

Stokes geometry for the quantized Hénon map

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半古典論の有効性は、カオス系の量子古典対応を議論する際の前提条件である。しかしながら、プランク定数を small parameter とする漸近展開は、そもそもが発散級数であることからその誤差評価等が難しいばかりか、適用・意味付けにおいて様々な問題点を抱える。ここでは、エノン写像を例に、完全 WKB 解析の観点からカオス系の半古典論を再構築する試みの一端を紹介する。

Our motivations to investigate Stokes geometry of the quantized Hénon map are; (1) to establish the theory of *chaotic tunneling* [1, 2], (2) to clarify *orbit correlations* in complex semiclassical treatment of chaotic systems. The Hénon map is a polynomial diffeomorphism given by:

$$f_a : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ y^2 - x + a \end{pmatrix}. \quad (1)$$

An alternative familiar form is obtained by making an affine change of variables as $(p, q) = (y - x, x - 1)$ together with a parameter $c = 1 - a$,

$$F_c : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q + p \\ p - V'(q + p) \end{pmatrix}. \quad (2)$$

Here the potential function $V(q)$ is given as $V(q) = -q^3/3 - cq$.

A standard recipe to formulate quantum mechanics of the symplectic mapping is first to construct the unitary operator generating the time evolution of quantum states. This is achieved by introducing discrete analog of the Feynman-type path integral:

$$I(q_n) \equiv \langle q_n | U^n | q_0 \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dq_1 dq_2 \cdots dq_{n-1} \exp \left[\frac{i}{\hbar} S(q_0, q_1, \dots, q_n) \right]. \quad (3)$$

Here we take the coordinate representation. The function $S(q_0, q_1, \dots, q_n)$ represents the discretized Lagrangian or the action functional given by

$$S(q_0, q_1, \dots, q_n) = \sum_{j=0}^{n-1} \frac{1}{2} (q_{j+1} - q_j)^2 - \sum_{j=1}^n V(q_j). \quad (4)$$

The action functional is derived so that applying the variational principle to $S(q_0, q_1, \dots, q_n)$ generates the symplectic map (2). In fact, we can easily see that the condition $\partial S / \partial q_j = 0$, $(1 \leq j \leq n-1)$ yields the classical map (2).

A usual (complex) semiclassical scheme is just to take the leading order contribution in evaluating the multiple integral $\langle q_n | U^n | q_0 \rangle$ by the stationary phase(or

saddle point) method. The resulting semiclassical formula is expressed as a sum over contributions of classical trajectories connecting the initial and final states:

$$\langle q_n | U^n | q_0 \rangle \approx \sum_{\gamma} A_{\gamma}(q_n, q_0) \exp\left\{\frac{i}{\hbar} S_{\gamma}(q_n, q_0) + i\mu_{\gamma} \frac{\pi}{2}\right\}, \quad (5)$$

where $A_{\gamma}(q_n, q_0)$ stands for the amplitude factor associated with quantum fluctuation around each classical path γ . $S_{\gamma}(q_n, q_0)$ is given by putting the data of the corresponding classical path γ into the action functional $S(q_0, q_1, \dots, q_n)$, and μ_{γ} represents the Maslov index. The summation is taken over such classical orbits that are located initially on the manifold $q_0 = \alpha$, and finally on $q_n = \beta$, where both α and β should take real values since they are observables in the representation under consideration.

We remark that even if there exist no *real orbits* connecting the initial and final manifolds $q_0 = \alpha$ and $q_n = \beta$, we always have *complex orbits*, which appear as saddle point solutions of $\partial S / \partial q_j = 0$, ($1 \leq j \leq n-1$) with the conditions $\alpha, \beta \in \mathbb{R}$. Physically, such complex orbits can be and should be regarded as *tunneling orbits* since the transition between initial and final manifolds is forbidden within *real* classical orbits. This type of tunneling transition is often called *dynamical tunneling* in the literature [3].

A special advantage to employ the Hénon map is that the theory of complex dynamical systems has well been developed for polynomial diffeomorphisms [4]. This is important because the saddle point solutions of the quantum propagator are just the classical trajectories in the complex plane, especially tunneling transitions are in question. It is thus crucial to know the nature of complex classical dynamics, and also in this respect, the Hénon map would be most suitable.

On the other hand, as is well known, in applying the saddle point method, one must take into account *Stokes phenomena*, that is, not all the saddle point solutions (=complex classical orbits) do not contribute to the final semiclassical superposition (5), but only the solutions controlled by the connection through Stokes phenomena do so. The aim of our work is to find a recipe to introduce proper Stokes geometry to the quantum propagator (3) of the Hénon map and how it should be treated.

