

Singular Solutions of the Briot-Bouquet Type Partial Differential Equations

Caritas College, Hiroshi Yamazawa

カリタス女子短期大学, 山澤浩司

1 Introduction

In this talk, we will study the following type of nonlinear singular first order partial differential equations:

$$t\partial_t u = F(t, x, u, \partial_x u) \quad (1.1)$$

where $(t, x) = (t, x_1, \dots, x_n) \in \mathbb{C}_t \times \mathbb{C}_x^n$, $\partial_x u = (\partial_1 u, \dots, \partial_n u)$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \dots, n$, and $F(t, x, u, v)$ with $v = (v_1, \dots, v_n)$ is a function defined in a polydisk Δ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_u \times \mathbb{C}_v^n$. Let us denote $\Delta_0 = \Delta \cap \{t = 0, u = 0, v = 0\}$.

The assumptions are as follows:

- (A1) $F(t, x, u, v)$ is holomorphic in Δ ,
- (A2) $F(0, x, 0, 0) = 0$ in Δ_0 ,
- (A3) $\frac{\partial F}{\partial v_i}(0, x, 0, 0) = 0$ in Δ_0 for $i = 1, \dots, n$.

Definition 1.1 ([2], [3]) *If the equation (1.1) satisfies (A1), (A2) and (A3) we say that the equation (1.1) is of Briot-Bouquet type with respect to t .*

Definition 1.2 ([2], [3]) *Let us define*

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0),$$

then the holomorphic function $\rho(x)$ is called the characteristic exponent of the equation (1.1).

Let us denote by

1. $\mathcal{R}(\mathbb{C} \setminus \{0\})$ the universal covering space of $\mathbb{C} \setminus \{0\}$,
2. $S_\theta = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |\arg t| < \theta\}$,
3. $S(\epsilon(s)) = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); 0 < |t| < \epsilon(\arg t)\}$ for some positive-valued function $\epsilon(s)$ defined and continuous on $\mathbb{R}$,

4. $D_R = \{x \in \mathbb{C}^n; |x_i| < R \text{ for } i = 1, \dots, n\}$,
5. $\mathbb{C}\{x\}$ the ring of germs of holomorphic functions at the origin of $\mathbb{C}^n$.

Definition 1.3 We define the set $\tilde{\mathcal{O}}_+$ of all functions $u(t, x)$ satisfying the following conditions;

1. $u(t, x)$ is holomorphic in $S(\epsilon(s)) \times D_R$ for some $\epsilon(s)$ and $R > 0$,
2. there is an $a > 0$ such that for any $\theta > 0$ and any compact subset K of D_R

$$\max_{x \in K} |u(t, x)| = O(|t|^a) \quad \text{as } t \rightarrow 0 \quad \text{in } S_\theta.$$

We know some results on the equation (1.1) of Briot-Bouquet type with respect to t . We concern the following result. Gérard R. and Tahara H. studied in [2] the structure of holomorphic and singular solutions of the equation (1.1) and proved the following result;

Theorem 1.4 (Gérard R. and Tahara H.) *If the equation (1.1) is of Briot-Bouquet type and $\rho(0) \notin \mathbb{N}^* = \{1, 2, 3, \dots\}$ then we have;*

- (1) (Holomorphic solutions) *The equation (1.1) has a unique solution $u_0(t, x)$ holomorphic near the origin of $\mathbb{C} \times \mathbb{C}^n$ satisfying $u_0(0, x) \equiv 0$.*
- (2) (Singular solutions) *Denote by S_+ the set of all $\tilde{\mathcal{O}}_+$ -solutions of (1.1).*

$$S_+ = \begin{cases} \{u_0(t, x)\} & \text{when } \operatorname{Re} \rho(0) \leq 0, \\ \{u_0(t, x)\} \cup \{U(\varphi); 0 \neq \varphi(x) \in \mathbb{C}\{x\}\} & \text{when } \operatorname{Re} \rho(0) > 0, \end{cases}$$

where $U(\varphi)$ is an $\tilde{\mathcal{O}}_+$ -solution of (1.1) having an expansion of the following form:

$$U(\varphi) = \sum_{i \geq 1} u_i(x) t^i + \sum_{i+2j \geq k+2, j \geq 1} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).$$

The purpose of this paper is to determine S_+ in the case $\rho(0) \in \mathbb{N}^*$.

