

A duality for quasi-Gorenstein singularities

渡辺公夫 (筑波大学 数学系)

Kimio Watanabe

Introduction

Let (X, x) be a germ of an n -dimensional normal isolated singularity, i.e., X is an n -dimensional normal Stein space and a point x is the only singularity of X . Let $\pi : (M, E) \rightarrow (X, x)$ be a resolution of the singularity, where $E = \pi^{-1}(x)$. Then for $1 \leq i \leq n-1$, $\dim_{\mathbb{C}} (R^i \pi_* \mathcal{O}_M)_x$ is finite and is independent of the choice of the resolution (for example, see Yau [Y, Theorem 2.6, p.434]). We write $\dim_{\mathbb{C}} (R^i \pi_* \mathcal{O}_M)_x = h^i(X, x)$ for $1 \leq i \leq n-2$ and define the geometric genus of (X, x) to be $p_g(X, x) = \dim_{\mathbb{C}} (R^{n-1} \pi_* \mathcal{O}_M)_x$.

The analytic local ring $\mathcal{O}_{X, x}$ is Cohen-Macaulay if and only if $h^i(X, x) = 0$ for $1 \leq i \leq n-2$. The analytic local ring $\mathcal{O}_{X, x}$ is Gorenstein if and only if it is Cohen-Macaulay and quasi-Gorenstein, i.e., the canonical line bundle is trivial in a deleted neighborhood of x in $X - \{x\}$ (see [H0, Theorem 1.6, p.421]).

The purpose of this paper is to show the following theorems:

Theorem A. Suppose that (X, x) is a quasi-Gorenstein singularity. Then

- (i) $h^i(X, x) = h^{n-(i+1)}(X, x)$ for $1 \leq i \leq n-2$,
- (ii) If $n = 4m + 3$, then $h^{2m+1}(X, x)$ is even.

Theorem B. If (X, x) is a Gorenstein singularity of dimension $n = 2m + 1$, then

$$p_g(X, x) = T_n(c_1, c_2, \dots, c_{n-1})[M],$$

where $T_n \in \mathbb{Q}[c_1, \dots, c_n]$ is the n -th Todd polynomial and $T_n[M]$ is the value of $T_n(c_1, \dots, c_n)$ on the fundamental class $[M] \in H_{2n}(M, \partial M)$.

Theorem B is proved by Looijenga [Lo, 4.1g, p.299] under the condition that the singularity (X, x) is smoothable.

The author gratefully acknowledges the encouragement and help of Professor Kei-ichi Watanabe during the investigation of these results. We would also like to thank Professor Masataka Tomari for pointing out to us that one can use a theorem of Hartshorne and Ogus [H-O, Theorem 1.1, p.417] to establish the first part of Theorem A. However, our proof is more direct and elementary.

(1.1) Let (X, x) be a germ of an n -dimensional normal isolated singularity. By a theorem of Artin [A], (X, x) can be realized as a Zariski open subset of a projective variety Y with x as its only singularity. Let $\pi : M \rightarrow X$ be a good resolution of the singular point. Then, in a natural manner, we get a desingularization $\rho : N \rightarrow Y$ of Y by letting N to be $(Y - \{x\}) \cup M$. Let $E = \pi^{-1}(\{x\})$ and denote by D_i ($i=1, \dots, r$) the irreducible components of E . These notations are used throughout the paper.

Note that M is a strongly pseudoconvex manifold and N is

a non-singular compactification of M . We may also assume that $N - M$ consists of non-singular divisors in normal crossing.

(1.2) Let D be a non-singular divisor of $M \subset N$, and let $d \in H^2(N, \mathbb{Z})$ be the cohomology class represented by the oriented $(2n-2)$ -cycle D . Denote by $[D]$ the line bundle defined by the integral divisor D . Then $c_1([D]) = d$.

The natural orientation of N defines an element of the $2n$ -dimensional integral homology group $H_{2n}(N, \mathbb{Z})$ called the fundamental cycle of N .

In general, following the notation in [H1], for $a = \sum_{k=0}^n a_k$ $\in H^*(N, \mathbb{C})$ with $a_k \in H^{2k}(N, \mathbb{C})$, we put

$$\kappa_n[a] = a_{2n},$$

$$\kappa_n(a) = a_{2n}[N] = \langle a_{2n}, [N] \rangle,$$

$[N]$ denoting the fundamental $2n$ -cycle of N .

