

Spaces having σ -compact-finite k -networks

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Introduction

Every CW-complex, more generally, every space dominated by locally separable metric spaces has a star-countable k -network. Also, every Lašnev space has a σ -hereditarily closure preserving (abbr, σ -HCP) k -network, and every space dominated by Lašnev subspaces has a σ -compact-finite k -network. We recall that spaces with a star-countable k -network, and spaces with a σ -HCP k -network have σ -compact-finite k -networks.

Spaces with a star-countable k -network are investigated in [IT], [LT1], [LT3], and [S]. Spaces with a σ -HCP k -network are investigated in [L], and spaces with a compact-countable k -network in [LT3] and [LT2].

In this paper, we shall investigate spaces with a σ -compact-finite k -network and around these spaces, and their applications.

All spaces are regular and T_1 , and maps are continuous and onto.

Definitions Let X be a topological space, and let $\mathcal{W}$ be a collection of subsets of X . We recall that $\mathcal{W}$ is compact-finite (star-countable) if for compact subset $K \subset X$ ($Q \in \mathcal{W}$), meets at most finitely (countably) many $P \in \mathcal{W}$. Let

$\mathcal{P}$ be a cover of X . Then $\mathcal{P}$ is called a k -network for X , if whenever $K \subset U$ with K compact and U open, then $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. Also, $\mathcal{P}$ is called a cs^* -network (cs -network) if whenever L is a sequence converging to a point $x \in X$ such that $x \in U$ with U open in X , then there exists $P \in \mathcal{P}$ such that $x \in P$, and

P contains a subsequence of L (L is eventually in P). If X has a σ -locally finite k -network (countable k -network), then X is called $\mathfrak{K}$ -space ($\mathfrak{K}_0$ -space).

Main Results

Theorem 1. Each of the following implies that Y has a σ -compact-finite k -network.

- (a) Y has a star-countable k -network.
- (b) Y has a σ -hereditarily closure-preserving k -network.
- (c) Y is dominated by spaces with a σ -compact-finite k -network.
- (d) Y is the closed image of a space X with a σ -compact-finite k -network, and one of the following properties holds.

- (1) X is a k -space.
- (2) X is a space with G_δ -points.
- (3) X is a normal, isocompact space.
- (4) X is realcompact.
- (5) Each $\partial f^{-1}(y)$ is Lindelöf.

Theorem 2. (CH) Let X be a k -space with a σ -compact-finite k -network. Then X is the topological sum of $\mathfrak{K}_0$ -spaces iff X is locally separable.

Theorem 3. (1) Let X be a k -space with a σ -compact-finite k -network. Then X has a star-countable k -network iff every metric closed subset of X is locally separable.

(2) Let X be a sequential space with a σ -compact-finite cs^* -network. Then X is the topological sum of $\mathfrak{K}_0$ -spaces iff every metric closed subset of X is locally separable.

Theorem 4. (1) Suppose that X is determined by a point-countable cover of locally separable metric subsets. If X has a σ -compact-finite k -network, then X has a star-countable k -network.

(2) Suppose that X is determined by a point-countable closed cover of locally separable metric subsets. If X has a point-countable cs-network, then X is a locally $\mathfrak{K}_0$ -space.

Theorem 5. (1) Let X be a separable space. Then each of the following implies that X is an $\mathfrak{K}_0$ -space.

(a) X is a Fréchet space with a point-countable k-network [GMT].

(b) (CH) X is a k-space with a σ -compact-finite k-network. (If X is meta-Lindelöf, or $\chi(X) \leq \omega_1$, (CH) can be omitted).

(2) Let X be a cosmic space (i.e space with a countable network). If X has a point-countable cs-network, then X is an $\mathfrak{K}_0$ -space.

We recall canonical spaces S_{ω_1} , S_ω , and S_2 . S_ω is called the sequential fan, and S_2 is the Arens' space. S_{ω_1} ; S_ω is respectively the space obtained from the topological sum of ω_1 ; ω many convergent sequences by identifying all limit points to a single point.

Theorem 6. Let X be a k-space with a σ -compact-finite k-network. Then X is the quotient s-image of a metric space iff X contains no closed copy of S_{ω_1} .

Theorem 7. (CH) Let X be a k-space with a σ -compact-finite k-network, Then X is weakly first countable iff X contains no closed copy of S_ω .

Theorem 8. (CH) Let X be a k-space with a σ -compact-finite k-network. Then X is a Lašnev space iff X contains no closed copy of S_2 .

References

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