The main result of this paper is;

Theorem 1.5 *If the equation (1.1) is of Briot-Bouquet type and if $\rho(0) = N \in \mathbb{N}^*$ and $\rho(x) \neq \rho(0)$, then*

$$S_+ = \{U(\varphi); \varphi(x) \in \mathbb{C}\{x\}\},$$

where $U(\varphi)$ is an $\tilde{\mathcal{O}}_+$ -solution of (1.1) having an expansion of the following form:

$$\begin{aligned} U(\varphi) = & u_1^0(x)t + u_0^{e_0}(x)\phi_N(t, x) + \sum_{\substack{i+|\beta| \geq 2, |\beta| < \infty \\ [\beta] \leq i+|\beta|-2}} u_i^\beta(x) t^i \Phi_N^\beta \\ & + w_{0,1,0}^0(x) t^{\rho(x)} + \sum_{\substack{i+j+|\beta| \geq 2, |\beta| < \infty \\ j \geq 1, [\beta] \leq i+j+|\beta|-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_N^\beta, \end{aligned}$$

where $u_N^0(x) \equiv 0$, $w_{0,1,0}^0(x) = \varphi(x)$ is an arbitrary holomorphic function and the other coefficients $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ are holomorphic functions determined by $w_{0,1,0}^0(x)$ and defined in a common disk, and

$$\begin{aligned} l &= (l_1, \dots, l_n) \in \mathbb{N}^n, \quad |l| = l_1 + \dots + l_n, \quad \beta = (\beta_l \in \mathbb{N}; \quad l \in \mathbb{N}^n), \\ |\beta| &= \sum_{|l| \geq 0} \beta_l, \quad |\beta|_p = \sum_{|l|=p} \beta_l \text{ for } p \geq 0, \quad [\beta] = \sum_{|l| \geq 2} (|l| - 1) \beta_l, \\ \Phi_N^\beta &= \prod_{|l| \geq 0} \left(\frac{\partial_x^l \phi_N}{l!} \right)^{\beta_l}, \quad \partial_x^l = \partial_1^{l_1} \dots \partial_n^{l_n}, \quad \phi_N(t, x) = \frac{t^{\rho(x)} - t^N}{\rho(x) - N}. \end{aligned}$$

The following lemma will play an important role in the proof of Theorem 1.5.

At first, we define some notations. We set for $l \in \mathbb{N}^n$, $e_l = (\beta_k; \quad k \in \mathbb{N}^n)$ with $\beta_l = 1$ and $\beta_k = 0$ for $k \neq l$ and for $p \in \{1, 2, \dots, n\}$, $e(p) = (i_1, \dots, i_n)$ with $i_p = 1$ and $i_q = 0$ for $q \neq p$, and define $l^1 < l^0$ is defined by $|l^1| < |l^0|$ and $l_i^1 \leq l_i^0$ for $i = 1, \dots, n$.

Lemma 1.6 *Let $\rho(x)$, ϕ_N and Φ_N^β be as in Theorem 1.5. Then we have;*

1. $\partial_p \Phi_N^\beta = \sum_{|l| \geq 0} \beta_l (l_p + 1) \Phi_N^{\beta - e_l + e_l + e(p)}$ for $i = 1, \dots, n$,
2. $t \partial_t \phi_N = \rho(x) \phi_N + t^N$,
3. $t \partial_t \Phi_N^\beta = |\beta| \rho(x) \Phi_N^\beta + \beta_0 t^N \Phi_N^{\beta - e_0} + \sum_{|l^0| \geq 1} \sum_{l^1 < l^0} \beta_{l^0} \frac{\partial_x^{(l^0 - l^1)}}{l^0 - l^1} \rho(x) \Phi_N^{\beta - e_{l^0} + e_{l^1}}.$