Let $j : D \rightarrow N$ be the embedding of D in N , and $c_i \in H^{2i}(N, \mathbb{Z})$ be the Chern classes of N . Every product $c_{j_1} c_{j_2} \cdots c_{j_r}$ of weight $n-s = j_1 + j_2 + \cdots + j_r$ defines an integer $c_{j_1} c_{j_2} \cdots c_{j_r} d^s[N]$, which is equal to $\langle j^*(c_{j_1} c_{j_2} \cdots c_{j_r} d^{s-1}), [D] \rangle$ if $s \geq 1$.

Denote the complex analytic tangent bundles of N , D by T_N , T_D . There is an exact sequence

$$0 \rightarrow T_D \rightarrow T_N|_D \rightarrow [D]|_D \rightarrow 0,$$

so we have

$$j^*c(T_N) = c(j^*T_N) = c(T_N|_D) = c(T_D)(1 + j^*d).$$

(multiplicity of the total Chern class)

Then any $j^*c_j(N)$ can be represented by the Chern classes of D

and j^*d . Thus $(c_{j_1} c_{j_2} \cdots c_{j_r} d^s)[N]$ is independent of the choice of the affine model of (X, x) and their non-singular compactifications if $s \geq 1$.

(1.3) Let f be a two dimensional cohomology class of $H^2(N, \mathbb{Z})$. Define $T(N, f)$ by

$$T(N, f) = \left\langle \kappa_n \left[e^f \prod_{i=1}^n \frac{\gamma_i}{1 - e^{-\gamma_i}} \right], [N] \right\rangle.$$

This formula is to be understood as follows: There is a formal factorization

$$1 + c_1 x + \cdots + c_n x^n = (1 + \gamma_1 x) \cdots (1 + \gamma_n x),$$

where $c_i \in H^{2i}(N, \mathbb{Z})$ are the Chern classes of N . Consider the term of degree n in f and the γ_i of the expression in square brackets. It is a symmetric function in the γ_i and is therefore a polynomial in f and the c_i with rational coefficients. If the multiplication is interpreted as the cup product in $H^*(N, \mathbb{Z})$, this polynomial defines as an element of $H^{2n}(N, \mathbb{Z}) \otimes \mathbb{Q}$. The value of this element on the $2n$ -dimensional cycle of N determined by the natural orientation is denoted by $T(N, f)$.

(1.4) Following Laufer [L], we consider the sheaf cohomology with support at infinity. Let F be a line bundle on M . The sequence

$$0 \rightarrow \Gamma(M, \mathcal{O}(F)) \rightarrow \Gamma_\infty(M, \mathcal{O}(F)) \rightarrow H_C^1(M, \mathcal{O}(F)) \rightarrow \cdots$$

is exact. By Siu [Si], p. 374, any section of F defined near the boundary of M has an analytic continuation to $M - E$. Therefore there is a natural isomorphism $\Gamma_\infty(M, \mathcal{O}(F)) \simeq \Gamma(M - E, \mathcal{O}(F))$. By Hartshorne [H], p. 225, there exists an isomorphism:

$$H_C^1(M, \mathcal{O}(F)) \simeq H^{n-1}(M, \mathcal{O}(K-F))$$

where K denotes the line bundle determined by canonical divisors. Since M is strongly pseudoconvex, $H^1(M, \mathcal{O}(K-F))$ is a finite dimensional vector space. Hence by the inequality

$$\begin{aligned} \dim \Gamma(M-E, \mathcal{O}(F)) / \Gamma(M, \mathcal{O}(F)) &\leq \dim H_C^1(M, \mathcal{O}(F)) \\ &= \dim H^{n-1}(M, \mathcal{O}(K-F)), \end{aligned}$$

we have $\dim \Gamma(M-E, \mathcal{O}(F)) / \Gamma(M, \mathcal{O}(F)) \leq +\infty$. We define the Euler-Poincaré characteristic $\chi(M, \mathcal{O}(F))$ by

$$\begin{aligned} \chi(M, \mathcal{O}(F)) &= \dim \Gamma(M-E, \mathcal{O}(F)) / \Gamma(M, \mathcal{O}(F)) \\ &\quad \sum_{q=1}^{\infty} (-1)^q \dim H^q(M, \mathcal{O}(F)). \end{aligned}$$

Now we have the following theorem of Riemann-Roch type.

Theorem 1.5 ([W]). For any integral divisor D with the first Chern class d on M , the equality

$$\chi(M, \mathcal{O}([D])) - \chi(M, \mathcal{O}) = T(N) - T(N, d)$$

holds.

(2.1) Let (X, x) be a normal n -dimensional isolated singularity. The geometric genus $p_g(X, x)$ is defined to be the dimension of $\dim_{\mathbb{C}}(R^{n-1}\pi_*\mathcal{O}_M)_x$ where $\pi : M \rightarrow X$ is a resolution of the singularity.

