

Buchsbaum Rings and Cone Singularities

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Let $S=k(X_0, \dots, X_N)$ be the polynomial ring. Let us regard the polynomial ring S as a graded ring in which the degree of X_j is one for every $0 \leq j \leq N$. Let $\mathfrak{m} = \bigoplus_{d \geq 1} S_d$. Let M be a finitely generated graded S -module of dimension $n+1$. Then M is called a Buchsbaum S -module (resp. a quasi-Buchsbaum S -module) if $M_{\mathfrak{m}}$ is a Buchsbaum $S_{\mathfrak{m}}$ -module (resp. a quasi-Buchsbaum $S_{\mathfrak{m}}$ -module). That is, M is a quasi-Buchsbaum module if and only if $\mathfrak{m}H_{\mathfrak{m}}^i(M) = 0$ for any $0 \leq i \leq n$. On the other hand, M is a Buchsbaum module if and only if the difference $l(M/qM) - e_q(M)$ is an invariant for any homogeneous parameter ideal $q (\subset \mathfrak{m})$ for M , where l , respectively e , denotes length, respectively multiplicity of q .

In this paper, we search for the condition that a graded S -module M is a Buchsbaum module. In particular, our purpose is to clarify the difference between Buchsbaum property and quasi-Buchsbaum property.

For this, we introduce the notion of quasi-Buchsbaum modules of type r in Section 1. Then we give a criterion (Theorem 1.7) for it, which explicitly describes "Surjectivity Criterion" through spectral sequence. In Corollary 1.10, we give a sufficient condition that M is Buchsbaum. It is described

only by the vanishing of its local cohomologies. This generalizes a result of Goto-Watanabe (4, Proposition 3.1). Also, the theorem enables us to calculate the length of $\text{Ext}_S^{n+1}(k, M)$ in Corollary 1.12.

In Section 2, we investigate the divisors of Segre products. In particular, we construct very concrete examples of quasi-Buchsbaum but not Buchsbaum rings.

§1. Quasi-Buchsbaum modules of type r

Definition 1.1. M is a quasi-Buchsbaum module of type r if, for every homogeneous system $f_1, \dots, f_{r-1}$ of parameters, $M/(f_1, \dots, f_k)M$ is a quasi-Buchsbaum module for every $k \leq r-1$.

Remark 1.2. In the above definition, we have only to take elements of degree one as homogeneous systems of parameters, by Goto-Suzuki (6, Theorem 1.9) or Suzuki (19, Theorem 3.6).

Remark 1.3. M is a quasi-Buchsbaum module of type $n+1$ if and only if M is a Buchsbaum module, by Stückrad-Vogel (14).

Let U_i be an open set $D_+(X_i)$ in $\text{Proj } S$ for $0 \leq i \leq N$. Then $\mathcal{U} = \{U_i\}_{0 \leq i \leq N}$ is an open covering of $\text{Proj } S$. Let us consider a Čech complex $C^\bullet = C^\bullet(\mathcal{U}; \tilde{M})$, where $\tilde{M}$ is the sheaf associated to the graded S -module M on $\text{Proj } S$. Then we define a complex

$L' = (0 \longrightarrow M \xrightarrow{\varepsilon} C'(-1))$, where ε is a natural map. (e.g. See Godement (2).)

Let K be the Koszul complex associated to $\langle X_0, \dots, X_N \rangle$, that is, $K_p = \bigwedge^p \left(\bigoplus_{k=0}^N Ae_k \right) = \bigoplus_{0 \leq i_1 < \dots < i_p \leq N} Ae_{i_1} \wedge \dots \wedge e_{i_p}$, where $\langle e_0, \dots, e_N \rangle$ is a free basis.

In this way, we have a double complex $K' = \text{Hom}(K, L')$. Now let us consider two "stupid" filtrations, that is,

$$K'_t = \sum_{p \geq t} K^{p,q} \quad \text{and} \quad K'_t = \sum_{q \geq t} K^{p,q}.$$

As usual, we get the spectral sequence for each filtered complex:

$$\begin{aligned} E_1^{p,q} &= \text{Ker } d^{-p,q} / \text{Im } d^{-p,q-1} \\ E_1^{p,q} &= \text{Ker } d^{-p,q} / \text{Im } d^{-p-1,q} \end{aligned} \quad \begin{aligned} &\Rightarrow \\ &\Rightarrow \end{aligned} \quad H^{p+q}(K' \cdot \cdot)$$

Lemma 1.4. Under the above conditions, we have the following things.

$$(1) \quad E_1^{p,q} = \bigwedge^p \left(\bigoplus_{k=0}^N H_m^q(M) e_k^* \right),$$

where $\langle e_0^*, \dots, e_N^* \rangle$ is the dual basis of $\langle e_0, \dots, e_N \rangle$, and

$$(2) \quad E_1^{p,q} = \begin{cases} \text{Ext}_S^p(k, M) & \text{if } q=0 \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $H^{p+q}(K' \cdot \cdot) = \text{Ext}_S^{p+q}(k, M)$.

