数理解析研究所講究録
第 1807 巻 2012 年 14-20



# Dynamics and weights of polynomial skew products on $\mathbb{C}^2$

Kohei Ueno（上野康平）
Toba National College of Maritime Technology

**Abstract**

We study the dynamics of polynomial skew products on $\mathbb{C}^2$. By using suitable weights, we prove the existence of several types of Green functions. Largely, continuity and plurisubharmonicity follow. Moreover, it relates to the dynamics of the rational extensions to weighted projective spaces.

## 1 Introduction

A polynomial skew product on $\mathbb{C}^2$ is a polynomial map of the form $f(z,w) = (p(z), q(z,w))$ such that $p(z) = az^\delta + O(z^{\delta-1})$ and $q(z,w) = b(z)w^d + O_z(w^{d-1})$. Let $\gamma = \deg b$. We assume that $\delta \geq 2$ and $d \geq 2$. Then we may assume that polynomials $p$ and $b$ are monic by taking an affine conjugate; $p(z) = z^\delta + O(z^{\delta-1})$ and $b(z) = z^\gamma + O(z^{\gamma-1})$. Let $\lambda = \max\{\delta, d\}$, which coincides with the dynamical degree of $f$.

The dynamics of $f$ consists of the dynamics on the base space and the dynamics on the fibers. The first component $p$ defines the dynamics on the base space $\mathbb{C}$. Note that $f$ preserves the set of vertical lines in $\mathbb{C}^2$. For this reason, we often use the notation $q_z(w)$ instead of $q(z,w)$. The restriction of $f^n$ to the vertical line $\{z\} \times \mathbb{C}$ can be viewed as the composition of $n$ polynomials on $\mathbb{C}$, $q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z$.

A useful tool in the study of the dynamics of $p$ on the base space is the Green function $G_p$ of $p$,

$$G_p(z) = \lim_{n \to \infty} \frac{1}{\delta^n} \log^+ |p^n(z)|.$$

It is well known that $G_p$ is defined, continuous and subharmonic on $\mathbb{C}$. More precisely, $G_p$ is harmonic and positive on $A_p$ and zero on $K_p$, and $G_p(z) = \log|z| + o(1)$ as $z \to \infty$. Here $A_p = \{z : p^n(z) \to \infty\}$ and $K_p = \{z : \{p^n(z)\}_{n\geq 1} \text{ bounded}\}$. By definition, $G_p(p(z)) = \delta G_p(z)$. In a similar fashion, we consider the fiberwise Green function of $f$,

$$G_z(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |Q_z^n(w)| \text{ or } G_z^\lambda(w) = \lim_{n\to\infty} \frac{1}{\lambda^n} \log^+ |Q_z^n(w)|,$$

where $Q_z^n = q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z$. By definition, $G_{p(z)}(q_z(w)) = dG_z(w)$ if it exists. Since $G_p$ exists on $\mathbb{C}$, the existence of $G_z$ implies that of the Green function $G_f$ of $f$,

$$G_f(z,w) = \lim_{n\to\infty} \frac{1}{\lambda^n} \log^+ |f^n(z,w)|,$$

where $|(z,w)| = \max\{|z|, |w|\}$. Favre and Guedj proved the existence of $G_z$ on $K_p \times \mathbb{C}$ in [1, Theorem 6.1], which is continuous and plurisubharmonic if $b^{-1}(0) \cap K_p = \emptyset$. Hence the remaining problem lies in investigating the existence of $G_z$ on $A_p \times \mathbb{C}$. In [2], with the assumption $\gamma = 0$, we studied the existence of $G_z$ and concluded that the weighted Green function $G_f^\alpha$ of $f$,

$$G_f^\alpha(z,w) = \lim_{n\to\infty} \frac{1}{\lambda^n} \log^+ |f^n(z,w)|_\alpha,$$

where $|(z,w)|_\alpha = \max\{|z|^\alpha, |w|\}$, is defined, continuous and plurisubharmonic on $\mathbb{C}^2$ for a suitable rational number $\alpha \geq 0$. Moreover, $f$ extends to an algebraically stable map on a weighted projective space, whose dynamics relates to $G_f^\alpha$.