2 Construction of formal solutions in the case $\rho(0) = 1$

By [2] (Gérard-Tahara), if the equation (1.1) is of Briot-Bouquet type with respect to t , then it is enough to consider the following equation:

$$Lu = t \partial_t u - \rho(x) u = a(x) t + G_2(x)(t, u, \partial_x u) \quad (2.1)$$

where $\rho(x)$ and $a(x)$ are holomorphic functions in a neighborhood of the origin, and the function $G_2(x)(t, X_0, X_1, \dots, X_n)$ is a holomorphic function in a neighborhood of the origin in $\mathbb{C}_x^n \times \mathbb{C}_t \times \mathbb{C}_{X_0} \times \mathbb{C}_{X_1} \times \dots \times \mathbb{C}_{X_n}$ with the following expansion:

$$G_2(x)(t, X_0, X_1, \dots, X_n) = \sum_{p+|\alpha| \geq 2} a_{p,\alpha}(x) t^p \{X_0\}^{\alpha_0} \{X_1\}^{\alpha_1} \dots \{X_n\}^{\alpha_n}$$

and we may assume that the coefficients $\{a_{p,\alpha}(x)\}_{p+|\alpha| \geq 2}$ are holomorphic functions on D_{R_0} for a sufficiently small $R_0 > 0$. Let $0 < R < R_0$. We put $A_{p,\alpha}(R) := \max_{x \in D_R} |a_{p,\alpha}(x)|$ for $p + |\alpha| \geq 2$. Then for $0 < r < R$

$$\sum_{p+|\alpha| \geq 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t^p X_0^{\alpha_0} X_1^{\alpha_1} \times \dots \times X_n^{\alpha_n} \quad (2.2)$$

is convergent in a neighborhood of the origin.

In this section, we assume $\rho(0) = 1$ and $\rho(x) \neq 1$ and we will construct formal solutions of the equation (2.1). In generally, we set $u(t, x) = \sum_{i=1}^{N-1} u_i(x)t^i + t^{N-1}w(t, x)$, and we consider an equation for $w(t, x)$.

Proposition 2.1 *If $\rho(0) = 1$ and $\rho(x) \neq 1$, the equation (2.1) has a family of formal solutions of the form:*

$$u = u_0^{e_0}(x)\phi_1 + \sum_{m \geq 2} \sum_{\substack{i+|\beta|=m \\ [\beta] \leq m-2}} u_i^\beta(x)t^i \Phi_1^\beta \\ + w_{0,1,0}^0(x)t^{\rho(x)} + \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ j \geq 1, [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x)t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \quad (2.3)$$

where $w_{0,1,0}^0(x)$ is an arbitrary holomorphic function and the other coefficients $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ are holomorphic functions determined by $w_{0,1,0}^0(x)$ and defined in a common disk.

Remark 2.2 *By the relation $[\beta] \leq m - 2$ in summations of the above formal solution, we have $\beta_l = 0$ for any $l \in \mathbb{N}^n$ with $|l| \geq m$.*

We define the following two sets U_m and W_m for $m \geq 1$ to prove Proposition 2.1.

Definition 2.3 *We denote by U_m the set of all functions u_m of the following forms:*

$$u_1 = u_1^0(x)t + u_0^{e_0}(x)\phi_1, \\ u_m = \sum_{\substack{i+|\beta|=m \\ [\beta] \leq m-2}} u_i^\beta(x)t^i \Phi_1^\beta \quad \text{for } m \geq 2, \quad (2.4)$$

and denote by W_m the set of all functions w_m of the following forms:

$$w_1 = w_{0,1,0}^0(x)t^{\rho(x)}, \\ w_m = \sum_{\substack{i+j+|\beta|=m \\ j \geq 1, [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x)t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \quad \text{for } m \geq 2 \quad (2.5)$$

where $u_i^\beta(x)$, $w_{i,j,k}^\beta(x) \in \mathbb{C}\{x\}$.

