

On a decay property of weak solution of the M.H.D. equations in a 3-dimensional exterior domain

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Introduction

The purpose of this report is to present the abstract theorems on the existence and a certain decay property of weak solution for a semilinear evolution equation of parabolic type. Then we shall show that these theorems are applicable to the M.H.D. equations in an exterior domain. In the case of the Navier-Stokes equations, the energy decay for a weak solution was shown by Masuda [3] and Sohr [6].

Let X be a separable Hilbert space. The abstract equation of evolution we consider has the form

$$du(t)/dt + Au(t) + Nu(t) = f(t) \quad t > 0,$$

(E)

$$u(0) = a.$$

Here A is a non-negative self-adjoint operator in X , while N is a non-linear operator in X specified later.

Our problem reads as follows.

Problem. Construct a weak solution $u = u(t)$ of (E) on
 $(0, \infty)$ such that $\|u(t)\| \rightarrow 0$ as $t \rightarrow \infty$. Here $\|\cdot\|$ denotes the
norm on X .

Concerning our problem, we follow Masuda [3] and Sohr [6].
 At the first step, we show that there exists a positive number α
 such that any weak solution u of (E) with $\int_0^\infty \|A^{1/2} u(\tau)\|^2 d\tau <$
 ∞ satisfies

$$\|(1 + A)^{-\alpha} u(t)\| \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (\text{W.D.})$$

An immediate consequence of (W.D.) is that

$$\lim_{t \rightarrow \infty} \int_t^{t+1} \|u(\tau)\|^2 d\tau = 0. \quad (0.1)$$

See Masuda [3, Corollary 1].

Hence in order to solve our problem, it is sufficient to
 construct a weak solution u of (E) satisfying the energy
 inequality of strong form:

$$\|u(t)\|^2 + 2 \int_s^t \|A^{1/2} u(\tau)\|^2 d\tau \leq \|u(s)\|^2 + 2 \int_s^t (f(\tau), u(\tau)) d\tau$$

(E.I.S.)

for almost all $s \geq 0$, including $s = 0$, and all $t > 0$ such

that $s < t$. $((\cdot, \cdot))$; the scalar product on X)

In fact, if f is a summable function on $(0, \infty)$ with values in X , we see by (E.I.S.) that the inequality

$$\|u(t)\|^2 + 2 \int_s^t \|A^{1/2} u(\tau)\|^2 d\tau \leq \|u(s)\|^2 + M_0 \int_s^\infty \|f(\tau)\| d\tau \quad (0.2)$$

holds for almost all $s \geq 0$ and all $t > s$. ($M_0 := \sup_{\tau > 0} \|u(\tau)\|$)

It follows from (0.1) and (0.2) that a weak solution u of (E) with (E.I.S.) satisfies $\lim_{t \rightarrow \infty} \|u(t)\|^2 = 0$.

In section 1, we state the general results of existence and decay property for weak solutions of (E) under some assumptions on A and N . Section 2 is devoted to apply the theorems obtained in section 1 to the magnetohydrodynamic (M.H.D.) equations in a three-dimensional exterior domain.

1 Existence and decay property (W.D.) of weak solutions of (E)

Let X be a separable Hilbert space. We denote by $((\cdot, \cdot))$ and $\|\cdot\|$ the scalar product and the norm on X . Suppose that A is a non-negative self-adjoint operator in X with the core D . we set

$$V = D(A^{1/2}). \quad (D(S); \text{domain of } S).$$

Equipped with the scalar product

$$((u, v)) = (u, v) + (A^{1/2}u, A^{1/2}v),$$

V is a Hilbert space. We denote by $||| \cdot |||$ the norm on V defined by $||| u ||| = ((u, u))^{1/2}$.

Let X^* and V^* denote the dual spaces X and V , respectively. Identifying X with X^* , we have the usual inclusions

$$V \subset X \equiv X^* \subset V^*,$$

where each space is dense in the following one and the injections are continuous.

