

ON THE UNIFICATION OF KUMMER AND ARTIN-SCHREIER-WITT THEORIES

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1. MOTIVATION

Our aim of this report is to give an explanation of the final version of our theory which unifies the Kummer theory and Artin-Schreier-Witt theory. The details of this report can be seen in the Bordeaux preprint [15].

First, we review the Kummer theory.

Let n be an integer with $n \geq 2$, and K be a field of characteristic q with $q \nmid n$ and $K \supset \mu_n = \{\zeta \mid \zeta^n = 1\}$.

Theorem 1.1 (Kummer Theory).

L/K : n -cyclic Galois extension

$$\iff \exists a \in K^* \text{ s.t. } L = K(\sqrt[n]{a})$$

$$\begin{array}{ccccc} L = K \otimes_{K[X, X^{-1}]} K[X, X^{-1}] & \leftarrow & K[X, X^{-1}] & & X^n \\ & & \uparrow & & \uparrow \\ \iff \exists a \in K^* \text{ s.t.} & & K & \leftarrow & K[X, X^{-1}] & & X \\ & & a & \leftarrow & 1 & & X \end{array}$$

$$\iff \exists f : \text{Spec } K \rightarrow \mathbb{G}_{m,K} \text{ s.t.} \quad \begin{array}{ccc} \text{Spec } L & \rightarrow & \mathbb{G}_{m,K} \\ \downarrow & \square & \downarrow \theta_n \\ \text{Spec } K & \xrightarrow{f} & \mathbb{G}_{m,K} \end{array},$$

where $\theta_n : \mathbb{G}_{m,K} \rightarrow \mathbb{G}_{m,K}; x \mapsto x^n$.

Namely, the Kummer theory implies that the following exact sequence (so-called the **Kummer exact sequence**) of sheaves on the fppf (or étale) site on $\text{Spec } K$ is essential in the world of cyclic coverings of K :

$$1 \rightarrow \mu_{n,K} \rightarrow \mathbb{G}_{m,K} \xrightarrow{\theta_n} \mathbb{G}_{m,K} \rightarrow 1$$

$$t \mapsto t^n$$

In fact, from the exact sequence, for any K -scheme X we can deduce the exact sequence:

$$\mathbb{G}_{m,K}(X) \xrightarrow{\theta_n} \mathbb{G}_{m,K}(X) \xrightarrow{\partial} H^1(X, \mu_{n,K}) \rightarrow H^1(X, \mathbb{G}_{m,K}) \rightarrow H^1(X, \mathbb{G}_{m,K}).$$

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Here

$H^1(X, \mu_{n,K})$ = the set of isomorphism classes of unramified μ_n coverings of X

$H^1(X, \mathbb{G}_{m,K}) = 0$ for suitable X 's by Hilbert Theorem 90

Next we review the Artin-Schreier-Witt theory.

Let k be a field of positive characteristic p .

$W_{n,k}$: the group scheme of Witt vectors of length n

$$\wp : W_{n,k} \rightarrow W_{n,k}; x \mapsto x^{(p)} - x$$

Theorem 1.2 (Artin-Schreier-Witt Theory).

K/k : p^n -cyclic Galois extension

$$\iff \exists a \in W_n(k) \text{ s.t. } K = k(\wp^{-1}(a))$$

$$\iff \exists a = (a_0, a_1, \dots, a_{n-1}) \in W_n(k) \text{ s.t.}$$

$K = k \otimes_{k[\mathbb{X}]} k[\mathbb{X}]$	$\leftarrow$	$k[\mathbb{X}]$
	$\uparrow$	$\uparrow \wp^*$
k	$\leftarrow$	$k[\mathbb{X}]$
a_i	$\leftarrow$	X_i

$$\iff \exists f : \text{Spec } k \rightarrow W_{n,k} \text{ s.t.}$$

$\text{Spec } K$	$\rightarrow$	$W_{n,k}$
$\downarrow$	$\square$	$\downarrow \wp$
$\text{Spec } k$	$\xrightarrow{f}$	$W_{n,k}$

where $\mathbb{X} = (X_0, X_1, \dots, X_{n-1})$.