We can rewrite the formal solution (2.3) as follows:

$$u = \sum_{m \geq 1} (u_m + w_m) \quad \text{where } u_m \in U_m, w_m \in W_m.$$

Let us show important relations of u_m and w_m for $m \geq 2$. By Lemma 1.6, we have

$$\begin{aligned} Lw_m = & \sum_{\substack{i+j+|\beta|=m \\ j \geq 1, |\beta| \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \left\{ \{i + (j + |\beta| - 1)\rho(x)\} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \right. \\ & + k w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^{k-1} \Phi_1^\beta + \beta_0 w_{i,j,k}^\beta(x) t^{i+j\rho(x)+1} \{\log t\}^k \Phi_1^{\beta-e_0} \\ & \left. + \sum_{|l^0|=1}^{m-1} \sum_{l^1 < l^0} \beta_{l^0} \frac{\partial_x^{l^0-l^1} \rho(x)}{(l^0-l^1)!} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^{\beta-e_{l^0}+e_{l^1}} \right\}. \end{aligned}$$

We show two lemmas.

Lemma 2.4 *If $u_m \in U_m$ and $w_m \in W_m$, then $Lu_m \in U_m$ and $Lw_m \in W_m$.*

Lemma 2.5 *If $u_m \in U_m$ and $w_m \in W_m$, then the following relations hold for $i, j = 1, \dots, n$*

1. $a(x)U_m \subset U_m$ and $a(x)W_m \subset W_m$ for any holomorphic function $a(x)$,
2. $tU_m, \phi_1 U_m \subset U_{m+1}$ and $t^\rho(x)U_m, tW_m, t^\rho(x)W_m, \phi_1 W_m \subset W_{m+1}$,
3. $u_m \times u_n, \partial_i u_m \times \partial_j u_n, \partial_i u_m \times u_n \in U_{m+n}$,
4. $w_m \times w_n, \partial_i w_m \times \partial_j w_n, \partial_i w_m \times w_n \in W_{m+n}$,
5. $u_m \times w_n, \partial_i u_m \times w_n, u_m \times \partial_j w_n, \partial_i u_m \times \partial_j w_n \in W_{m+n}$.

Let us show that u_m and w_m are determined inductively on $m \geq 1$. By substituting $\sum_{m \geq 1} (u_m + w_m)$ into (2.1), we have

$$(1 - \rho(x))u_1^0(x) + u_0^{e_0}(x) = a(x), \quad (2.6)$$

and for $m \geq 2$

$$Lu_m = \sum_{\substack{p+|\alpha| \geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_0,h_0} \prod_{j=1}^n \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_j,h_j}, \quad (2.7)$$

$$\begin{aligned} Lw_m = & \sum_{\substack{p+|\alpha| \geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} (u_{m_0,h_0} + w_{m_0,h_0}) \prod_{j=1}^n \prod_{h_j=1}^{\alpha_j} \partial_j (u_{m_j,h_j} + w_{m_j,h_j}) \\ & - \sum_{\substack{p+|\alpha| \geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_0,h_0} \prod_{j=1}^n \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_j,h_j}, \end{aligned} \quad (2.8)$$

where $|m_n| = \sum_{i=0}^n m_i(\alpha_i)$ and $m_i(\alpha_i) = m_{i,1} + \dots + m_{i,\alpha_i}$ for $i = 0, 1, \dots, n$.

We take any holomorphic function $\varphi(x) \in \mathbb{C}\{x\}$ and put $w_{0,1,0}^0(x) = \varphi(x)$, and by (2.6), we put $u_1^0(x) \equiv 0$ and $u_0^{e_0}(x) = a(x)$.

For $m \geq 2$, let us show that u_m and w_m are determined by induction. By Lemma 2.5, the right side of (2.7) belongs to U_m and the right side of (2.8) belongs to W_m . Further by $m_{j,h_j} \geq 1$, we have $m_{j,h_j} < m$ for $h_j = 1, \dots, \alpha_j$ and $j = 0, \dots, n$. Then for $m \geq 2$, we compare with the coefficients of $t^i \Phi_1^\beta$ and $t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta$ respectively for (2.7) and (2.8), then put

