

Carleson inequalities in weighted harmonic Bergman spaces, $0 < p \leq 1$

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ABSTRACT. We give a necessary and sufficient condition for positive measures μ and ν on the upper half-space of $\mathbb{R}^n$ to satisfy the inequality

$$\int |D^\alpha u|^p d\mu \leq C \int |D_y^n u|^p d\nu$$

for all u in a subclass of a harmonic Bergman space when $0 < p \leq 1$, $d\nu = \omega dV$, and ω satisfies a certain condition.

1. Introduction

Let H be the upper half-space of the n -dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$), that is, $H = \{z = (x, y) \in \mathbb{R}^n; y > 0\}$, where we have written a point $z \in \mathbb{R}^n$ as $z = (x, y)$ with $x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. For $0 < p < \infty$, let $b^p = b^p(H, dV)$ be the class of all harmonic functions u on H such that

$$\|u\|_p = \left(\int_H |u|^p dV \right)^{1/p} < \infty$$

where dV denotes the Lebesgue volume measure on H . The class b^p is called the harmonic Bergman space. Properties of functions in the harmonic Bergman space on the upper half-space were studied by Ramey and Yi [13] when $1 \leq p < \infty$, and by the author [15] when $0 < p \leq 1$.

Let μ and ν be σ -finite positive Borel measures on H . We consider conditions on μ and ν for which there exists a constant $C > 0$ such that $\int |u| d\mu \leq C \int |D_y u| d\nu$ for all u in a subclass of b^1 , where D_y denotes the differentiation operator with respect to y . (Our consideration is more general.) Such inequalities on the unit disk in the complex plane were studied by Stegenga, and multipliers of the Dirichlet space were characterized [14]. When $d\nu = (1 - |\zeta|)^r dA$ and $r \geq 1$, Stegenga proved that finite positive Borel measures μ and ν on the unit disk satisfy the inequality $\int |f|^2 d\mu \leq C \int |f'|^2 d\nu$ for all holomorphic functions f , $f(0) = 0$ if and only if there is a constant K such that $\mu(S_I) \leq K|I|^r$ for any interval I in the unit circle, where dA denotes the Lebesgue area measure, $|I|$ denotes the normalized arc length of I , and S_I is the corresponding Carleson square over I . It was also proved that when $0 \leq r < 1$ such measures are those satisfying $\mu(\cup S_{I_j}) \leq K \text{Cap}(\cup I_j)$ for all finite disjoint collections of intervals $\{I_j\}$, where Cap is an appropriate Bessel capacity (if $r < 0$ any finite Borel measure satisfies this

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inequality). It is known that these characterizations can be generalized to the case of $p > 1$ (see also [14]). When $0 < p \leq 1$, $d\nu = (1 - |\zeta|)^r dA$, and $-1 < r \leq p - 1$, Ahern and Jevtić [1] proved that there is a constant $C > 0$ such that $\int |f|^p d\mu \leq C \int |f|^p d\nu$ if and only if $\mu(S_I) \leq K|I|^{2-p+r}$. Using this result, Ahern and Jevtić characterized inner multipliers of the Besov space in case $0 < p \leq 1$. Such investigations for several variables are in [4]. In these investigations, when $p > 1$ necessary and sufficient conditions were not obtained completely. It was also shown that, in general, the above condition is not necessary. When $0 < p \leq 1$ and $d\nu = y^r dV$, such a inequality on the upper half-space was studied by author [15]. On the unit disk of the complex plane, for more general measures μ and ν , the properties of measures satisfying a inequality $\int |f|^p d\mu \leq C \int |f|^p d\nu$ were studied in [8], [9], and [12], and partial results were obtained.

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of nonnegative integers with order ℓ , then D^α denotes the partial differentiation operator $\partial^\ell / \partial x_1^{\alpha_1} \dots \partial x_{n-1}^{\alpha_{n-1}} \partial y^{\alpha_n}$. We now state our main result in this paper.

THEOREM 1. *Let $0 < p \leq 1$ and ℓ, m be nonnegative integers. Suppose that μ is a σ -finite positive Borel measure on H , $d\nu = \omega dV$ and ω satisfies the $(A_q)_\partial$ -condition for some $1 < q < \infty$. Then, the following (1) $\sim$ (3) are equivalent.*

(1) *There is a constant $C > 0$ such that*

$$\int_H |D^\alpha u|^p d\mu \leq C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$ and multi-indices α of order ℓ ,

(2) *There is a constant $C > 0$ such that*

$$\int_H |D_y^\ell u|^p d\mu \leq C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$.