Theorem 2.2 (Laufer-Yau[Y]). Let (X, x) be a normal n -dimensional isolated singularity. Suppose that x is the only singularity of X and X is a Stein space. Let $\pi : M \rightarrow X$ be a resolution of the singularity. Then

$$\dim H^{n-1}(M, \mathcal{O}) = \dim \Gamma(M - E, \mathcal{O}(K)) / \Gamma(M, \mathcal{O}(K))$$

where $E = \pi^{-1}(\{x\})$.

Definition 2.3. Let (X, x) be a normal isolated singularity. We say (X, x) is quasi-Gorenstein if there exists a holomorphic n -form ω defined on a deleted neighborhood of x , which is nowhere vanishing on this neighborhood.

(2.4) Assume that (X, x) is a quasi-Gorenstein singularity. Then there exists a nowhere vanishing holomorphic n -form ω defined on $X - \{x\}$. Let K_∞ be the part of the divisor of $\pi^*\omega$ on N which is supported on $N - M$. Then $(\omega) \sim K + K_\infty$. Let $k, k_\infty \in H^2(N, \mathbb{Z})$ be the cohomology class represented by the cycle K, K_∞ respectively.

(3.1) Let $\{T_k(c_1, \dots, c_k)\}$ be the multiplicative sequence with characteristic power series

$$Q(x) = \frac{x}{1 - e^{-x}}.$$

The polynomial T_k are called Todd polynomial. For small n ,

$$\begin{aligned} T_1 &= \frac{1}{2}c_1, \\ T_2 &= \frac{1}{12}(c_2 + c_1^2), \\ T_3 &= \frac{1}{24}c_1c_2. \end{aligned}$$

Lemma 3.2. Let n be a positive integer, then

$$\sum_{k=0}^{n-1} \frac{(-c_1)^{n-k}}{(n-k)!} T_k(c_1, \dots, c_k) = \{(-1)^{n-1}\} T_n(c_1, \dots, c_n).$$

Proof.

$$\begin{aligned}
\sum_{k=0}^n \frac{(-c_1)^{n-k}}{(n-k)!} T_k(c_1, \dots, c_n) &= \kappa_n \left[e^{-c_1} \prod_{i=1}^n \frac{\gamma_i}{1 - e^{-\gamma_i}} \right] \\
&= \kappa_n \left[e^{-(\gamma_1 + \dots + \gamma_n)} \prod_{i=1}^n \frac{\gamma_i}{1 - e^{-\gamma_i}} \right] = \kappa_n \left[\prod_{i=1}^n \frac{\gamma_i}{e^{\gamma_i} - 1} \right] \\
&= \kappa_n \left[\prod_{i=1}^n \frac{-\gamma_i}{1 - e^{-(-\gamma_i)}} \right] = T_n(-c_1, c_2, \dots, (-1)^1 c_1, \dots, (-1)^n c_n) \\
&= (-1)^n T_n(c_1, \dots, c_n)
\end{aligned}$$

Corollary 3.3 ([Hi]). $T_k(c_1, \dots, c_n)$ is divisible by c_1 for k odd.

Lemma 3.4. $T(N) - T(N, k) = \{1 - (-1)^n\} T_n(-k, c_2, \dots, c_n)[N]$.

Proof. By definition $T(N) = T_n(c_1, \dots, c_n)[N]$ and $T(N, k) = \langle \kappa_n \left[e^k \prod_{i=1}^n \frac{\gamma_i}{1 - e^{-\gamma_i}} \right], [N] \rangle$, and hence it suffices to show

$$\begin{aligned}
T_n(c_1, \dots, c_n) - \kappa_n \left[e^k \prod_{i=1}^n \frac{\gamma_i}{1 - e^{-\gamma_i}} \right] \\
&= T_n(c_1, \dots, c_n) - \sum_{j=0}^n \frac{k^{n-j}}{(n-j)!} T_j(c_1, \dots, c_j) \\
&= - \sum_{j=0}^{n-1} \frac{k^{n-j}}{(n-j)!} T_j(c_1, \dots, c_j) \\
&= - \sum_{j=0}^{n-1} \frac{k^{n-j}}{(n-j)!} T_j(-k - k_\infty, \dots, c_j) \\
&= - \sum_{j=0}^{n-1} \frac{k^{n-j}}{(n-j)!} T_j(-k, \dots, c_j) \quad [k \cdot k_\infty = 0] \\
&= \{1 - (-1)^n\} T_n(-k, \dots, c_n).
\end{aligned}$$