For $T > 0$, we set $\mathcal{Y}_T = L^\infty(0, T; X) \cap L^2(0, T; V)$. $\mathcal{Y}_T$ is the Banach space with norm $|\cdot|_T$:

$$|u|_T = \sup_{0 < t < T} \|u(t)\| + \left\{ \int_0^T \|A^{1/2}u(\tau)\|^2 d\tau \right\}^{1/2}.$$

We can now introduce the assumptions on N .

N is a continuous mapping from V into V^* satisfying the following conditions.

Assumption 1. There exists a monotone increasing function $L_1 = L_1(\lambda)$ such that the inequality

$$|\langle Nu, \phi \rangle| \leq L_1(\|u\|)(1 + \|A^{1/2}u\|)\|A^{1/2}u\|\|A^{1/2}\phi\|$$

holds for all u and ϕ in V . Here and hereafter $\langle \cdot, \cdot \rangle$

denotes the duality between V^* and V .

Assumption 2. There exist a constant $0 \leq p < 2$ and a monotone increasing function $L_2 = L_{2,\phi}(\lambda)$ depending only on ϕ in D such that the inequality

$$|\langle Nu, \phi \rangle| \leq L_2(\|u\|)(1 + \|A^{1/2}u\|)^p$$

holds for all u in V .

Assumption 3. For each $T > 0$, there exist a monotone increasing function $L_3 = L_{3,v,\phi}(\lambda)$ depending only on $v \in \mathcal{Y}_T$ and $\phi = \phi(t) = h(t)\phi$ ($h \in C_0^1([0, T])$, $\phi \in D$), an integer $K = K_{\varepsilon,v,\phi}$ depending only on $\varepsilon > 0$, v and ϕ as above, positive numbers $\alpha_i (i=1, \dots, K)$ and functions $\psi_i (i=1, \dots, K)$ in $C^0([0, T]; X)$ such that the inequality

$$\left| \int_0^T \langle Nu(\tau) - Nv(\tau), \phi(\tau) \rangle d\tau \right| \leq L_3(\|u\|_T) \left\{ \varepsilon + \int_0^T \left(\sum_{i=1}^K |(u(\tau) - v(\tau), \psi_i(\tau))| \right)^{\alpha_i} d\tau \right\}$$

holds for all $u \in \mathcal{Y}_T$.

For a weak solution of (E), we give the following definition.

Definition. Let the initial data a be in X and let f be in $L^1(0, \infty; X)$. Suppose that the assumptions 1, 2 and 3 hold. u is called a weak solution of (E), if (i) and (ii) hold:

$$(i) \quad u \in L^\infty(0, \infty; X) \cap L^2_{loc}(0, \infty; V).$$

(ii) For all $\Phi \in C^1_0([0, \infty); V)$, the equality

$$\begin{aligned} & \int_0^\infty \{-(u(t), \partial_t \Phi(t)) + (A^{1/2} u(t), A^{1/2} \Phi(t)) + \langle Nu(t), \Phi(t) \rangle\} dt \\ &= (a, \Phi(0)) + \int_0^\infty (f(t), \Phi(t)) dt \end{aligned}$$

is satisfied.

Our result on existence of weak solutions reads:

Theorem 1. Let the initial data a be in X and let f be in $L^1(0, \infty; X)$. Suppose that the assumptions 1, 2 and 3 hold. If $\langle Nu, u \rangle \geq 0$ for all $u \in V$, then there exists a weak solution u of (E) such that the inequality

$$\|u(t)\|^2 + 2 \int_0^t \|A^{1/2} u(\tau)\|^2 d\tau \leq \|a\|^2 + 2 \int_0^t (f(\tau), u(\tau)) d\tau$$

holds for all $t > 0$.