(3) *There are constants $K > 0$ and $0 < \varepsilon < 1$ such that $\mu(S(w)) \leq K t^{(\ell-m)p} \nu(D_\varepsilon(w))$ for all $w = (s, t) \in H$.*

In §2, we give some lemmas for investigations of Theorem 1. In §3, the necessity of the condition is shown. In §4, we define the notion of the $(A_p)_\partial$ -condition on the upper half-space, and study some properties of the $(A_p)_\partial$ -condition. The $(A_p)_\partial$ -condition on the unit disk of the complex plane is defined in [12]. In the definition of the $(A_p)_\partial$ -condition on the unit disk, the normalized reproducing kernel in the Bergman space is used. However, on the upper half-space of $\mathbb{R}^n$, we can not use arguments in the complex plane. Therefore, we will extend the notion of the $(A_p)_\partial$ -condition using another function. In §5, the sufficiency of the condition is contained.

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

2. Preliminary lemmas

Recall that a point $z \in H$ will be written as $z = (x, y)$ with $x \in \mathbb{R}^{n-1}$ and $y > 0$. We use the absolute value symbol $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$. For $z = (x, y)$, let

$\bar{z} = (x, -y)$. The pseudohyperbolic metric ρ in H is defined by $\rho(z, w) = |w - z|/|\bar{w} - z|$. It is clear that ρ is invariant under horizontal translations. Let $D_\varepsilon(w) = \{z \in H; \rho(z, w) < \varepsilon\}$ when $w = (s, t) \in H$ and $0 < \varepsilon < 1$. $D_\varepsilon(w)$ is a Euclidean ball whose center and radius are $(s, \frac{1+\varepsilon^2}{1-\varepsilon^2}t)$ and $\frac{2\varepsilon t}{1-\varepsilon^2}$ respectively. It follows that there is a constant $C = C_\varepsilon > 0$ such that $C^{-1}t^n \leq V(D_\varepsilon(w)) \leq Ct^n$ for all $w \in H$. The following lemma is stated in [15].

LEMMA 1. *Let $0 < \varepsilon < 1$. Then, the following are true.*

- (1) *If z, w, ζ are in H and $\rho(z, w) < \varepsilon$, then $C^{-1}|\bar{\zeta} - z| \leq |\bar{\zeta} - w| \leq C|\bar{\zeta} - z|$ with a positive constant C depending only on ε .*
- (2) *If $z = (x, y), w = (s, t)$ are in H and $\rho(z, w) < \varepsilon$, then $C^{-1}y \leq t \leq Cy$ with a positive constant C depending only on ε .*
- (3) *If $0 < \varepsilon < 1/2$ then there exist a positive integer N and a sequence $\{\zeta_j\}$ in H satisfying the following conditions: (a) $H = \cup D_\varepsilon(\zeta_j)$, (b) any point in H belongs to at most N of the sets $D_{2\varepsilon}(\zeta_j)$.*

For a function u on H and $\delta > 0$, let $\tau_\delta u$ denote the function on H defined by $\tau_\delta u(x, y) = u(x, y + \delta)$, and let $\mathcal{T}^p = \{\tau_\delta u; u \in b^p, \delta > 0\}$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of nonnegative integers with order ℓ , then D^α denotes the partial differentiation operator $\partial^\ell / \partial x_1^{\alpha_1} \dots \partial x_{n-1}^{\alpha_{n-1}} \partial y^{\alpha_n}$. The following lemma is stated in [15].

LEMMA 2. *Let $0 < p \leq 1$. Then, the following are true.*

- (1) *For any $u \in b^p$, there is a constant $C > 0$ such that $|D^\alpha u(s, t)| \leq C/t^{n/p+|\alpha|}$ for all $(s, t) \in H$.*
- (2) *For any $u \in b^p$, there is a constant $C > 0$ such that $|(D^\alpha \tau_\delta u)(s, t)| \leq C/(t + \delta)^{n/p+|\alpha|}$ for all $(s, t) \in H$.*

Let $w = (s, t) \in H$. The Poisson kernel P_w is the function on $\mathbb{R}^{n-1}$ given by $P_w(x) = P(s - x, t) = \gamma_n t / (|s - x|^2 + t^2)^{n/2}$ (γ_n is the positive constant $\gamma_n = 2/(nV(\mathbb{B}_n))$, where $\mathbb{B}_n$ denotes the unit ball in $\mathbb{R}^n$). The harmonic extension of this function to H is $P(s - x, t + y)$. If $z = (x, y) \in H$, then we may write $P_w(z)$. We note that $P_w(z) = \gamma_n(t + y)/|\bar{w} - z|^n$, $|D_z^\alpha P_w(z)| \leq C/|\bar{w} - z|^{n+|\alpha|-1}$, and $D_z^\alpha P_w(z) = (-1)^{\alpha_1 + \dots + \alpha_{n-1}} D_w^\alpha P_w(z)$. The following lemma is useful and stated in [13, Lemma 3.1]

LEMMA 3. *Let $0 < c < 1$. Then, there is a constant $C > 0$ depending on c and n such that*

$$\int_H \frac{y^{-c}}{|w - \bar{z}|^n} dV(z) = Ct^{-c}$$

for all $w = (s, t) \in H$.