(4.1) From Lemma 3.4, applying Theorem 1.5 to the case $[D] = K$, we have the following:

Corollary 4.2. Let (X, x) be a normal isolated singularity of dimension n . If (X, x) is quasi-Gorenstein, then

$$\begin{aligned} & \{1 - (-1)^n\} \{p_g(X, x) - T_n(-k, c_2, \dots, c_{n-1})[N]\} \\ & = h^1(X, x) - h^2(X, x) + \dots + (-1)^{n-1} h^{n-2}(X, x). \end{aligned}$$

Proof. $\chi(M, K) - \chi(M) =$

$$\begin{aligned} & = p_g(X, x) - \{h^1(X, x) - h^2(X, x) + \dots + (-1)^n h^{n-1}(X, x)\} \\ & = \{1 - (-1)^n\} p_g(X, x) - \{h^1(X, x) - h^2(X, x) + \dots + (-1)^{n-1} h^{n-2}(X, x)\} \end{aligned}$$

On the other hand, from Lemma 3.4

$$T(N) - T(N, k) = \{1 - (-1)^n\} T_n(-k, \dots, c_n)[N].$$

Hence we obtain the corollary by Theorem 1.5.

Corollary 4.3. Let (X, x) be a normal isolated singularity of dimension $2m + 1$. If (X, x) is quasi-Gorenstein, then

$$\begin{aligned} & 2\{p_g(X, x) - T_{2m+1}(-k, c_2, \dots, c_{2m}, c_{2m+1})[N]\} \\ & = h^1(X, x) - h^2(X, x) + \dots + h^{2m-1}(X, x). \end{aligned}$$

Corollary 4.4. Let (X, x) be a normal isolated singularity of odd dimension. If (X, x) is Gorenstein, then

$$p_g(X, x) = T_n(-k, c_2, \dots, c_{n-1})[N].$$

Proof. As is well known, $h^i(X, x) = 0$ for $1 \leq i \leq n-1$; [Ya, Theorem 2.6, p.434].

Corollary 4.5. Let (X, x) be a normal isolated singularity of even dimension. If (X, x) is quasi-Gorenstein, then

$$h^1(X, x) - h^2(X, x) + \dots - h^{n-2}(X, x) = 0,$$

i.e., $\chi(M, \mathcal{O}) = h^{n-1}(X, x) = p_g(X, x)$.

Corollary 4.6. Let (X, x) be a normal isolated singularity of dimension 3. If (X, x) is quasi-Gorenstein, then

$$2\{ p_g(X, x) - \frac{-k \cdot c_2}{24}[N] \} = h^1(X, x),$$

i.e., the dimension of the second local cohomology group of $\mathcal{O}_{X, x}$ is even.

Corollary 4.7. Let (X, x) be a normal isolated singularity of dimension 4. If (X, x) is quasi-Gorenstein, then

$$h^1(x, x) = h^2(X, x).$$

Remark 4.8. A quasi-homogeneous cone over a three dimensional abelian variety satisfies the condition of this Corollary.

Theorem 5.1. If (X, x) is a quasi-Gorenstein normal isolated singularity of dimension n , then $h^i(X, x) = h^{n-(i+1)}(X, x)$.

Proof. Let $\pi : M \rightarrow X$ be a resolution of the singularity. By (2,2) of [La], we have the following exact sequence

$$\begin{aligned} H^1(M, \mathcal{O}(K)) &\rightarrow H_{\infty}^1(M, \mathcal{O}(K)) \rightarrow \\ H_C^2(M, \mathcal{O}(K)) &\rightarrow H^2(M, \mathcal{O}(K)) \rightarrow \dots \\ &\dots \rightarrow H_{\infty}^{n-2}(M, \mathcal{O}(K)) \rightarrow \\ H_C^{n-1}(M, \mathcal{O}(K)) &\rightarrow H^{n-1}(M, \mathcal{O}(K)) . \end{aligned}$$

By the vanishing theorem of Grauert-Riemenschneider, $H^1(M, \mathcal{O}(K))$

= 0 for $i \geq 1$. Therefore

$$H_{\omega}^i(M, \mathcal{O}(K)) \simeq H_C^{i+1}(M, \mathcal{O}(K)) \quad \text{for } 1 \leq i \leq n-2.$$

Since $H^i(M, \mathcal{O})$ is Serre dual to $H_C^{n-i}(M, \mathcal{O}(K))$,

$$H_{\omega}^i(M, \mathcal{O}(K)) \simeq \left(H^{n-(i+1)}(M, \mathcal{O}) \right)^* \quad \text{for } 1 \leq i \leq n-1.$$