From now on we will treat the first filtration. So we

write $E_r^{p,q}$ for $E_r^{p,q}$.

Lemma 1.5. The spectral sequence $(E_r^{p,q})$ does not depend on the choice of coordinates $\langle X_0, \dots, X_N \rangle$.

Now then we will study the relation between the spectral sequence and Buchsbaum property.

Proposition 1.6. Notations being above, the following conditions are equivalent.

- (a) M is a quasi-Buchsbaum S -module.
- (b) $d_1^{p,q} : E_1^{p,q} \longrightarrow E_1^{p+1,q}$ is a zero map for $q \leq n$.
- (c) $d_1^{0,q} : E_1^{0,q} \longrightarrow E_1^{1,q}$ is a zero map for $q \leq n$.

Proof. By (1.4.1.), $E_1^{p,q} = \wedge^p \left(\bigoplus_{k=0}^N H_m^q(M) e_k^* \right)$. From the construction of the double complex $K^{\bullet}$, we see that

$$d_1^{p,q}(e_{i_1}^* \wedge \dots \wedge e_{i_p}^*) = \sum_{j=0}^N X_j e_j^* \wedge e_{i_1}^* \wedge \dots \wedge e_{i_p}^*.$$

Thus the assertion follows from the definition of quasi-Buchsbaum module.

Suppose that M is a quasi-Buchsbaum module and that every subset I of $\{X_0, \dots, X_N\}$ such that $\#I = n+1$ makes a system of parameters of M . Setting $\bar{M} = M/X_j M$, we have the following exact sequence:

$$0 \longrightarrow (0:X_j)_M \longrightarrow M \xrightarrow{\cdot X_j} M \longrightarrow \bar{M} \longrightarrow 0.$$

Since $H_m^q((0:X_j)_M) = 0$ for $q \geq 1$ and M is a quasi-Buchsbaum module, we have the following short exact sequence:

$$0 \longrightarrow H_m^{q-1}(M) \longrightarrow H_m^{q-1}(\bar{M}) \longrightarrow H_m^q(M) \longrightarrow 0.$$

for $1 \leq q \leq n$. Thus we have the following commutative diagram with exact rows:

$$(1.7.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H_m^{q-1}(M) & \longrightarrow & H_m^{q-1}(\bar{M}) & \xrightarrow{\alpha} & H_m^q(M) \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow \bar{\psi} & & \downarrow \phi' \\ 0 & \longrightarrow & H_m^{q-1}(M) & \xrightarrow{\beta} & H_m^{q-1}(\bar{M}) & \longrightarrow & H_m^q(M) \longrightarrow 0 \end{array}$$

for $1 \leq q \leq n$, where ϕ , ϕ' and $\bar{\psi}$ are multiplication maps $\cdot X_i$'s. Since M is a quasi-Buchsbaum module, ϕ and ϕ' are zero maps. By snake lemma, we get a graded S -homomorphism of degree 2 $\psi : H_m^q(M) \longrightarrow H_m^{q-1}(M)$ such that $\psi \cdot \alpha = \beta \cdot \bar{\psi}$. Let us write $(X_i \wedge X_j)$ for this map ψ . Since $\psi = 0$ is equivalent $\bar{\psi} = 0$, we see that $\bar{M}$ is a quasi-Buchsbaum module if and only if $(X_i \wedge X_j)$ is a zero map for any i . In other words, M is a quasi-Buchsbaum module of type 2 if and only if and only if $(X_i \wedge X_j)$ are zero map for any i and j .

Continuing the similar steps, we have the following theorem.

Theorem 1.7. Let S be the polynomial ring $k(X_0, \dots, X_N)$.

Let M be a finitely generated graded S -module with dimension $n+1$.

Suppose that M is a quasi-Buchsbaum module of type $r-1$. Then

$$(X_I) = (X_{i_1} \wedge \dots \wedge X_{i_r}) : H_m^q(M) \longrightarrow H_m^{q-r+1}(M)$$

is well-defined.

Furthermore, M is a quasi-Buchsbaum module of type r if and only if (X_I) defined above is a zero map for any I .

Now there still remains to link between the spectral sequence $(E_r^{p,q})$ and the maps (X_I) 's constructed above.