Let m be a nonnegative integer and let $c_m = (-2)^m/m!$. The following Lemma 4 is given in [15].

LEMMA 4. *Let $0 < p \leq 1$. If $u \in \mathcal{T}^p$, then*

$$u(w) = -2c_{m+k} \int_H y^{m+k} (D_y^m u)(z) D_y^{k+1} P_w(z) dV(z)$$

for all $m, k \geq 0$ and $w \in H$.

We show that Lemma 4 is also valid for $u \in b^p$ when the integer k is sufficiently large.

LEMMA 5. Let $0 < p \leq 1$ and k be a nonnegative integer such that $k > n/p$. If $u \in b^p$, then

$$u(w) = -2c_{m+k} \int_H y^{m+k} (D_y^m u)(z) D_y^{k+1} P_w(z) dV(z)$$

for all $m \geq 0$ and $w \in H$.

3. (μ, ν) -Carleson inequality

We give a sufficient condition for measures μ and ν which satisfy the (μ, ν) -Carleson inequality with derivatives.

PROPOSITION 2. Let $0 < p \leq 1$, $1 < q < \infty$, and $k > n/p$. Suppose that ℓ, m be nonnegative integers. Assume that μ is a σ -finite positive Borel measure on H and $d\nu = \omega dV$ such that $\omega \in L_{loc}^1(H, dV)$. If there are constants $K > 0$ and $0 < \varepsilon < 1$ such that

$$\int_H \left(\int_{D_\varepsilon(w)} \omega^{\frac{-1}{q-1}} dV \right)^{q-1} \frac{t^{p(n+m+k)-nq}}{|w - \bar{z}|^{p(n+\ell+k)}} d\mu(z) \leq K,$$

for all $w = (s, t) \in H$, then there is a constant $C > 0$ such that

$$\int_H |D^\alpha u|^p d\mu \leq C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$ and multi-indices α of order ℓ .

We will also give a necessary condition for the (μ, ν) -Carleson inequality. We need the following lemma, and Lemma 6 is stated in [15].

LEMMA 6. Let k be a nonnegative integer. Then, there exist constants $0 < \sigma \leq 1$ and $C > 0$ such that $|D_y^k P_w(z)| \geq C/t^{n+k-1}$ for all $w = (s, t) \in H$ and $z \in S(s, \sigma t)$.

PROPOSITION 3. Let $0 < p \leq 1$. Suppose that ℓ, m be nonnegative integers. Assume that μ and ν are σ -finite positive Borel measures on H . If there is a constant $C > 0$ such that

$$\int_H |D_y^\ell u|^p d\mu \leq C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$, then there are constants $0 < \sigma \leq 1$ and $K = K_\sigma > 0$ such that

$$\mu(S(s, \sigma t)) \leq K t^{p(\ell+n+k-1)} \int_H \frac{1}{|\bar{w} - z|^{p(m+k+n-1)}} d\nu$$

for all $w = (s, t) \in H$.

4. $(A_p)_\partial$ -condition

Let $1 < p < \infty$, and ω be a non-negative L^1_{loc} function on H . We say that the function ω satisfies the $(A_p)_\partial$ -condition if there exists a constant $\gamma > 0$ such that for every $w = (s, t) \in H$,

$$\int_H \frac{t^n}{|\bar{w} - z|^{2n}} \omega dV(z) \left(\int_H \frac{t^n}{|\bar{w} - z|^{2n}} \omega^{\frac{-1}{p-1}} dV(z) \right)^{p-1} \leq \gamma.$$