$$\begin{aligned} & \{i + (|\beta| - 1)\rho(x)\} u_i^\beta(x) \\ & + (\beta_0 + 1) u_{i-1}^{\beta+e_0}(x) + \sum_{|l^0|=1}^{m-1} \sum_{0 \leq l^1 < l^0} (\beta_{l^0} + 1) \frac{\partial_x^{l^0-l^1} \rho(x)}{(l^0 - l^1)!} u_i^{\beta+e_{l^0}-e_{l^1}}(x) \\ & = f_i^\beta(\{a_{p,\alpha}\}_{2 \leq p+|\alpha| \leq m}, \{u_{i'}^{\beta'}(x)\}_{i'+|\beta'| < m}) \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} & \{i + (j + |\beta| - 1)\rho(x)\} w_{i,j,k}^\beta(x) + (k + 1) w_{i,j,k+1}^\beta(x) \\ & + (\beta_0 + 1) w_{i-1,j,k}^{\beta+e_0}(x) + \sum_{|l^0|=1}^{m-1} \sum_{0 \leq l^1 < l^0} (\beta_{l^0} + 1) \frac{\partial_x^{l^0-l^1} \rho(x)}{(l^0 - l^1)!} w_{i,j,k}^{\beta+e_{l^0}-e_{l^1}}(x) \\ & = g_{i,j,k}^\beta(\{a_{p,\alpha}\}_{2 \leq p+|\alpha| \leq m}, \{u_{i'}^{\beta'}(x)\}_{i'+|\beta'| < m}, \{w_{i',j',k'}^{\beta'}(x)\}_{i'+j'+|\beta'| < m}). \end{aligned} \quad (2.10)$$

Hence we obtain Proposition 2.1. Q.E.D.

3 Convergence of the formal solutions in the case $\rho(0) = 1$

In this section, we show that the formal solution (2.3) converges in $\tilde{O}_+$.

Proposition 3.1 *Let γ satisfy $0 < \gamma < 1$ and let λ be sufficiently large. Then for any sufficiently small $r > 0$ we have the following result;*

For any $\theta > 0$ there is an $\epsilon > 0$ such that the formal solution (2.3) converges in the following region:

$$\begin{aligned} & \{(t, x) \in \mathbf{C}_t \times \mathbf{C}_x^n; |\eta(t, \lambda)t| < \epsilon, |\eta(t, \lambda)^2 t^{\rho(x)}| < \epsilon, \\ & |\eta(t, \lambda)t^\gamma| < \epsilon, t \in S_\theta \text{ and } x \in D_r\}, \end{aligned}$$

where $\eta(t, \lambda) = \max\{ |(\log t)/\lambda|, 1 \}$.

In this section, we put $w_{i,0,0}^\beta(x) = u_i^\beta(x)$ and $w_{i,0,k}^\beta(x) \equiv 0$ for $k \geq 1$ in the formal solution (2.3). Then the formal solution (2.3) is as follows:

$$\begin{aligned} u &= w_{0,0,0}^{e_0}(x) \phi_1 + w_{0,1,0}^0(x) t^{\rho(x)} \\ &+ \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ |\beta| \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta. \end{aligned} \quad (3.1)$$

Let us define the following set V_m for (3.1).

Definition 3.2 We denote by V_m the set of all the functions v_m of the following forms:

$$v_m = u_m + w_m \quad \text{for } u_m \in U_m \quad \text{and } w_m \in W_m. \quad (3.2)$$

We define the following estimate for the function v_m .

Definition 3.3 For the function (3.2), we define

$$\begin{aligned} \|v_1\|_{r,c,\lambda} &= \|v_1\|_{r,c} := \frac{\|w_{0,0,0}^{e_0}\|_r}{c} + \|w_{0,1,0}^0\|_r, \\ \|v_m\|_{r,c,\lambda} &:= \sum_{\substack{i+j+|\beta|=m \\ [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{\|w_{i,j,k}^\beta\|_r \lambda^k}{c^{<\beta>}} \quad \text{for } m \geq 2 \end{aligned} \quad (3.3)$$

for $c > 0$ and $\lambda > 0$, where

$$\|w_{i,j,k}^\beta\|_r = \max_{x \in D_r} |w_{i,j,k}^\beta(x)| \quad \text{and } <\beta> = \sum_{|l| \geq 0} (|l| + 1) \beta_l.$$