Namely, the Artin-Schreier-Witt theory implies that the following exact sequence (so-called the **Artin-Schreier-Witt exact sequence**) of sheaves on the fppf (or étale) site on the $\text{Spec } k$ is essential:

$$0 \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow W_{n,k} \xrightarrow{\wp} W_{n,k} \rightarrow 0.$$

$$x \longmapsto x^{(p)} - x$$

In fact, from the exact sequence, for any k -scheme X we can deduce the exact sequence:

$$W_{n,k}(X) \xrightarrow{\wp} W_{n,k}(X) \xrightarrow{\partial} H^1(X, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow H^1(X, W_{n,k}) \rightarrow H^1(X, W_{n,k}).$$

Here

$H^1(X, \mathbb{Z}/p^n\mathbb{Z})$ = the set of isomorphism classes of unramified $\mathbb{Z}/p^n\mathbb{Z}$ coverings of X

$H^1(X, W_{n,k}) = 0$ for affine schemes X

Therefore, the **Kummer theory** implies that in the world of unramified p^n -cyclic coverings in characteristic 0, the **Kummer exact sequence** is the **Buddha**, and any such coverings is deduced from the sequence. On the other hand, the **Artin-Schreier-Witt theory** implies that in the world of unramified p^n -cyclic coverings in characteristic p , the

Artin-Schreier-Witt exact sequence is the **Buddha**, and any such covering is deduced from the sequence. But our religion asserts that every Buddha should be deduced from the unique essential **Buddha** (**Mahāvairocanaḥ**). Hence, behind the two Buddhas, there should exist a more essential Buddha unifying them.

So we arrive at the following problems:

- Search for the Buddha unifying the Kummer and ASW sequences.
- Construct the deformations of the group schemes of Witt vectors of finite length to tori.
- Such deformations should keep the filtrations of the group schemes of Witt vectors.

2. 1 DIMENSIONAL CASE

Let $(A, \mathfrak{m})$ be a DVR with f.f. $A = K$ and $A/\mathfrak{m} = k$, and $\lambda \in \mathfrak{m} \setminus \{0\}$. Now we look at the plane curve over A :

$$C: Y^2Z - \lambda XYZ - X^3 = 0 \subset \mathbb{P}^2,$$

whose generic fibre is a nodal curve and the special fibre is a cuspidal curve. Therefore the Picard scheme of the curve gives a deformation of an additive group scheme to a torus:

$$\mathrm{Pic}^0(C/A) \cong \mathrm{Spec} A[X, 1/(1 + \lambda X)],$$

with group law $x \cdot y = \lambda xy + x + y$. Hereafter we denote this group scheme by $\mathcal{G}^{(\lambda)}$:

$$\mathcal{G}^{(\lambda)} := \mathrm{Spec} A[X, 1/(1 + \lambda X)].$$

The important fact is that any deformations of $\mathbb{G}_a$ to $\mathbb{G}_m$ over A are only the type of $\mathcal{G}^{(\lambda)}$'s. In fact, we have the following.

Theorem 2.1 ([17, Th. 2.5]). *Let $\mathcal{G}$ be a flat group scheme over $\mathrm{Spec} A$ with gereric fibre $\mathbb{G}_m$ and special fibre $\mathbb{G}_a$. Then there exists a non-zero element λ of $\mathfrak{m}$, uniquely up to unit factors, such that*

$$\mathcal{G} \cong \mathcal{G}^{(\lambda)}.$$

3. HIGHER DIMENSIONAL CASE

If we obtain a deformation $\mathcal{W}_{n-1}$ of W_{n-1} to $\mathbb{G}_{m,K}^{n-1}$, then since the Witt vectors has the filtration

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z}/p & \rightarrow & \mathbb{Z}/p^n & \rightarrow & \mathbb{Z}/p^{n-1} \rightarrow 0 \\ & & \cap & & \cap & & \cap \\ 0 & \rightarrow & \mathbb{G}_{a,k} & \rightarrow & W_{n,k} & \rightarrow & W_{n-1,k} \rightarrow 0, \end{array}$$

we can expect the next one fits into an extension

$$0 \rightarrow \mathcal{G}^{(\lambda)} \rightarrow \mathcal{W}_{n+1} \rightarrow \mathcal{W}_n \rightarrow 0 \in \mathrm{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}).$$