In this report, assuming $\gamma \neq 0$, we investigate the existence of $G_z$ on $A_p \times \mathbb{C}$, which implies the existence of $G_f$ and $G_f^\alpha$. Although the dynamics becomes much more difficult without the condition $\gamma = 0$, the idea of imposing suitable weights is still effective. We also show the existence of other Green functions such as

$$G_z^\alpha(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \left| \frac{Q_z^n(w)}{p^n(z)^\alpha} \right| \text{ and } G(z,w) = \lim_{n\to\infty} \frac{1}{d^n} \log \left| \frac{Q_z^n(w)}{p^n(z)^{n\frac{\gamma}{d}}} \right|$$

in the cases $\delta \neq d$ and $\delta = d$, respectively.

# 2 Weights

Let $q(z,w) = z^\gamma w^d + \sum_j c_j z^{n_j} w^{m_j}$, where $c_j \neq 0$, and define $\alpha$ as

- $\min\{l \in \mathbb{Q} : l\delta \geq \gamma + ld$ and $l\delta \geq n_j + lm_j$ for any $j\}$ if $\delta > d$,
- $\min\{l \in \mathbb{Q} : \gamma + ld \geq l\delta$ and $\gamma + ld \geq n_j + lm_j$ for any $j\}$ if $\delta < d$,
- $\inf\{l \in \mathbb{Q} : \gamma + ld \geq n_j + lm_j$ for any $j\}$ if $\delta = d$.

Since $q$ has only finitely many terms, we can take the minimum if $\delta \neq d$. In the case $\delta = d$, we can replace inf by min if $q(z,w) \neq b(z)w^d$, and $\alpha = -\infty$ if $q(z,w) = b(z)w^d$. Let $W_R = \{|z| > R, |w| > R|z|^\alpha\}$. We then obtain the following main lemma from the definition of $\alpha$.

**Lemma 2.1.** *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, or $\delta \leq d$, then $q(z,w) \sim z^\gamma w^d$ on $W_R$ for large $R > 0$; that is, the ratio of $q$ and $z^\gamma w^d$ on $W_R$ tends to $1$ as $R \to \infty$. Moreover, $f$ preserves $W_R$; that is, $f(W_R) \subset W_R$.*

Since $p(z) \sim z^\delta$ as $z \to \infty$, this lemma implies that $f(z,w) \sim (z^\delta, z^\gamma w^d)$ on $W_R$ as $R \to \infty$. Let $A_f = \cup_{n \geq 0} f^{-n}(W_R)$. This lemma induces the existence, continuity and pluriharmonicity of the Green functions on $A_f$; the results are written in the next section.

We explain the importance of $\alpha$ and Lemma 2.1 in terms of the weighted homogeneous part of $q$. Let us define the weight of a monomial $z^n w^m$ as $n + \alpha m$, and let $h$ be the weighted homogeneous part of $q$ of highest weight. In the case $\delta > d$, the polynomial $h$ may not contain $z^\gamma w^d$. However, if $\alpha = \gamma/(\delta - d)$, then $h$ contains $z^\gamma w^d$. On the other hand, $h$ always contains $z^\gamma w^d$ in the case $\delta \leq d$.

In addition, it is useful to consider the dynamics of the rational extensions of $f$ to weighted projective spaces. Let $r$ and $s$ be any two positive integers. We denote a point in the weighted projective space $\mathbb{P}(r,s,1)$ by weighted homogeneous coordinates $[z : w : t]$. It follows that $f$ extends to a rational map $\tilde{f}$ on $\mathbb{P}(r,s,1)$,

$$\tilde{f}[z : w : t] = \left[ p\left(\frac{z}{t^r}\right) t^{\lambda r} : q\left(\frac{z}{t^r}, \frac{w}{t^s}\right) t^{\lambda s} : t^\lambda \right].$$