Lemma 1.8. Let M be a quasi-Buchsbaum graded S -module of type $r-1$ and with dimension $n+1$. Then we have

$$(1) \quad E_r^{p,q} = \bigwedge^p \left(\bigoplus_{k=0}^N H_m^q(M) e_k^* \right) \quad \text{for any } q \neq n+1.$$

By (1), we can write $(d_r^{p,q})_{J,K} : H_m^q(M) \longrightarrow H_m^{q-r+1}(M)$ for the map from e_K^* -component to $e_J \wedge e_K^*$ -component of the map $d_r^{p,q}$.

Then we have

$$(2) \quad (d_r^{p,q})_{J,K} = (-1)^{(r-1)(p+q-r)} (X_J), \quad \text{where } (X_J) \text{ is the map defined in (1.7).}$$

Then we have the following theorem.

Theorem 1.9. Notations being above, the following conditions are equivalent.

- (a) M is a quasi-Buchsbaum S -module of type r .

(b) $d_s^{p,q} : E_s^{p,q} \longrightarrow E_s^{p+s, q-s+1}$ is a zero map for $s \leq r$ and $q \leq n$.

(c) $d_s^{0,q} : E_s^{0,q} \longrightarrow E_s^{s, q-s+1}$ is a zero map for $s \leq r$ and $q \leq n$.

Corollary 1.10. Let M be a finitely generated graded S -module. Let us define

$$\mathcal{G} = \{(i, \ell) \mid 0 \leq i \leq n, \ell \in \mathbb{Z}, H_m^i(M)_\ell \neq 0\}.$$

Suppose that $\mathcal{G}$ satisfies the following condition (*).

(*) For any (i, ℓ) and (j, m) of $\mathcal{G}$, if $i \geq j$, then $i + \ell + 1 \neq j + m$.

Then M is a Buchsbaum module.

Proof. Since $(E_1^{p,q})_\ell = \oplus H_m^q(M)_\ell$, $(E_1^{p+s, q-s+1})_{\ell+s} = \oplus H_m^{q-s+1}(M)_{\ell+s}$ and $d_s^{p,q}$ is a graded homomorphism of degree s , the assumption gives that $d_s^{p,q}$ is zero for any $p, q (\leq n)$ and s . By Theorem 1.9 (or (1.7)), M is a Buchsbaum module.

Remark 1.11. (1.10) is a generalization of Goto-Watanabe (4) and Schenzel (13, Theorem 3.1) and is "best possible" in the following sense. Unless $\mathcal{G}$ satisfies (*), we cannot see only by the vanishing of its local cohomologies whether or not M is a Buchsbaum module. In fact, Goto abstractly and systematically constructed such examples in his papers (3, 5). Moreover, his method gives the construction

of quasi-Buchsbaum ring of type r but not of type $r+1$ by virtue of Evans-Griffith (1).

Now then, let us calculate the length of $\text{Ext}_S^{n+1}(k, M)$ written by $r(M)$. By (1.9), we have the following result.

Corollary 1.12. Suppose that M is a Buchsbaum S -algebra, we have

$$\sum_{j=0}^n \binom{N+1}{n+1-j} \dim H_m^j(M) + 1 \leq r(M) \leq \sum_{j=0}^n \binom{N+1}{n+1-j} \dim H_m^j(M) + \mu(K_M),$$

where $K_M = \text{Ext}_S^{N-n}(M, S)$ and $\mu(K_M)$ is the minimal number of generators of K_M .

§2. Divisors on Segre products

Let k be a field. Let X be an arithmetically Cohen-Macaulay subscheme of $\mathbb{P}_k^N = \text{Proj } S$, that is, its affine cone $C(X) = \text{Spec } S/J$ is locally Cohen-Macaulay, where $J = \bigoplus_{t \in \mathbb{Z}} r(\mathcal{I}_{X/P}(t))$.

Let V be a subscheme of X such that $0 < \dim V = n < \dim X$. Let A be the coordinate ring of V . The following proposition shows that the results of Section 1 can be applied to geometric case.

Proposition 2.1. Under the above conditions, we have

$$\tau_0^{n+1} \text{R}\Gamma\left(\bigoplus_{t \in \mathbb{Z}} \mathcal{I}_{V/X}(t)\right) \simeq \tau_0^{n+1} \text{R}\Gamma_m(A)$$

in the derived category $D_h^+(S)$ of complexes bounded below of

graded S -modules.

Example 2.2. Let $X = \mathbb{P}_k^r \times \mathbb{P}_k^s$ be Segre embedding in $\mathbb{P} = \mathbb{P}_k^{rs+r+s}$. Let V be a divisor of X corresponding to $\mathcal{O}_X(a, b) = \mathcal{O}_{\mathbb{P}_k^r}(a) \otimes \mathcal{O}_{\mathbb{P}_k^s}(b)$.

(1) V is an arithmetically Cohen-Macaulay subscheme of $\mathbb{P}$ if and only if $a-r \leq b \leq a+s$.

(2) The following things are equivalent.