The $(A_p)_\partial$ -condition on the unit disk Δ of the complex plane is defined in [12]. In the definition of the $(A_p)_\partial$ -condition on the unit disk, the normalized reproducing kernel in the Bergman space is used. The B_p -condition is defined in [3] for characterizing the boundedness of a projection from $L^p(\omega)$ onto $L^p_a(\omega)$. And the C_p -condition is defined in [10]. For $z, w \in \Delta$, let $k_w(z) = \frac{1 - |w|^2}{(1 - \bar{w}z)^2}$ and $\phi_w(z) = \frac{w - z}{1 - \bar{w}z}$. The functions $k_w(z)$ and $\phi_w(z)$ are called the normalized reproducing kernel of the Bergman space on Δ and the Möbius mapping of Δ , respectively. Let $S_w = \{z \in \Delta; 1 - |w| < |z| < 1, |\arg z - \arg w| < 1 - |w|\}$ and $\Delta_w = \Delta_{w,\varepsilon} = \{z \in \Delta; |\phi_z(w)| < \varepsilon\}$. The $(A_p)_\partial$, B_p , and C_p -conditions on the unit disk Δ are the following.

The $(A_p)_\partial$ -condition: there exists a constant $\gamma > 0$ such that for every $w \in \Delta$,

$$\int_\Delta |k_w(z)|^2 \omega dA(z) \left(\int_\Delta |k_w(z)|^2 \omega^{\frac{-1}{p-1}} dA(z) \right)^{p-1} \leq \gamma.$$

The B_p -condition: there exists a constant $\gamma > 0$ such that for every $w \in \Delta$,

$$\frac{1}{A(S_w)} \int_{S_w} \omega dA(z) \left(\frac{1}{A(S_w)} \int_{S_w} \omega^{\frac{-1}{p-1}} dA(z) \right)^{p-1} \leq \gamma.$$

The C_p -condition: there exists a constant $\gamma > 0$ such that for every $w \in \Delta$,

$$\frac{1}{A(\Delta_w)} \int_{\Delta_w} \omega dA(z) \left(\frac{1}{A(\Delta_w)} \int_{\Delta_w} \omega^{\frac{-1}{p-1}} dA(z) \right)^{p-1} \leq \gamma.$$

In general, it is easy to see that

$$\frac{1}{A(\Delta_w)} \int_{\Delta_w} \omega dA(z) \leq C \frac{1}{A(S_w)} \int_{S_w} \omega dA(z) \leq C' \int_\Delta |k_w(z)|^2 \omega dA(z).$$

On the upper half-space H , it is also easy to see that there is a constant $C > 0$ such that $\frac{1}{V(D_\varepsilon(w))} \int_{D_\varepsilon(w)} \omega dV(z) \leq C \frac{1}{V(S(w))} \int_{S(w)} \omega dA(z)$. However, we do not know that the second inequality is satisfied or not. For $z = (x, y)$, $w = (s, t) \in H$, let

$$R_w(z) = \frac{4}{nV(B)} \frac{n(y+t)^2 - |\bar{w} - z|^2}{|\bar{w} - z|^{n+2}}$$

and

$$r_w(z) = \frac{(2t)^{\frac{n}{2}}}{\sqrt{n-1}} \frac{n(y+t)^2 - |\bar{w} - z|^2}{|\bar{w} - z|^{n+2}}.$$

The functions $R_w(z)$ and $r_w(z)$ are called the reproducing kernel and the normalized reproducing kernel of the harmonic Bergman space, respectively.

THEOREM 2. *Let ω be a non-negative L^1_{loc} function on H . Then, the following (1) and (2)*

(1) There are constants $0 < \sigma \leq 1$ and $C > 0$ such that for every $w = (s, t) \in H$,

$$\frac{1}{V(S(s, \sigma t))} \int_{S(s, \sigma t)} \omega dV(z) \leq C \int_H |r_w(z)|^2 \omega dV(z).$$

(2) There is a constant $C > 0$ such that for every $w \in H$,

$$\frac{1}{V(S(w))} \int_{S(w)} \omega dV(z) \leq C \int_H \frac{t^n}{|\bar{w} - z|^{2n}} \omega dV(z).$$

By Theorem 2, we obtain the following result.

THEOREM 3. Let $1 < p < \infty$ and ω be a non-negative L^1_{loc} function on H . Then, the following (1) and (2) are hold.

(1) If ω satisfies the $(A_p)_\delta$ -condition on H , then there is a constant $C > 0$ such that for every $w \in H$,

$$C^{-1} \int_H |r_w(z)|^2 \omega dV(z) \leq \int_H \frac{t^n}{|\bar{w} - z|^{2n}} \omega dV(z) \leq C \int_H |r_w(z)|^2 \omega dV(z).$$

(2) If ω satisfies the $(A_p)_\delta$ -condition on H , then ω satisfies the B_p -condition on H , and hence ω satisfies the C_p -condition on H .

5. Proof of Theorem 1

In this section, we give a proof of Theorem 1.