Let $L_\infty$ be the line at infinity $\{t = 0\}$, and let $I_{\tilde{f}}$ be the indeterminacy set of $\tilde{f}$. Because $\gamma \neq 0$, the point $p_\infty = [0 : 1 : 0]$ is always an indeterminacy point. In the case $\delta > d$, the point $p_\infty$ is the unique indeterminacy point, and $\tilde{f}$ is algebraically stable if $s/r \geq \alpha$. More precisely, if $s/r = \alpha$ then the dynamics on $L_\infty - \{p_\infty\}$ is induced by the polynomial $h(1, w)$, and if $s/r > \alpha$ then $\tilde{f}$ contracts $L_\infty - \{p_\infty\}$ to the attracting fixed point $[1 : 0 : 0]$. On the other hand, $\tilde{f}$ contracts $L_\infty - I_{\tilde{f}}$ to $p_\infty$ in the case $\delta \leq d$. Therefore, if $\delta > d$ and $\alpha = \gamma/(\delta - d)$, or if $\delta \leq d$, then $p_\infty$ is attracting in some sense. For these cases, $W_R$ is included in the attracting basin of $p_\infty$, and $A_f$ is the restriction of the attracting basin of $p_\infty$ to $A_p \times \mathbb{C}$.

# 3 Results on Green functions

Now we state our results on the existence, continuity and plurisubharmonicity of the Green functions of $f$. This section divides into three subsections: the cases $\delta > d$, $\delta < d$ and $\delta = d$. Since $\alpha$ can be negative unlike the case $\gamma = 0$, we redefine $|(z, w)|_\alpha$ as $\max\{|z|^{\max\{\alpha, 0\}}, |w|\}$. See [3] for the proofs and more details.

## 3.1 The case $\delta > d$

Observing the dynamics of $\tilde{f}$ on $\mathbb{P}(r, s, 1)$, where $s/r > \alpha$, we obtain the following upper estimate of $G_z^\lambda$.

**Proposition 3.1.** *If $\delta > d$, then $\tilde{G}_z^\lambda \leq \alpha G_p$ on $A_p \times \mathbb{C}$, where $\tilde{G}_z^\lambda = \limsup_{n \to \infty} \lambda^{-n} \log^+ |Q_z^n|$.*

**Corollary 3.2.** *If $\delta > d$, then $G_f^\alpha = \alpha G_p$ on $\mathbb{C}^2$.*

Now we apply Lemma 2.1 to show the existence of $G_z^\alpha$.

**Theorem 3.3.** *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, then the limit $G_z^\alpha$ is defined, continuous and pluriharmonic on $A_f$. Moreover,*

$$\left|G_z^\alpha(w) - \log\left|z^{-\alpha} w\right|\right| < C_R,$$

*where $C_R$ tends to $0$ as $R \to \infty$, $G_z^\alpha \sim \log|w|$ as $w \to \infty$ for fixed $z$ in $A_p$, and $G_z^\alpha$ tends to $0$ as $(z, w)$ in $A_f$ tends to $\partial A_f - J_p \times \mathbb{C}$.*

Let $B_f = A_p \times \mathbb{C} - A_f$. Since $G_z^\alpha = 0$ on $B_f$, this theorem implies the existence of $G_z^\alpha$ on $A_p \times \mathbb{C}$.

**Corollary 3.4.** *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, then $G_z^\alpha$ is defined, continuous and plurisubharmonic on $A_p \times \mathbb{C}$. Moreover, it is pluriharmonic on $A_f$ and int$B_f$.*

Theorem 3.3 also implies the existence of $G_z^\lambda$ and $G_f$ on $A_f$.

**Corollary 3.5.** *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, then $G_z^\lambda = \alpha G_p$ and $G_f = \max\{\alpha, 1\}G_p$ on $(K_p \times \mathbb{C}) \cup A_f$.*

We end this subsection with a claim on the uniform convergence to $G_z^\alpha$ and the asymptotics of $G_z^\alpha$ near infinity. Let $h(c) := h(1, c)$. Because $h$ and $G_h$ have some symmetries related to the denominator of $\alpha$, the Green function $G_h(z^{-\alpha}w)$ is well defined.

**Proposition 3.6.** *If $\delta > d$ and $\alpha = \gamma/(\delta - d)$, then the convergence to $G_z^\alpha$ is uniform on $V \times \mathbb{C}$, where $\overline{V} \subset A_p$, and $G_z^\alpha(w) = G_h(z^{-\alpha}w) + o(1)$ as $z \to \infty$.*

## 3.2 The case $\delta < d$

Lemma 2.1 induces the existence of $G_z$.