Definition 3.1. Let $(A, \mathfrak{m})$ be a DVR, and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathfrak{m} \setminus \{0\}$. If $\mathcal{W}_n$ is given by the extensions

$$\begin{aligned} 0 \rightarrow \mathcal{G}^{(\lambda_2)} \rightarrow \mathcal{W}_2 \rightarrow \mathcal{G}^{(\lambda_1)} \rightarrow 0 &\in \text{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)}) \\ 0 \rightarrow \mathcal{G}^{(\lambda_3)} \rightarrow \mathcal{W}_3 \rightarrow \mathcal{W}_2 \rightarrow 0 &\in \text{Ext}^1(\mathcal{W}_2, \mathcal{G}^{(\lambda_3)}) \\ \dots\dots\dots \\ 0 \rightarrow \mathcal{G}^{(\lambda_n)} \rightarrow \mathcal{W}_n \rightarrow \mathcal{W}_{n-1} \rightarrow 0 &\in \text{Ext}^1(\mathcal{W}_{n-1}, \mathcal{G}^{(\lambda_n)}), \end{aligned}$$

we call it a **group scheme of type $(\lambda_1, \lambda_2, \dots, \lambda_n)$** .

To compute the group $\text{Ext}^1(\mathcal{W}_\ell, \mathcal{G}^{(\lambda_{\ell+1})})$ for a group scheme $\mathcal{W}_\ell$ of type $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$, the following exact sequence of sheaves on each the small Zariski, fppf or étale site on $\text{Spec } A$ is essential:

$$\begin{array}{ccccccc} 0 \rightarrow & \mathcal{G}^{(\lambda)} & \xrightarrow{\alpha^{(\lambda)}} & \mathbb{G}_{m,A} & \xrightarrow{\rho^{(\lambda)}} & \iota_* \mathbb{G}_{m,A/\lambda} & \rightarrow 0. \\ & x & \mapsto & 1 + \lambda x & & & \\ & & & t & \mapsto & t \bmod \lambda & \end{array}$$

where $\iota : \text{Spec}(A/\lambda) \hookrightarrow \text{Spec } A$ is the canonical inclusion.

By an explicit computation of cocycles, we have

Proposition 3.1.

$$\text{Ext}^1(\mathcal{G}^{(\lambda)}, \mathbb{G}_{m,A}) = 0.$$

Therefore inductively we have

$$\text{Ext}^1(\mathcal{W}_\ell, \mathbb{G}_{m,A}) = 0,$$

for any group scheme of type $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$.

Hence, by using the above exact sequence we obtain the following.

Theorem 3.2. Let $\mathcal{W}_n$ be a group scheme of type $(\lambda_1, \lambda_2, \dots, \lambda_n)$, and $\lambda \in \mathfrak{m} \setminus \{0\}$. Then we have

$$\text{Ext}^1(\mathcal{E}, \mathcal{G}^{(\lambda)}) \cong \text{Hom}(\mathcal{E}, \iota_* \mathbb{G}_{m,A/\lambda}) / (\rho^{(\lambda)})_* (\text{Hom}(\mathcal{E}, \mathbb{G}_{m,A})).$$

From this theorem, we can deduce the following.

Theorem 3.3. Let $\mathcal{W}_n$ be a group scheme of type $(\lambda_1, \lambda_2, \dots, \lambda_n)$. Then there exists a homomorphism

$$D_\ell : \mathcal{W}_\ell \rightarrow \iota_* \mathbb{G}_{m,A/\lambda_{\ell+1}}$$

for each ℓ ($2 \leq \ell \leq n-1$), and each $\mathcal{W}_\ell$ is given by

$$\begin{aligned} \mathcal{W}_\ell \cong \text{Spec } A[X_0, \dots, X_{\ell-1}, \frac{1}{1 + \lambda_1 X_0}, \\ \frac{1}{D_1(X_0) + \lambda_2 X_1}, \dots, \frac{1}{D_{\ell-1}(X_0, \dots, X_{\ell-2}) + \lambda_\ell X_{\ell-1}}]. \end{aligned}$$

Moreover, the group law of $\mathcal{W}_\ell$ is the one which makes the morphism

$$\begin{aligned} \alpha^{(\ell)} : \mathcal{W}_\ell &\rightarrow (\mathbb{G}_{m,A})^\ell \\ (X_0, \dots, X_{\ell-1}) &\mapsto (1 + \lambda_1 X_0, D_1(X_0) + \lambda_2 X_1, \\ &\quad \dots, D_{\ell-1}(X_0, \dots, X_{\ell-2}) + \lambda_\ell X_{\ell-1}) \end{aligned}$$

a group-schematic homomorphism.

