

On Termination of One-Rule String Rewriting Systems

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Let Σ be a finite alphabet. The free monoid and the free semigroup generated by Σ are denoted by Σ^* and Σ^+ , respectively. The length of a word x in Σ^* is denoted by $|x|$. For $x, y \in \Sigma^*$, we set $\text{OVL}(x, y) = \{z \in \Sigma^+ \mid x = uz, y = zv \text{ for some } u, v \in \Sigma^+\}$. A *rewriting system* R on Σ is a subset of $\Sigma^* \times \Sigma^*$. An element (l, r) in R is denoted by $l \rightarrow r$. If R contains only one element, R is said to be a *one-rule rewriting system*. A *single step reduction relation* $\rightarrow$ induced by R is the following relation on Σ^* : For any $x, y \in \Sigma^*$, $x \rightarrow y$ if and only if there exists $(l, r) \in R$ such that $x = ulv$, $y = urv$ for some $u, v \in \Sigma^*$. $\rightarrow^*$ is the reflexive and transitive closure of $\rightarrow$.

A rewriting system R is said to be *confluent* if for any $w, x, y \in \Sigma^*$, $w \rightarrow^* x$ and $w \rightarrow^* y$ imply $x \rightarrow^* z$ and $y \rightarrow^* z$ for some $z \in \Sigma^*$. R is *terminating* (or *noetherian*) if there is no infinite sequence $x_1, x_2, \dots$ such that $x_1 \rightarrow x_2 \rightarrow \dots$. A confluent and terminating rewriting system is said to be *complete*.

It is not known whether the completeness is decidable for one-rule rewriting systems. Let $R = \{l \rightarrow r\}$ be a one-rule rewriting system. If $r \in \Sigma^*l\Sigma^*$ then R is always non-terminating. If $|l| \geq |r|$ and $l \neq r$ then R is always terminating.

Result 1 [3] *It is decidable whether or not a one-rule rewriting system is confluent.*

Result 2 [2] *For a confluent one-rule rewriting system $R = \{l \rightarrow r\}$ with $|l| < |r|$, we can effectively construct a rewriting system $R' = \{l' \rightarrow r\}$ such that:*

- (1) $|l'| < |r'|$ and $\text{OVL}(l', l') = \emptyset$.
- (2) R' is terminating if and only if R is terminating.

Hence the completeness problem for one-rule systems is reduced to the termination problem for one-rule systems $R = \{l \rightarrow r\}$ with $\text{OVL}(l, l) = \emptyset$. It is not difficult to see that if $\text{OVL}(r, l) = \emptyset$ or $\text{OVL}(l, r) = \emptyset$ then R is terminating. In this note, we consider the case where $\text{OVL}(r, l) = \{p\}$, a singleton.

For each $s \in \text{OVL}(l, r)$, we determine $\bar{s} \in \Sigma^*$ by $l = \bar{s}s$. The decidability of the terminating problem for such one-rule systems is given as follows.

Theorem 1. *Let $R = \{l \rightarrow r\}$ be a one-rule rewriting system such that $\text{OVL}(l, l) = \emptyset$ and $\text{OVL}(r, l) = \{p\}$. Let $l = px$, $r = y\bar{s}_k \cdots \bar{s}_1 p$, where $s_1, \dots, s_k \in \text{OVL}(l, r)$ and $y \notin \Sigma^* \bar{s}$ for any $s \in \text{OVL}(l, r)$.*

(1) *If there is a reduction of length $|r|^2$ starting with $(y\bar{s}_k \cdots \bar{s}_1)^3$ then R is non-terminating.*

(2) *Assume that the maximal length of reductions starting with $(y\bar{s}_k \cdots \bar{s}_1)^3$ is N with $N < |r|^2$. If $|x| > |y|$ and there is a reduction of length $2N + 1$ starting with $(y\bar{s}_k \cdots \bar{s}_1)^4$ then R is non-terminating, otherwise, R is terminating.*

The exact characterization of non-terminating one-rule systems is given as follows.

Theorem 2. *Let $R = \{l \rightarrow r\}$ be a one-rule rewriting system such that $\text{OVL}(l, l) = \emptyset$ and $\text{OVL}(r, l) = \{p\}$. Then R is non-terminating if and only if one of the following conditions is satisfied.*

(k, m, n are positive integers. $x, y, z, w \in \Sigma^+$ and $u, v \in \Sigma^*$.)

(1) $l \in \Sigma^* r \Sigma^*$.

(2) $r = s_k u \bar{s}_k \cdots \bar{s}_1 p$, $s_1, \dots, s_k \in \text{OVL}(l, r) \cap (xu)^* x$.

(3) $l = p(ux)^n$, $r = (xu)^{n+m} \bar{s}_k \cdots \bar{s}_1 p$, $s_1, \dots, s_k \in \text{OVL}(l, r) \cap (xu)^* x$.

(4) $l = p(ux)^n$, $r = (xu)^{n+m} \bar{s}_k \cdots \bar{s}_1 p$, $s_1, \dots, s_k \in \text{OVL}(l, r)$,

$s_1, \dots, s_{j-1} \in (xu)^* x$, $s_j \in (xu)^i x$, $1 \leq j \leq k$, $1 \leq i \leq 2m$.

(5) $l = pxy$, $r = y(xyz)^m \bar{s}_k \cdots \bar{s}_1 p$,

$s_1, \dots, s_k \in \text{OVL}(l, r)$, $s_1 = y(xyz)^m$, $xyz = wxy$.

(6) $l = xy((zx^m ky)^{m-1} zy)^{n+1}$, $r = y((zx^m ky)^{m-1} zy)^{n+1} zx^m ky$

(7) $l = zxyxv (z^{2m+n+1} xyxv)^m x$, $pu = zxyxv$,

$r = xyxv (z^{2m+n+1} xyxv)^m xuz^{2m+n} p$.

(8) $l = pu((pu)^k z)^{2m+n} pu((pu)^k z)^{2m+n} p^{m-1} x$, $pu = zxyxv$.

$r = xyxvz((pu)^k z)^{2m+n-1} pu((pu)^k z)^{2m+n} p^{m-1} xu((pu)^k z)^{2m+n} p$.

(9) $l = zx(yx)^{k-1} (yz^{m+n+1} x(yx)^{k-1})^m yx$,

$r = x(yx)^{k-1} (yz^{m+n+1} x(yx)^{k-1})^m yxyz^{m+n+1} x(yx)^{k-1}$.

References

- [1] R.V. Book, F. Otto, *String-Rewriting Systems*, Texts and Monographs in Computer Science, Springer-Verlag (1993).
- [2] K. Shikishima-Tsuji, M. Katsura, Y. Kobayashi, Termination of one-rule string rewriting systems, submitted for publication.
- [3] C. Wrathall, Confluence of one-rule Thue systems, *Word Equations and Related Topics*, Lecture Notes in Computer Science 572 (1992), 237-246.

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