We will make use of

Lemma 3.4 For a holomorphic function $f(x)$ on D_{R_0} , we have

$$\|\partial_x^\alpha f\|_R \leq \frac{\alpha!}{(R_0 - R)^{|\alpha|}} \|f\|_{R_0} \quad \text{for } 0 < R < R_0.$$

Proof. By Cauchy's integral formula, we have the desired result. Q.E.D

Lemma 3.5 If a holomorphic function $f(x)$ on D_R satisfies

$$\|f\|_r \leq \frac{C}{(R-r)^p} \quad \text{for } 0 < r < R$$

then we have

$$\|\partial_i f\|_r \leq \frac{Ce(p+1)}{(R-r)^{p+1}} \quad \text{for } 0 < r < R, \quad i = 1, \dots, n.$$

For the proof, see Hörmander ([5] lemma 5.1.3)

Let us show the following estimate for the function Lv_m .

Lemma 3.6 Let $0 < R < R_0$. Then there exists a positive constant σ such that for $m \geq 2$, if $v_m \in V_m$ we have

$$\|Lv_m\|_{r,c,\lambda} \geq \frac{\sigma}{2} m \|v_m\|_{r,c,\lambda} \quad \text{for } 0 < r \leq R$$

for sufficiently small $c > 0$ and sufficiently large $\lambda > 0$.

Let us estimate the function $\partial_i v_m$.

Definition 3.7 For the function $v_m \in V_m$ we define

$$D_p v_m := \sum_{\substack{i+j+|\beta|=m \\ |\beta| \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \partial_p w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta$$

for $p = 1, \dots, n$.

Lemma 3.8 If $v_m \in V_m$, then for $i = 1, \dots, n$, we have

$$\|\partial_i v_m\|_{r,c,\lambda} \leq \|D_i v_m\|_{r,c,\lambda} + c_0 \lambda m \|v_m\|_{r,c,\lambda} + \frac{3m-2}{c} \|v_m\|_{r,c,\lambda} \quad \text{for } 0 < r \leq R. \quad (3.4)$$

Therefore by the relations (2.7), (2.8) and Lemma 3.8, we have the following lemma.

Lemma 3.9 If $u = \sum_{m \geq 1} v_m$ is a formal solution of the equation (2.1) constructed in Section 2, we have the following inequality for v_m ($m \geq 2$):

$$\begin{aligned} \|Lv_m\|_{r,c,\lambda} &\leq \sum_{\substack{p+|\alpha| \geq 2 \\ p+|m_n|=m}} \|a_{p,\alpha}\|_r \prod_{h_0=1}^{\alpha_0} \|v_{m_0,h_0}\|_{r,c,\lambda} \\ &\times \prod_{i=1}^n \prod_{h_i=1}^{\alpha_i} \left\{ \|D_i v_{m_i,h_i}\|_{r,c,\lambda} + c_0 \lambda m_{i,h_i} \|v_{m_i,h_i}\|_{r,c,\lambda} + \frac{3m_{i,h_i}-2}{c} \|v_{m_i,h_i}\|_{r,c,\lambda} \right\}. \end{aligned}$$

Let us define a majorant equation to show that the formal solution (3.1) converges.

We take A_1 so that

$$\begin{aligned} \frac{\|w_{0,0,0}^{e_0}\|_R}{c} + \|w_{0,1,0}^0\|_R &\leq A_1, \\ \frac{\|\partial_i w_{0,0,0}^{e_0}\|_R}{c} + \|\partial_i w_{0,1,0}^0\|_R &\leq A_1 \end{aligned}$$

for $i = 1, \dots, n$.

Then we consider the following equation:

$$\begin{aligned} \frac{\sigma}{2} Y &= \frac{\sigma}{2} A_1 t_1 \\ &+ \frac{1}{R-r} \sum_{p+|\alpha| \geq 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t_1^p Y^{\alpha_0} \prod_{i=1}^n \left(eY + c_0 \lambda Y + \frac{3}{c} Y \right)^{\alpha_i}. \quad (3.5) \end{aligned}$$