Definition 3.2. Suppose that A dominates $\mathbb{Z}_{(p)}[\mu_{p^n}]$, and put $\lambda = \lambda_{(1)}$. We call a group scheme $\mathcal{W}_1 = \mathcal{G}^{(\lambda)}, \mathcal{W}_2, \dots, \mathcal{W}_n$ over A of type $(\lambda)^n = \overbrace{(\lambda, \lambda, \dots, \lambda)}^n$ a **KASW group scheme** over A , if there exists an inclusion $i_\ell : \mathbb{Z}/p^\ell \hookrightarrow \mathcal{W}_\ell$ for each ℓ satisfying a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & (\mathbb{Z}/p)_A & \rightarrow & (\mathbb{Z}/p^\ell)_A & \rightarrow & (\mathbb{Z}/p^{\ell-1})_A \rightarrow 0 \\ & & \downarrow i_1 & & \downarrow i_\ell & & \downarrow i_{\ell-1} \\ 0 & \rightarrow & \mathcal{G}^{(\lambda)} & \rightarrow & \mathcal{W}_\ell & \xrightarrow{r_\ell} & \mathcal{W}_{\ell-1} \rightarrow 0. \end{array}$$

Once we obtain a KASW group scheme, then it embodies the unified Kummer-Artin-Schreier-Witt theory.

Theorem 3.4 (KASW theory). Let $\mathcal{W}_n$ be a KASW group scheme over A . Let B and C are local flat A -algebras such that C is an unramified p^n -cyclic covering over B . Then there exists an A -morphism $f : \text{Spec } B \rightarrow \mathcal{W}_n/(\mathbb{Z}/p^n)$, and the covering $\text{Spec } C \rightarrow \text{Spec } B$ is given by the fibre product

$$\begin{array}{ccc} \text{Spec } C & \xrightarrow{\quad} & \mathcal{W}_n \\ \downarrow & \square & \downarrow \\ \text{Spec } B & \xrightarrow{f} & \mathcal{W}_n/(\mathbb{Z}/p^n). \end{array}$$

By these argument, our work is concentrated upon the calculation of $\text{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)})$, namely of $\text{Hom}(\mathcal{W}_n, \mathbb{G}_{m,A/\lambda})$.

4. DETERMINATION OF $\text{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)})$

We provide some notations.

$(A, \mathfrak{m})$: DVR dominating $\mathbb{Z}_{(p)}$, $\lambda \in \mathfrak{m} \setminus \{0\}$

$\Phi_n(\mathbb{T}) = T_0^{p^n} + pT_1^{p^{n-1}} + \dots + p^n T_n$: Witt polynomial

$\tilde{a} := (a, 0, 0, \dots) \in W(A)$ for $A \in A$

$[p] : W_A \rightarrow W_A$; $[p]\mathbf{b} := (0, b_0^p, b_1^p, \dots)$ for $\mathbf{b} = (b_0, b_1, \dots)$

$V : W_A \rightarrow W_A$: Verschiebung endomorphism

$F : W_A \rightarrow W_A$: the generalized Frobenius endomorphism

$F^{(\lambda)} := F - (\lambda^{p-1})^\sim$

For $\mathbf{a} \in W(A)$, we define $T_{\mathbf{a}} : W(A) \rightarrow W(A)$ by

$$\Phi_n(T_{\mathbf{a}}\mathbf{x}) = a_0^{p^n} \Phi_n(\mathbf{x}) + pa_1^{p^{n-1}} \Phi_{n-1}(\mathbf{x}) + \dots + p^n a_n \Phi_0(\mathbf{x}) \quad (n \geq 0)$$

for $x \in W(A)$. Then we have $T_a = \sum_{k \geq 0} V^k \cdot \tilde{a}_k$.