The Stokes phenomenon for the 1-step quantum propagator is almost trivial, since the single integral $I(q_2)$ can be transformed into a canonical form of the Airy integral by an appropriate change of variables. For $n \geq 3$, the object we have to analyze is multiple integrals. As easily anticipated, difficulties to understand Stokes phenomena in multiple integrals much escalate. Recent progresses in the exact WKB analysis, however, provide us promising approach to such issues [5]. In particular, we should mention the work by Aoki, Kawai and Takei [6], in which they have provided a prescription to analyze Stokes phenomena in higher-order differential equations, say $P(x, \eta^{-1}d/dx)\psi(x) = 0$, within the exact WKB framework [6]. Their work contains not only a mathematical justification of the preceding work [7] in which *new Stokes curves* should be introduced in order to recover the univaluedness around crossing points of ordinary Stokes curves in an ad-hoc way, but also claims that *virtual turning points* (they are originally called new turning points in [6]) should first be taken into account to construct complete Stokes geometry. They also clarified that new Stokes curves play essentially the same part in the Stokes geometry.

The argument of [6] starts with defining virtual turning points as self-intersection points of bicharacteristic curves for the Borel transformed differential equation. Here self-intersection points are obtained by projecting full bicharacteristic curves onto (x, y) plane, where the variables y denotes the variable dual to a large parameter η . Then new Stokes curves are defined as the ones emanating from virtual turning points.

The first task in our multiple-integrals was to establish the definition of turning points and Stokes curves, which are the most relevant ingredients to construct the Stokes geometry. In order to do this, we have connected our problem to the treatment of Stokes phenomena in higher-order differential equations by deriving differential operators acting on our multiple integrals. We can show, though not explicitly presented here, that deriving differential equations is equivalent to solving an initial value problem of the Hénon map [8]. Virtual turning points as well as new Stokes curves were then introduced.

This looks somewhat a redundant way, since in usual cases integral representations carry much more information than differential equations. However, in multiple integrals, even if they certainly take integral forms it is not trivial at all to see how the saddle point method should be applied or in which co-dimensional space Stokes phenomena occurs. In particular, although what we need is to know how the connection occurs in $I(q_n)$ as a function of q_n , little is known about how the Stokes geometry should be constructed in such a situation.

With these settings, Stokes graphs for 2 and 3 step Hénon map propagators were drawn, and the corresponding Stokes geometry was discussed under the principle of the univaluedness condition on given Stokes graphs [8]. Several concrete examples provide unique geometry although, in principle, arbitrary combinations of turning points and Stokes lines do not necessarily fix Stokes geometry uniquely.

In order to verify that resulting Stokes geometry is correct, hyperasymptotic expansions were considered [8]. A set of algebraic equations with controllable errors was used to explore the Riemann sheet structure of the Borel transform or equivalently adjacency relation of saddles of our multiple integrals [9]. The results were entirely consistent with those determined through the univaluedness condition.

We should mention that our study is still a first step towards our final goal, and a lot of significant but unsolved problems remain. The most urgent issue would be to make clear the relation between classical dynamics generating chaos and the corresponding Stokes geometry. In the classical side, stretching and folding is a key mechanism generating chaos, and the number of folding points of the Lagrangian manifold increases exponentially as a function of time. The folding points of the Lagrangian manifold manifest themselves as the turning points in Stokes geometry. Three Stokes curves emanate from a simple turning point, and local Stokes geometry in the vicinity of folding points can be well understood within conventional arguments. However, because of the crossing of Stokes curves and resulting complicated Stokes graphs, we cannot avoid taking into account global aspects of geometry, and it is difficult to give an intuitive mapping relation connecting the structure of Lagrangian manifolds and the corresponding Stokes geometry.

In order to go beyond it, analyzing Stokes geometry in the anti-integrable limit would

be the first target to be investigated. If the situation for the anti-integrable limit is understood, one way to step forward is to trace Stokes geometry as a function of the system parameter, and focus on bifurcation phenomena [10, 8]. In the horseshoe case, all the turning points are located on the real plane, but as the nonlinear parameter c decreases, some of them fall into purely imaginary plane. Such an event occurs as a result of coalescence of turning points. If we know how the Stokes graph changes when such a bifurcation phenomenon occurs, the Stokes geometry in a generic parameter value can be traced from the anti-integrable limit in principle. This is exactly the same strategy to study the *pruning* of the horseshoe structure.

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