Our result on the decay property of weak solutions reads:

Theorem 2. Let a be in X and let f be in $L^1(0, \infty; X)$. Suppose that the assumptions 1, 2 and 3 hold. If zero is not an eigenvalue of A , then any weak solution of (E) with

$$\int_0^\infty \|A^{1/2} u(\tau)\|^2 d\tau < \infty \quad \text{satisfies}$$

$$\lim_{t \rightarrow \infty} \|(1 + A)^{-1/4} u(t)\| = 0.$$

For the proof of Theorems 1 and 2, see Kozono [2].

2 Application

In this section, we apply the theorems obtained in the preceding section to the M.H.D. equations in a three-dimensional exterior domain. In the case of the Navier-Stokes equations, such a result was obtained by Masuda [3] and Sohr [6]. The main work is to check whether or not the assumptions 1, 2 and 3 hold.

Let O be a bounded domain in R^3 with smooth boundary ∂O . We set $\Omega = R^3 - O$. For simplicity, we assume that Ω is simply connected. In $Q := \Omega \times (0, \infty)$, we consider the following magnetohydrodynamic (M.H.D.) equations:

$$\begin{aligned}
\partial_t v - \Delta v + (v, \nabla)v + B \times \text{rot} B + \nabla \pi &= b \quad \text{in } Q, \\
\partial_t B - \Delta B + (v, \nabla)B - (B, \nabla)v &= 0 \quad \text{in } Q, \\
\text{div } v &= 0, \quad \text{div } B = 0 \quad \text{in } Q, \\
v &= 0, \quad B \cdot v = 0, \quad \text{rot} B \times v = 0, \quad \text{on } \partial\Omega \times (0, \infty), \\
v|_{t=0} &= v_0, \quad B|_{t=0} = B_0.
\end{aligned}$$

Here $v = v(x, t) = (v^1(x, t), v^2(x, t), v^3(x, t))$, $B = B(x, t) = (B^1(x, t), B^2(x, t), B^3(x, t))$ and $\pi = \pi(x, t)$ denote respectively the unknown velocity field of the fluid, magnetic field and pressure of the fluid, $b = b(x, t) = (b^1(x, t), b^2(x, t), b^3(x, t))$ denotes the given external force, $v_0 = v_0(x) = (v_0^1(x), v_0^2(x), v_0^3(x))$ and $B_0 = B_0(x) = (B_0^1(x), B_0^2(x), B_0^3(x))$ denote the given initial data and v denotes the unit outward normal on $\partial\Omega$.

We introduce some function spaces.

Let $C_{0,\sigma}^\infty(\Omega)$ be the set of all C^∞ vector functions ϕ with compact support in Ω , such that $\text{div } \phi = 0$ ($x \in \Omega$). $L_\sigma^2(\Omega)$ denotes the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega) := L^2(\Omega)^3$.

We denote by P the orthogonal projection from $L^2(\Omega)$ onto $L_\sigma^2(\Omega)$.

We define the operators A_D and A_N on $L_\sigma^2(\Omega)$ as follows:

$$D(A_D) = H^2(\Omega)^3 \cap \{v \in H_0^1(\Omega)^3; \operatorname{div} v = 0\},$$

$$A_D v = -P \Delta v \quad \text{for } v \in D(A_D),$$

$$D(A_N) = \{B \in H^2(\Omega)^3; B \cdot v = 0, \operatorname{rot} B \times v = 0 \text{ on } \partial\Omega\} \cap L_\sigma^2(\Omega),$$

$$A_N B = -\Delta B \quad \text{for } B \in D(A_N).$$

Note that $-\Delta B \in L_\sigma^2(\Omega)$ if and only if $B \in D(A_N)$. With the aid of Miyakawa [5, Theorem 1.8] and [4, Theorem 3.8], we see that A_D and A_N are non-negative self-adjoint operators in $L_\sigma^2(\Omega)$. Moreover, we have

$$\|A_D^{1/2} v\|_{L^2} = \|\nabla v\|_{L^2} \quad \text{for } v \in D(A_D^{1/2}), \quad (2.1)$$

$$\|A_N^{1/2} B\|_{L^2} = \|\operatorname{rot} B\|_{L^2} \quad \text{for } B \in D(A_N^{1/2}). \quad (2.2)$$