If A is a ring (not necessarily a $\mathbb{Z}_{(p)}$ -algebra),

$$\widehat{W}_n(A) = \left\{ (a_0, a_1, \dots, a_{n-1}) \in W_n(A) ; a_i \text{ is nilpotent for all } i \right\}$$

and

$$\widehat{W}(A) = \left\{ (a_0, a_1, a_2, \dots) \in W_n(A) ; \begin{array}{l} a_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

Moreover we need to deform the Artin-Hasse exponential series

$$\begin{aligned} E_p(X) &:= \exp \left(X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \dots \right) \\ &= e^X e^{\frac{X^p}{p}} e^{\frac{X^{p^2}}{p^2}} \dots \in \mathbb{Z}_{(p)}[[X]]. \end{aligned}$$

The well-known formula $\lim_{\lambda \rightarrow 0} (1 + \lambda x)^{\alpha/\lambda} = e^{\alpha x}$ can be seen that $(1 + \lambda x)^{\alpha/\lambda}$ is a deformation of $e^{\alpha x}$. From this point of view, we obtain the deformations of Artin-Hasse exponential series:

$$\begin{aligned} E_p(U, \Lambda; X) &:= (1 + \Lambda X)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} \left(1 + \Lambda^{p^k} X^{p^k} \right)^{\frac{1}{p^k} \left(\left(\frac{U}{\Lambda} \right)^{p^k} - \left(\frac{U}{\Lambda} \right)^{p^{k-1}} \right)} \\ &\in \mathbb{Z}_{(p)}[U, \Lambda][[X]]. \end{aligned}$$

Moreover for a Witt vector $\mathbf{a} \in W(A)$, we define a formal power series as follows:

$$\begin{aligned} E_p(\mathbf{a}, \lambda; X) &:= \prod_{k=0}^{\infty} E_p(a_k, \lambda^{p^k}; X^{p^k}) \\ &= (1 + \lambda X)^{\frac{a_0}{\lambda}} \prod_{k=1}^{\infty} \left(1 + \lambda^{p^k} X^{p^k} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)} \mathbf{a})}. \end{aligned}$$

The boundary of this power series $E_p(\mathbf{a}, \lambda; X)$ is given by the following.

$$\begin{aligned} (\partial E_p(\mathbf{a}, \lambda; \cdot))(X, Y) &= \frac{E_p(\mathbf{a}, \lambda; X) E_p(\mathbf{a}, \lambda; Y)}{E_p(\mathbf{a}, \lambda; X + Y + \lambda XY)} \\ &= \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)} \mathbf{a})}. \end{aligned}$$

Now replacing $F^{(\lambda)} \mathbf{a}$ with a Witt vector $\mathbf{b} = (b_0, b_1, \dots)$ in the right hand side of this equation, we define a cocycle as follows.

$$\begin{aligned} F_p(\mathbf{b}, \lambda; X, Y) &:= \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(\mathbf{b})} \\ &\in \mathbb{Z}_{(p)}[\mathbf{b}, \lambda][[X, Y]]. \end{aligned}$$

Using these deformed Artin-Hasse exponential series, we can obtain the following.

Theorem 4.1 (Explicit Formula in 1 Dimensional Case).

$$\begin{array}{ccc} \xi_0^1 : \text{Ker} \left(\widehat{W}(A/\lambda_2) \xrightarrow{F^{(\lambda_1)}} \widehat{W}(A/\lambda_2) \right) & \xrightarrow{\sim} & \text{Hom}(\mathcal{G}^{(\lambda_1)}, {}_{\mathcal{L}}\mathbb{G}_{m,A/\lambda_2}), \\ \mathbf{a} & \mapsto & E_p(\mathbf{a}, \lambda_1; X) \\ \xi_1^0 : \text{Coker} \left(\widehat{W}(A/\lambda_2) \xrightarrow{F^{(\lambda_1)}} \widehat{W}(A/\lambda_2) \right) & \xrightarrow{\sim} & H_0^2(\mathcal{G}^{(\lambda_1)}, {}_{\mathcal{L}}\mathbb{G}_{m,A/\lambda_2}). \\ \mathbf{b} & \mapsto & F_p(\mathbf{b}, \lambda_1; X, Y) \end{array}$$