PROOF OF THEOREM 1. (1) $\Rightarrow$ (2) is trivial. We show that (2) $\Rightarrow$ (3). We suppose that (2) is hold. Then, Proposition 3 implies that there are constants $0 < \sigma \leq 1$ and $K = K_\sigma > 0$ such that $\mu(S(s, \sigma t)) \leq K t^{p(\ell+n+k-1)} \int_H 1/|\bar{w} - z|^{p(m+k+n-1)} d\nu$ for all $w = (s, t) \in H$. Since $|\bar{w} - z| \geq t$, We have $\mu(S(s, \sigma t)) \leq K t^{p(\ell-m)+n} \int_H t^n/|\bar{w} - z|^{2n} d\nu$. Moreover, since ω satisfies the $(A_q)_\delta$ -condition, we obtain $\mu(S(s, \sigma t)) \leq K t^{p(\ell-m)} \nu(D_\varepsilon(s, \sigma t))$. Since s and t are arbitrary, we can replace t by t/σ . This implies that $\mu(S(w)) \leq C t^{p(\ell-m)} \nu(D_\varepsilon(w))$. We will show (3) $\Rightarrow$ (1). Let $c = p(\ell - m)$ and suppose that $\mu(S(\zeta)) \leq K \eta^c \nu(D_\varepsilon(\zeta))$ for all $\zeta = (\xi, \eta) \in H$. Since ω satisfies the $(A_q)_\delta$ -condition, the sufficient condition in Proposition 2 is equivalent to the condition $\int_H t^{p(n+m+k)}/|\bar{w} - z|^{p(n+\ell+k)} d\mu(z) \leq K \nu(D_\varepsilon(w))$. Therefore, it is enough to prove that $\int_H 1/|\bar{w} - z|^\gamma d\mu(z) \leq C t^{c-\gamma} \nu(D_\varepsilon(w))$ for all $w = (s, t) \in H$, where $\gamma = p(n + \ell + k)$ and k is sufficiently large. Let $w \in H$. Clearly, if $z \notin S(s, 2^{j-1}t)$, then $|w - \bar{z}| \geq 2^{j-1}t$ ($j \geq 1$). Therefore, the hypothesis implies that

$$\begin{aligned} \int_H \frac{1}{|w - \bar{z}|^\gamma} d\mu(z) &\leq t^{-\gamma} \int_{S(s, t)} d\mu + t^{-\gamma} \sum_{j=1}^{\infty} \frac{1}{2^{\gamma(j-1)}} \int_{S(s, 2^j t) \setminus S(s, 2^{j-1} t)} d\mu \\ &\leq t^{-\gamma} \mu(S(s, t)) + t^{-\gamma} \sum_{j=1}^{\infty} \frac{1}{2^{\gamma(j-1)}} \mu(S(s, 2^j t)) \\ &\leq K t^{c-\gamma} \nu(D_\varepsilon(s, t)) + K t^{-\gamma} \sum_{j=1}^{\infty} \frac{1}{2^{\gamma(j-1)}} (2^j t)^c \nu(D_\varepsilon(s, 2^j t)) \\ &= K t^{c-\gamma} \left(\nu(D_\varepsilon(s, t)) + 2^\gamma \sum_{j=1}^{\infty} \frac{1}{2^{(\gamma-c)j}} \nu(D_\varepsilon(s, 2^j t)) \right) \end{aligned}$$

Since ω satisfies the $(A_q)_\partial$ -condition, $\nu = \omega dV$ satisfies the doubling condition. Therefore, there is a constant $\lambda > 0$ such that $\nu(D_\varepsilon(s, 2t)) \leq 2^\lambda \nu(D_\varepsilon(s, t))$. Hence, we have

$$\begin{aligned} \int_H \frac{1}{|w - \bar{z}|^\gamma} d\mu(z) &\leq K t^{c-\gamma} \left(\nu(D_\varepsilon(w)) + 2^\gamma \sum_{j=1}^{\infty} \frac{1}{2^{(\gamma-c)j}} 2^{\lambda j} \nu(D_\varepsilon(w)) \right) \\ &= K t^{c-\gamma} \left(1 + 2^\gamma \sum_{j=1}^{\infty} \frac{1}{2^{(\gamma-c-\lambda)j}} \right) \nu(D_\varepsilon(w)). \end{aligned}$$

If we choose an integer k such that $\gamma - c - \lambda = p(n + m + k) - \lambda > 0$, then we obtain $\int_H 1/|\bar{w} - z|^\gamma d\mu(z) \leq C t^{c-\gamma} \nu(D_\varepsilon(w))$.

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