In the context of section 1, we set

$$X = L_\sigma^2(\Omega) \times L_\sigma^2(\Omega), \quad D = D(A_D) \times D(A_N)$$

and

$$A = \begin{pmatrix} A_D & 0 \\ 0 & A_N \end{pmatrix} \quad \text{with the domain } D(A) = D(A_D) \times D(A_N).$$

Then we have $V = D(A_D^{1/2}) \times D(A_N^{1/2})$.

We define $Nu \in V^*$ for $u = {}^t[v, B] \in V$ by

$$\langle Nu, \Phi \rangle := \int_{\Omega} ((v, \nabla)v + B \times \text{rot} B) \cdot \Phi \, dx + \int_{\Omega} ((v, \nabla)B - (B, \nabla)v) \cdot \Psi \, dx$$

for $\Phi = {}^t[\phi, \psi] \in V$.

By integration by parts, we have $\langle Nu, u \rangle = 0$ for all $u \in V$.

By the Holder inequality, the Gagliardo-Nirenberg inequality and Duvaut-Lions [4, Chapter 7 Theorem 6.1], the inequalities

$$\begin{aligned} & \left| \int_{\Omega} ((v, \nabla)v + B \times \text{rot} B) \cdot \Phi \, dx \right| \\ & \leq (\|v\|_{L^3} \|\nabla v\|_{L^2} + \|B\|_{L^3} \|\text{rot} B\|_{L^2}) \|\Phi\|_{L^6} \\ & \leq C(\|v\|_{L^2}^{1/2} \|\nabla v\|_{L^2}^{3/2} + \|B\|_{L^3}^{1/2} \|B\|_{H^1}^{1/2} \|\text{rot} B\|_{L^2}) \|\nabla \Phi\|_{L^2} \\ & \leq C\{(1+\|v\|_{L^2})(1+\|A_D^{1/2} v\|_{L^2})\|A_D^{1/2} v\|_{L^2} + (1+\|B\|_{L^2})(1+\|A_N^{1/2} B\|_{L^2}) \times \\ & \quad \times \|A_N^{1/2} B\|_{L^2}\} \|A_D^{1/2} \Phi\|_{L^2}, \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega} ((v, \nabla)B - (B, \nabla)v) \cdot \Psi \, dx \right| \\ & = \left| \int_{\Omega} \text{rot}(B \times v) \cdot \Psi \, dx \right| = \left| \int_{\Omega} B \times v \cdot \text{rot} \Psi \, dx \right| \quad (\text{by integration by parts}) \\ & \leq \|B\|_{L^3} \|v\|_{L^6} \|\text{rot} \Psi\|_{L^2} \\ & \leq C\|B\|_{L^2}^{1/2} \|B\|_{H^1}^{1/2} \|\nabla v\|_{L^2} \|\text{rot} \Psi\|_{L^2} \\ & \leq C(\|B\|_{L^2} + \|\text{rot} B\|_{L^2}) \|\nabla v\|_{L^2} \|\text{rot} \Psi\|_{L^2} \end{aligned}$$

$$\leq C(1 + \|B\|_{L^2})(1 + \|A_N^{1/2} B\|_{L^2})\|A_D^{1/2} v\|_{L^2}\|A_N^{1/2} \psi\|_{L^2}$$

hold for all $u = {}^t[v, B] \in V$ and all $\phi = {}^t[\phi, \psi] \in V$.

Hence the assumption 1 holds.