Therefore

$$\xi_0^1 : \frac{\text{Ker} \left(\widehat{W}(A/\lambda_2) \xrightarrow{F^{(\lambda_1)}} \widehat{W}(A/\lambda_2) \right)}{\langle \widetilde{\lambda_1} \rangle} \xrightarrow{\sim} \frac{\text{Hom}(\mathcal{G}^{(\lambda_1)}, {}_{\mathcal{L}}\mathbb{G}_{m,A/\lambda_2})}{\text{Hom}(\mathcal{G}^{(\lambda_1)}, \mathbb{G}_{m,A})} \xrightarrow{\sim} \text{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)}).$$

In higher dimensional case, we need more notations. For a vector $\mathbb{U} = (U_0, U_1, \dots)$, we define

$$\begin{aligned} [p]E_p(\mathbb{U}, \Lambda; X) &:= E_p([p]\mathbb{U}, \Lambda; X), \\ [p]F_p(\mathbb{U}, \Lambda; X, Y) &:= F_p([p]\mathbb{U}, \Lambda; X, Y). \end{aligned}$$

Moreover

$$\begin{aligned} H(X, Y) &:= \frac{1}{\Lambda_2} \{ F_p(\mathbb{U}, \Lambda_1; X, Y) - 1 \}, \\ G_p(\mathbb{A}, \Lambda_2; E) &:= \prod_{\ell \geq 1} \left(\frac{1 + (E - 1)^{p^\ell}}{[p]^\ell E} \right)^{\frac{1}{p^\ell \Lambda_2^{p^\ell}} \Phi_{\ell-1}(\mathbb{A})}, \\ G_p(\mathbb{A}, \Lambda_2; F) &:= \prod_{\ell \geq 1} \left(\frac{1 + (F - 1)^{p^\ell}}{[p]^\ell F} \right)^{\frac{1}{p^\ell \Lambda_2^{p^\ell}} \Phi_{\ell-1}(\mathbb{A})} \\ &\in \mathbb{Z}_{(p)}[\mathbb{A}, \frac{\mathbb{U}}{\Lambda_2}, \Lambda_1, \Lambda_2][[X, Y]] \end{aligned}$$

For a series of variables $\Lambda_1, \Lambda_2, \dots$, and a series of vectors $\mathbb{A}_j^i = (A_{j0}^i, A_{j1}^i, \dots)$ ($1 \leq i; 1 \leq j \leq i$), we denote

$$\mathbb{A}^i = (\mathbb{A}_\ell^i)_{1 \leq \ell \leq i} := \begin{pmatrix} \mathbb{A}_1^i \\ \mathbb{A}_2^i \\ \vdots \\ \mathbb{A}_i^i \end{pmatrix} \quad \text{and} \quad (\Lambda_\ell)_{1 \leq \ell \leq i} := \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_i \end{pmatrix}.$$

We define vectors $\mathbb{B}_j^i$ ($1 \leq j < i$) inductively by

$$\mathbb{B}_1^2 := \frac{1}{\Lambda_2} F^{(\Lambda_1)} \mathbb{A}_1^1,$$

and for $k \geq 2$,

$$\begin{cases} \mathbb{B}_j^{k+1} := \frac{1}{\Lambda_{k+1}} \left(F^{(\Lambda_j)} \mathbb{A}_j^k - \sum_{\ell=j+1}^k T_{\mathbb{B}_j^\ell} \mathbb{A}_\ell^k \right) & 1 \leq j \leq k-1 \\ \mathbb{B}_k^{k+1} := \frac{1}{\Lambda_{k+1}} F^{(\Lambda_k)} \mathbb{A}_k^k. \end{cases}$$

Using these symbols, we define triangle matrices U^n 's by

$$U^n := \begin{pmatrix} F^{(\Lambda_1)} & -T_{\mathbf{B}_1^2} & -T_{\mathbf{B}_1^3} & \cdots & -T_{\mathbf{B}_1^n} \\ \mathbf{0} & F^{(\Lambda_2)} & -T_{\mathbf{B}_2^3} & \cdots & -T_{\mathbf{B}_2^n} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & -T_{\mathbf{B}_{n-1}^n} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & F^{(\Lambda_n)} \end{pmatrix}.$$

