappa$ while the following conjecture has seemed reasonable comparing with supercompactness:

Conjecture 5.2. *Suppose that $\text{cf}(\lambda) < \kappa$ and κ is (almost) λ -ineffable. Then, $\lambda^{<\kappa} = \lambda^+$ and κ is λ^+ -ineffable.*

Lemma 5.3. *If $\lambda^{<\kappa} = 2^\lambda$, then there exists $X \in \text{WNS}_{\kappa, \lambda}^*$ such that $I_{\kappa, \lambda} \restriction X = \text{NS}_{\kappa, \lambda} \restriction X$.*

Proof. Let ${}^{\lambda \times \lambda} \lambda = \{f_s : s \in P_\kappa \lambda\}$ be an enumeration and set $X = \{x \in P_\kappa \lambda : x \in \bigcap \{C_{f_s} : s \prec x\}\}$. For every $s \in P_\kappa \lambda$ $C_{f_s} \in \text{NS}_{\kappa, \lambda}^* \subset \text{WNS}_{\kappa, \lambda}^*$ hence $X \in \text{WNS}_{\kappa, \lambda}^*$. Let $Y \notin I_{\kappa, \lambda} \restriction X$. For every club $C \subset P_\kappa \lambda$ there exists $s \in P_\kappa \lambda$ such that $C_{f_s} \subset C$. We have $x \in Y \cap X$ with $s \prec x$. Then, $x \in C_{f_s} \cap Y \cap X \subset C \cap Y \cap X$ hence $Y \cap X \in \text{NS}_{\kappa, \lambda}^+$. $\square$

Since $\text{WNS}_{\kappa, \lambda} \subset \text{NAIn}_{\kappa, \lambda}$ we have:

Theorem 5.4. $NAIn_{\kappa,\lambda} = NIn_{\kappa,\lambda}$ if $\lambda^{<\kappa} = 2^\lambda$.

By lemma 4.12 we have the following:

Lemma 5.5. If κ is λ -ineffable, then $\{x \in P_\kappa\lambda : P_{x \cap \kappa}x \in NIn_{x \cap \kappa, x}\} \notin NIn_{\kappa,\lambda}$.

Theorem 5.6. Suppose that $\lambda^{<\kappa} = 2^\lambda$ and κ is λ^+ -ineffable. Then, $\{x \in P_\kappa\lambda^+ : x \cap \kappa \text{ is } o.t.(x \cap \lambda)\text{-ineffable, not } o.t.(x)\text{-ineffable, and } o.t.(x) = o.t.(x \cap \lambda)^+\} \notin NIn_{\kappa,\lambda^+}$.

Proof. We know $A = \{x \in P_\kappa\lambda^+ : x \cap \kappa \text{ is almost } o.t.(x)\text{-ineffable, } o.t.(x) = o.t.(x \cap \lambda)^+, cf(o.t.(x \cap \lambda)) < x \cap \kappa, \text{ and } o.t.(x \cap \lambda)^{<x \cap \kappa} = 2^{o.t.(x \cap \lambda)}\} \in NIn_{\kappa,\lambda^+}^*$ ([12], [2]). Every $x \in A$ is almost $o.t.(x \cap \lambda)$ -ineffable hence $o.t.(x \cap \lambda)$ -ineffable by the previous theorem. Now the conclusion follows by lemma 5.5. $\square$

We conclude by the negation of $I_{\kappa,\lambda}^+ \rightarrow (I_{\kappa,\lambda}^+)^2$ with $\lambda^{<\kappa} = 2^\lambda$.

Definition 5.7. For $X \subset P_\kappa\lambda$ let $[X]^2 = \{(x, y) \in X \times X : x \subsetneq y\}$. We say $I_{\kappa,\lambda}$ has the partition property if for every $X \notin I_{\kappa,\lambda}$ and $F : [X]^2 \rightarrow 2$ there exists $H \notin I_{\kappa,\lambda}$ such that $F \upharpoonright [H]^2$ is constant.

Theorem 5.8. If $\lambda^{<\kappa} = 2^\lambda$, then $I_{\kappa,\lambda}$ does not have the partition property.

Proof. Suppose otherwise and let $I = I_{\kappa,\lambda} \upharpoonright X = NS_{\kappa,\lambda} \upharpoonright X$ with X as in [?]. For every $X \in I^+$ and $F : [X]^2 \rightarrow 2$ there exists $H \notin NS_{\kappa,\lambda}$ such that $F \upharpoonright [H]^2$ is constant. Hence $NIn_{\kappa,\lambda} \subset I$ ([15]). However $I \subset WNS_{\kappa,\lambda} \subsetneq NIn_{\kappa,\lambda}$. Contradiction. $\square$

Remark 5.9. P. Matet proved that I_{κ,κ^+} does not have the partition property if $2^\kappa = \kappa^+$. While M. Shioya [17] constructed the model in which $I_{\kappa,\lambda}$ has the partition property with κ supercompact.

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