Consider the following exact sequence

$$H_C^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}) \longrightarrow H_{\omega}^1(M, \mathcal{O}) \longrightarrow$$

... ..

$$\dots \longrightarrow H^{n-2}(M, \mathcal{O}) \longrightarrow H_{\omega}^{n-2}(M, \mathcal{O}) \longrightarrow H_C^{n-1}(M, \mathcal{O}) .$$

By Serre duality, we know that $H_C^i(M, \mathcal{O})$ is the strong dual of $H^{n-i}(M, \mathcal{O}(K))$ which is zero by the vanishing theorem of Grauert-Riemenschneider for $i \neq n$. So $H_C^i(M, \mathcal{O}) = 0$ for $i \neq n$. It follows that

$$H^i(M, \mathcal{O}) \simeq H_{\omega}^i(M, \mathcal{O}) \quad \text{for } 1 \leq i \leq n-1 .$$

Since the singularity is quasi-Gorenstein, there exists a holomorphic n -form ω defined on a deleted neighborhood of $x \in X$, which is nowhere vanishing on this neighborhood. Cupping, or wedging, with $\tilde{\omega} = \pi^* \omega$, we have a morphism

$$H_{\omega}^i(M, \mathcal{O}) \longrightarrow H_{\omega}^i(M, \mathcal{O}(K)).$$

The morphism is an isomorphism, since "at ω " $\tilde{\omega} = \omega$ doesn't vanish. Therefore $h^i(X, x) = h^{n-(i+1)}(X, x)$.

Proposition 5.2. If (X, x) is a quasi-Gorenstein singularity of dimension $n = 4m + 3$, then $T_n(-k, c_2, \dots, c_n)[N]$ is an integer.

Proof. By Corollary 4.3

$$2\{p_g(X, x) - T_n(-k, c_2, \dots, c_n)[N]\} = h^1 - h^2 + \dots + h^{4m+1}.$$

Then, since $h^i = h^{4m+1}$,

$$2\{p_g(X, x) - T_n(-k, c_2, \dots, c_n)[N]\} \\ = 2\{h^1 - h^2 + \dots - h^{2m}\} + h^{2m+1}$$

On the other hand, $H^{2m+1}(M, \mathcal{O})$ has a non-degenerate skew-symmetric bilinear form, then the dimension of $H^{2m+1}(M, \mathcal{O})$ is even. Thus the number $T_n(-k, c_2, \dots, c_n)[N]$ is an integer.

References

- [A] M. Artin, Algebraic approximation of structure over complete local ring. Publ. Math. I.H.E.S. 36 (1969)
- [G] A. Grothendieck, Local Cohomology, Lecture Notes in Mathematics 41, Springer, Berlin, Heidelberg, New York, 1967
- [Ha] R. Hartshorne, Ample subvarieties of algebraic varieties. Berlin, Heidelberg, New York:Springer 1970
- [Hi] F. Hirzebruch, Topological Methods in Algebraic Geometry. Grundlehren 131, Springer-Verlag, Heidelberg (3rd ed., 1966)
- [K] M. Kato, Riemann-Roch theorem for strongly pseudoconvex manifolds of dimension 2. Math. Ann. 222, 243-250 (1976)
- [K-M-M] Y. Kawamata, K. Matsuda, K. Matsuki, Minimal Model Problem. Advanced Studies in Pure Mathematics 10, 1987 Algebraic Geometry, Sendai, 1985.
- [La] H. B. Laufer, On rational singularities. Amer. J. Math. 94, 597-608 (1972)
- [Lo] E. Looijenga, Riemann-Roch and smoothings of singularities, Topology Vol. 25, No. 3, pp. 293-302, 1986
- [Sa] F. Sakai, Kodaira dimension of complements of divisors. Complex analysis and algebraic geometry: a collection of papers dedicated to K. Kodaira. Tokyo: Iwanami 1977

- [Si] Y. T. Siu, Absolute Gap-sheaves and extensions of coherent analytic sheaves. Trans. Amer. Math. Soc. **141**, 361-376 (1969)
- [W] K. Watanabe, On Plurigenera of Normal Isolated Singularities, I. Math. Ann. **250**, 65-94 (1980)
- [Ya] S.S.-T. Yau, Sheaf cohomology on 1-convex manifolds. Recent developements in several complex variables. Annals of mathematics studies; 100.

Kimio Watanabe
Institute of Mathematics
University of Tsukuba
Ibaraki, 305
Japan