The equation (3.5) has a unique holomorphic solution $Y = Y(t_1)$ with $Y(0) = 0$ at $(Y, t_1) = (0, 0)$ by implicit function theorem. By an easy calculation, the solution $Y = Y(t_1)$ has the following form:

$$Y = \sum_{m \geq 1} Y_m t_1^m \text{ with } Y_m = \frac{C_m}{(R-r)^{m-1}}$$

where $Y_1 = C_1 = A_1$ and $C_m \geq 0$ for $m \geq 1$.
Then we have;

Lemma 3.10 *For $m \geq 1$, we have*

$$m \|v_m\|_{r,c,\lambda} \leq Y_m \text{ for } 0 < r < R. \quad (3.6)$$

Let us show that the formal solution (3.1) converges by using (3.6) in Lemma 3.10. We rewrite v_m as follows:

$$v_m = \sum_{\substack{i+j+|\beta|=m \\ [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{w_{i,j,k}^\beta(x) \lambda^k}{c^{<\beta>}} t^{i+j\rho(x)} \left(\frac{\log t}{\lambda} \right)^k \Psi_1^\beta,$$

where

$$\Psi_1^\beta = \prod_{|l| \geq 0} \left(c^{|l|+1} \frac{\partial_x^l \phi_1}{l!} \right)^{\beta_l}. \quad (3.7)$$

Firstly let us estimate (3.7). For $\|\phi_1\|_R$, we have the following lemma.

Lemma 3.11 *For any γ with $0 < \gamma < 1$, there is an $R > 0$ such that*

$$\|\phi_1\|_R = O(|t|^\gamma) \text{ as } t \rightarrow 0 \text{ in } S_\theta$$

holds for any $\theta > 0$.

By Lemma 3.11, there exists a positive constant c_1 such that

$$\|\phi_1\|_R \leq c_1 |t|^\gamma \text{ in } S_\theta. \quad (3.8)$$

By Lemma 3.4 and (3.8), we have

$$\|\Psi_1^\beta\|_r \leq \prod_{|l| \geq 0} \left(c^{|l|+1} \frac{c_1}{(R-r)^{|l|}} |t|^\gamma \right)^{\beta_l} = \left(\frac{c}{R-r} \right)^{<\beta>} (c_1(R-r) |t|^\gamma)^{|\beta|} \quad (3.9)$$

for $0 < r < R < R_0$ in S_θ .

Let us estimate $t^{i+j\rho(x)} \left(\frac{\log t}{\lambda} \right)^k \Psi_1^\beta$.

We put $\eta(t, \lambda) = \max \left\{ \left| \frac{\log t}{\lambda} \right|, 1 \right\}$, $c_2 = \max \left\{ \frac{c}{R-r}, 1 \right\}$ and $c_3 = c_1(R-r)$. Since

we have $[\beta] \leq m - 2 < m = i + j + |\beta|$, $\langle \beta \rangle \leq 2|\beta| + [\beta] \leq i + j + 3|\beta|$ and $k \leq i + |\beta|_0 + |\beta|_1 + 2(j - 1) \leq i + |\beta| + 2j$, we obtain

$$\left\| t^{i+j\rho(x)} \left(\frac{\log t}{\lambda} \right)^k \Psi_1^\beta \right\|_r \leq \{ |c_2 \eta(t, \lambda) t| \}^i \{ \|c_2 \eta(t, \lambda)^2 t^{\rho(x)}\|_r \}^j \{ |(c_2)^3 c_3 \eta(t, \lambda) t^\gamma| \}^{|\beta|}$$

in S_θ . For any sufficiently small $\epsilon > 0$, there exists a sufficiently small $\delta > 0$ such that for any $t \in S_\theta$ with $0 < |t| < \delta$ we have

$$|c_2 \eta(t, \lambda) t| < \epsilon, \|c_2 \eta(t, \lambda)^2 t^{\rho(x)}\|_r < \epsilon, |(c_2)^3 c_3 \eta(t, \lambda) t^\gamma| < \epsilon.$$

Then by Lemma 3.10, we have

$$\|u\|_r \leq \sum_{m \geq 1} Y_m \epsilon^m \quad (3.10)$$

for sufficiently small $|t|$ in S_θ . Hence the formal solution (3.1) converges for $x \in D_r$ and sufficiently small $|t|$ in S_θ . Q.E.D.

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