(a) V is an arithmetically Buchsbaum subscheme of $\mathbb{P}$.

(b) V is an arithmetically quasi-Buchsbaum subscheme of $\mathbb{P}$.

(c) $a-r-1 \leq b \leq a+s+1$.

Proof. It is given, for example, by Goto-Watanabe (7), Stückrad-Vogel (18) or Schenzel (13, Proposition 5.1). We will prove it, however, because our proof indicates the motivation of the next example.

Now let us assume $a \geq b$. First of all, let us find the numbers $1 \leq i < r+s$ and $\ell \in \mathbb{Z}$ satisfying $H^i(\mathcal{I}_{V/X}(\ell)) = 0$. By the way $\mathcal{I}_{V/X}$ is isomorphic to $\mathcal{O}_X(-a, -b)$. By Künneth formula, we have $H^i(\mathcal{I}_{V/X}(\ell)) = 0$ if and only if $i=r$ and $b \leq \ell \leq a-r-1$. This shows (1) and "(c) $\Rightarrow$ (a)" of (2). It is trivial that (a) implies (b). Thus it still remains to prove that if $a-r-1 > b$, then V is not an arithmetically quasi-Buchsbaum subscheme of $\mathbb{P}$.

Let us take $\mathbb{P}^r \times \mathbb{P}^s = \text{Proj } k(X_0, \dots, X_r) \times \text{Proj } k(Y_0, \dots, Y_s)$.

Then we can show

$$\cdot X_j: H^r(\mathcal{O}_{\mathbb{P}^r}(\ell-a)) \longrightarrow H^r(\mathcal{O}_{\mathbb{P}^r}(\ell+1-a))$$

is surjective and

$$\cdot Y_j: H^0(\mathcal{O}_{\mathbb{P}^s}(\ell-b)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^s}(\ell+1-b))$$

is injective. On the other hand we see

$$H^r(\mathcal{I}_{V/X}(\ell)) \cong H^r(\mathcal{O}_{\mathbb{P}^r}(\ell-a)) \oplus H^0(\mathcal{O}_{\mathbb{P}^s}(\ell-b)).$$

Hence we have that $\bigoplus_{\ell \in \mathbb{Z}} H^i(\mathcal{I}_{V/X}(\ell))$ is not a k -vector space if

$a-r-1 > b$. This gives V is not an arithmetically Buchsbaum subscheme of $\mathbb{P}$.

In Example 2.2, we need not distinguish Buchsbaum rings from quasi-Buchsbaum rings. However, taking $X = \mathbb{P}_k^r \times \mathbb{P}_k^r \times \mathbb{P}_k^r$, we can find the examples clarifying the difference between them.

Theorem 2.3. Let $X = \mathbb{P}_k^r \times \mathbb{P}_k^r \times \mathbb{P}_k^r$ be Segre embedding in $\mathbb{P} = \mathbb{P}_k^{(r+1)^3-1}$. Let V be a divisor of X corresponding to $\mathcal{O}_X(a-r-1, a, a+r+1)$. Then V is an arithmetically quasi-Buchsbaum subscheme of type r in $\mathbb{P}$ but not an arithmetically quasi-Buchsbaum subscheme of type $r+1$.

Proof. For simplicity, we will prove only in case $r=1$. Let $\langle X, Y \rangle \times \langle Z, W \rangle \times \langle U, T \rangle$ be a coordinate of $X = \mathbb{P}_k^1 \times \mathbb{P}_k^1 \times \mathbb{P}_k^1$. By Künneth formula, we have

$$H^2\left(\bigoplus_{\ell \in \mathbb{Z}} \mathcal{F}_{V/X}(\ell)\right) = k \cdot 1 \otimes k \cdot \frac{1}{Z \cdot W} \otimes \left(k \cdot \frac{1}{U^3 T} + k \cdot \frac{1}{U^2 T^2} + k \cdot \frac{1}{U T^3}\right)$$

and

$$H^1\left(\bigoplus_{\ell \in \mathbb{Z}} \mathcal{F}_{V/X}(\ell)\right) = (k \cdot X^2 + k \cdot XY + k \cdot Y^2) \otimes k \cdot 1 \otimes k \cdot \frac{1}{U \cdot T}.$$

Then we see

$$(X \otimes Z \otimes U) \wedge (Y \otimes W \otimes T) : H^2\left(\bigoplus_{\ell \in \mathbb{Z}} \mathcal{F}_{V/X}(\ell)\right) \longrightarrow H^1\left(\bigoplus_{\ell \in \mathbb{Z}} \mathcal{F}_{V/X}(\ell)\right)$$

is not zero. By Theorem 1.7, V is not arithmetically quasi-Buchsbaum of type 2. Further, it is easy to show V is arithmetically quasi-Buchsbaum. Thus the assertion is proved.

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