**Theorem 3.7.** *If $\delta < d$, then the limit $G_z$ is defined, continuous and pluriharmonic on $A_f$. Moreover,*

$$\left|G_z(w) - \log\left|z^{\gamma/(d-\delta)}w\right|\right| < C_R \text{ on } W_R,$$

*where $C_R$ tends to $0$ as $R \to \infty$, $G_z \sim \log|w|$ as $w \to \infty$ for fixed $z$ in $A_p$, and $G_z$ tends to $0$ as $(z, w)$ in $A_f$ tends to $\partial A_f - J_p \times \mathbb{C}$.*

Since $G_z = 0$ on $B_f$, this theorem implies the following corollary.

**Corollary 3.8.** *If $\delta < d$, then $G_z$ is defined on $\mathbb{C}^2$, which is continuous and plurisubharmonic on $A_p \times \mathbb{C}$. Moreover, $G_z = G_f = G_f^\alpha$ on $\mathbb{C}^2$, and $G_z = G_z^\alpha$ on $A_p \times \mathbb{C}$.*

The convergence to $G_z$ seems not to be uniform on $W_R$. However, one can prove that the convergence to $G_z^\alpha$ is uniform on $W_R$ if $\alpha = \gamma/(\delta - d)$.

**Proposition 3.9.** *If $\delta < d$ and $\alpha = \gamma/(\delta - d)$, then the convergence to $G_z^\alpha$ is uniform on $V \times \mathbb{C}$, where $\overline{V} \subset A_p$, and $G_z^\alpha(w) = G_h(z^{-\alpha}w) + o(1)$ as $z \to \infty$.*

## 3.3 The case $\delta = d$

The dynamics of $f$ differs depending on whether $\gamma = 0$ or $\gamma \neq 0$. See [2] for the case $\gamma = 0$; if $\delta = d$ and $\gamma = 0$ then $f$ extends to holomorphic maps on weighted projective spaces. Assuming $\gamma \neq 0$, we obtain the following three theorems from Lemma 2.1.

**Theorem 3.10.** *If $\delta = d$ and $\gamma \neq 0$, then $G_z = \infty$ on $A_f$ and $\tilde{G}_z \leq \max\{\alpha, 0\} G_p$ on $B_f$, where $\tilde{G}_z = \limsup_{n\to\infty} d^{-n} \log^+ |Q_z^n|$.*

**Corollary 3.11.** *If $\delta = d$ and $\gamma \neq 0$, then*

$$G_f^{\alpha}(z,w) = \begin{cases} \infty & \text{on } A_f \\ \max\{\alpha, 0\} G_p(z) & \text{on } B_f. \end{cases}$$

**Theorem 3.12.** *If $\delta = d$ and $\gamma \neq 0$, then*

$$\lim_{n\to\infty} \frac{1}{n\gamma d^{n-1}} \log^+ |f^n(z,w)| = \lim_{n\to\infty} \frac{1}{n\gamma d^{n-1}} \log^+ |Q_z^n(w)|$$

$$= \begin{cases} G_p(z) & \text{on } (K_p \times \mathbb{C}) \cup A_f \\ 0 & \text{on } B_f. \end{cases}$$

**Theorem 3.13.** *If $\delta = d$ and $\gamma \neq 0$, then the limit $G$ is defined, continuous and pluriharmonic on $A_f$. Moreover,*

$$|G(z,w) - \log|w|| < C_R \text{ on } W_R,$$

*where the constant $C_R > 0$ tends to $0$ as $R \to \infty$, $G \sim \log|w|$ as $w \to \infty$ for fixed $z$ in $A_p$, and $G$ tends to $-\infty$ as $(z,w)$ in $A_f$ tends to any point in $\partial A_f - J_p \times \mathbb{C}$.*

Since $G = -\infty$ on $B_f$, this theorem implies the following corollary.

**Corollary 3.14.** *If $\delta = d$ and $\gamma \neq 0$, then $G$ is defined and plurisubharmonic on $A_p \times \mathbb{C}$ if we admit minus infinity.*

# References


[1] C. Favre and V. Guedj, *Dynamique des applications rationnelles des espaces multiprojectifs*, Indiana Univ. Math. J. **50** (2001), 881-934.

[2] K. Ueno, *Weighted Green functions of nondegenerate polynomial skew products on* $\mathbb{C}^2$, Discrete Contin. Dyn. Syst. **31** (2011), 985-996.

[3] K. Ueno, *Green functions and weights of polynomial skew products on* $\mathbb{C}^2$, preprint.