We define inductively a series of formal power series $D_k(X_0, X_1, \dots, X_{k-1})$'s by

$$\begin{aligned} D_0 &= 1, \\ D_1(X_0) &= E_p(\mathbb{A}_1^1, \Lambda_1; X_0), \end{aligned}$$

and for $k \geq 1$,

$$\begin{aligned} D_{k+1}(X_0, X_1, \dots, X_k) &= E_p(\mathbb{A}^{k+1}, (\Lambda_\ell)_{1 \leq \ell \leq k+1}; X_0, X_1, \dots, X_k) \\ &:= \prod_{i=1}^{k+1} E_p(\mathbb{A}_i^{k+1}, \Lambda_i; \frac{X_{i-1}}{D_{i-1}(X_0, \dots, X_{i-1})}). \end{aligned}$$

Hereafter, we put $\mathbb{X} = (X_0, X_1, \dots)$, $\mathbb{Y} = (Y_0, Y_1, \dots)$ and $\Sigma := \mathbb{X} \dot{+} \mathbb{Y} \in W$. We define

$$\begin{aligned} F^{(k)} &:= \partial(D_k(\mathbb{X})) = \frac{D_k(\mathbb{X})D_k(\mathbb{Y})}{D_k(\Sigma)} \\ H_k(\mathbb{X}, \mathbb{Y}) &:= \frac{1}{\Lambda_{k+1}}(F^{(k)} - 1) \\ F_p(\mathbb{V}_1, \Lambda_1; \mathbb{X}, \mathbb{Y}) &:= F_p(\mathbb{V}_1, \Lambda_1; X_0, Y_0) \\ F_p((\mathbb{V}_i)_{1 \leq i \leq n}, (\Lambda_i)_{1 \leq i \leq n}; \mathbb{X}, \mathbb{Y}) \\ &= \prod_{i=1}^n F_p(\mathbb{V}_i, \Lambda_i; \frac{X_{i-1}}{D_{i-1}(\mathbb{X})}, \frac{Y_{i-1}}{D_{i-1}(\mathbb{Y})}) \\ &\quad \times \prod_{i=2}^n F_p(\mathbb{V}_i, \Lambda_i; H_{i-1}, \frac{X_{i-1}}{D_{i-1}(\mathbb{X})} \dot{+} \frac{Y_{i-1}}{D_{i-1}(\mathbb{Y})}) \\ &\quad \times \prod_{i=2}^n G_p(\mathbb{V}_i, \Lambda_i; F^{(i-1)})^{-1}. \end{aligned}$$

Then the important thing is the following result.

Theorem 4.2. *For each $n \geq 1$, we have*

$$F^{(n)} = \frac{D_n(\mathbb{X})D_n(\mathbb{Y})}{D_n(\Sigma)} = F_p(U^n \mathbb{A}^n, (\Lambda_i)_{1 \leq i \leq n}; \mathbb{X}, \mathbb{Y}).$$

By using this theorem, we can obtain the explicit determination of $\text{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)})$. in fact, let $(A, \mathfrak{m})$ be a DVR dominating $\mathbb{Z}_{(p)}$, and $\lambda, \lambda_1, \lambda_2, \dots$ be non-zero elements of $\mathfrak{m}$. We choose Witt vectors

$$\bar{\mathbf{a}}^i = (\bar{\mathbf{a}}_j^i)_{1 \leq j \leq i} \in \text{Ker} \left(U^i : \widehat{W}(A/\lambda_{i+1})^i \rightarrow \widehat{W}(A/\lambda_{i+1})^i \right)$$

inductively by the following recursive conditions:

$$\begin{aligned} U^1 &= F^{(\lambda_1)}, \\ \bar{a}^1 &= \bar{a}_1^1 \in \text{Ker} \left(U^1 : \widehat{W}(A/\lambda_2) \rightarrow \widehat{W}(A/\lambda_2) \right), \\ b_1^2 &= \frac{1}{\lambda_2} a_1^1, \quad U^2 = \begin{pmatrix} F^{(\lambda_1)} & -T_{b_1^2} \\ 0 & F^{(\lambda_2)} \end{pmatrix}, \end{aligned}$$

and for $k \geq 2$, we choose

$$\bar{a}^k = (\bar{a}_i^k)_{1 \leq i \leq k} \in \text{Ker} \left(U^k : \widehat{W}(A/\lambda_{k+1})^k \rightarrow \widehat{W}(A/\lambda_{k+1})^k \right),$$