Moreover, for each $\phi \in D(A_D)$ and each $\psi \in D(A_N)$ the inequalities

$$\begin{aligned} & \left| \int_{\Omega} ((v, \nabla)v + B \times \text{rot} B) \cdot \phi \, dx \right| \\ & \leq \text{ess sup}_{x \in \Omega} |\phi(x)| (\|v\|_{L^2} \|\nabla v\|_{L^2} + \|B\|_{L^2} \|\text{rot} B\|_{L^2}) \\ & = \text{ess sup}_{x \in \Omega} |\phi(x)| (\|v\|_{L^2} \|A_D^{1/2} v\|_{L^2} + \|B\|_{L^2} \|A_N^{1/2} B\|_{L^2}) \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega} ((v, \nabla)B - (B, \nabla)v) \cdot \psi \, dx \right| \\ & \leq \text{ess sup}_{x \in \Omega} |\psi(x)| (\|v\|_{L^2} (\|B\|_{L^2} + \|A_N^{1/2} B\|_{L^2}) + \|B\|_{L^2} \|A_D^{1/2} v\|_{L^2}) \end{aligned}$$

hold for all ${}^t[v, B] \in V$.

Taking $p = 1$ and

$$L_{2, \phi}(\lambda) := (\text{ess sup}_{x \in \Omega} |\phi(x)| + \text{ess sup}_{x \in \Omega} |\psi(x)|)(1 + \lambda^2)$$

for each $\phi = {}^t[\phi, \psi] \in D \subset L^\infty(\Omega)^3 \times L^\infty(\Omega)^3$, we see that the assumption 2 holds.

We obtain the assumption 3 by using Masuda's beautiful technique. See Masuda [3, Lemma 2.5].

Finally, it remains to show that zero is not an eigenvalue of A . In fact, by (2.1) it is easy to see that zero is not an eigenvalue of A_D . Suppose that $A_N B = 0$ for $B \in D(A_N)$. Then by (2.2), $\operatorname{rot} B = 0$ in Ω . Since $\operatorname{div} B = 0$ in Ω and since $B \cdot \nu = 0$ on $\partial\Omega$, it follows from the classical potential theory that there is a scalar function p with $p \in L^2_{\text{loc}}(\Omega)$, $\nabla p \in L^2(\Omega)$ and

$$\Delta p = 0 \quad \text{in } \Omega, \quad \partial p / \partial \nu = 0 \quad \text{on } \partial\Omega$$

such that $B = \nabla p$.

According to Miyakawa [5, Lemma 1.4], such p must satisfy that $\nabla p = 0$ and hence $B = 0$. Thus zero is not an eigenvalue of A_N .

Remark. Theorems 1 and 2 are applicable to the following system of semilinear parabolic equations:

$$\partial_t u^i - \sum_{j,k=1}^L \partial_{x_j} (a_{jk}(x) \partial_{x_k} u^i) + |u|^p \sum_{j=1}^L b_j^i(x) u^j = f^i \quad \text{in } Q, \\ (i = 1, \dots, L)$$

$$u^i = 0 \quad \text{on } \partial\Omega \times (0, \infty),$$

$$u^i|_{t=0} = a^i,$$

where $Q = \Omega \times (0, \infty)$ (Ω ; bounded or unbounded domain in $\mathbb{R}^n$) and $|u| = \{ \sum_{j=1}^L (u^j)^2 \}^{1/2}$.

Suppose that $3 \leq n \leq 6$ and $1 \leq \rho \leq 6/n$.

We assume that a_{ij} and b_j^i are in $\mathcal{B}^\infty(\bar{\Omega})$ and $a_{ij} = a_{ji}$ for all $i, j = 1, \dots, L$. Moreover, we assume that there is a positive constant δ such that

$$\sum_{j,k=1}^L a_{jk}(x) \xi^j \xi^k \geq \delta |\xi|^2, \quad \sum_{i,j=1}^L b_j^i(x) \xi^i \xi^j \geq \delta |\xi|^2$$

hold for all $x \in \bar{\Omega}$ and all $\xi = (\xi^1, \dots, \xi^L) \in \mathbb{R}^L$.

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