and we define

$$\begin{aligned} \begin{cases} b_j^{k+1} := \frac{1}{\lambda_{k+1}} \left(F^{(\lambda_j)} a_j^k - \sum_{\ell=j+1}^k T_{b_j^k} a_\ell^k \right) & 1 \leq j \leq k-1 \\ b_k^{k+1} := \frac{1}{\lambda_{k+1}} F^{(\lambda_k)} a_k^k, \end{cases} \\ U^{k+1} := \begin{pmatrix} F^{(\lambda_1)} & -T_{b_1^k} & -T_{b_2^k} & \cdots & -T_{b_k^k} \\ 0 & F^{(\lambda_2)} & -T_{b_2^k} & \cdots & -T_{b_k^k} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & -T_{b_k^k} \\ 0 & 0 & \cdots & 0 & F^{(\lambda_{k+1})} \end{pmatrix}, \\ \bar{a}^{k+1} = (\bar{a}_i^{k+1})_{1 \leq i \leq k+1} \in \text{Ker} \left(U^{k+1} : \widehat{W}(A/\lambda_{k+2})^{k+1} \rightarrow \widehat{W}(A/\lambda_{k+2})^{k+1} \right). \end{aligned}$$

We define formal power series $D_k(\mathbb{X}) = D_k(X_0, \dots, X_{k-1})$ ($k \geq 1$) by

$$\begin{aligned} D_0 &= 1, \\ D_1(X_0) &= E_p(a_1^1, \lambda_1; X_0), \end{aligned}$$

and for $k \geq 1$,

$$\begin{aligned} D_{k+1}(X_0, X_1, \dots, X_k) &= E_p(a^{k+1}, (\lambda_\ell)_{1 \leq \ell \leq k+1}; X_0, X_1, \dots, X_k) \\ &:= \prod_{i=1}^{k+1} E_p(a_i^{k+1}, \lambda_i; \frac{X_{i-1}}{D_{i-1}(X_0, \dots, X_{i-1})}). \end{aligned}$$

We put

$$\mathcal{W}_n := \text{Spec } A[X_0, \dots, X_{n-1}, \frac{1}{1 + \lambda_1 X_0}, \frac{1}{D_1(X_0) + \lambda_2 X_1}, \dots, \frac{1}{D_{n-1}(\mathbb{X}) + \lambda_n X_{n-1}}].$$

Theorem 4.3 (Explicit Formula in General Case). *Let $B = A/\lambda$. Then we have*

$$\begin{aligned} \xi_0^n : \text{Ker}(\widehat{W}(B)^n \xrightarrow{U^n} \widehat{W}(B)^n) &\xrightarrow{\sim} \text{Hom}(\mathcal{W}_{n,B}, \mathbb{G}_{m,B}); \\ \bar{v}^n &= (\bar{v}_i^n)_{1 \leq i \leq n} \mapsto E_p(\bar{v}^n, (\lambda_i)_{1 \leq i \leq n}; X_0, X_1, \dots, X_{n-1}) \\ \xi_1^n : \text{Coker}(\widehat{W}(B)^n \xrightarrow{U^n} \widehat{W}(B)^n) &\xrightarrow{\sim} H_0^2(\mathcal{W}_{n,B}, \mathbb{G}_{m,B}). \\ \bar{w}^n &= (\bar{w}_i^n)_{1 \leq i \leq n} \mapsto F_p(\bar{w}^n, (\lambda_i)_{1 \leq i \leq n}; \mathbb{X}, \mathbb{Y}) \end{aligned}$$

Theorem 4.4.

$$\bar{\xi}_0^n : \frac{\text{Ker}(U^n : \widehat{W}(B)^n \rightarrow \widehat{W}(B/\lambda)^n)}{\langle \mathbf{c}^0, \mathbf{c}^1, \dots, \mathbf{c}^{n-1} \rangle} \xrightarrow{\sim} \text{Ext}^1(\mathcal{W}_{n,B}, \mathcal{G}_B^{(\lambda)}),$$

where $\langle \mathbf{c}^0, \mathbf{c}^1, \dots, \mathbf{c}^{n-1} \rangle$ is the subgroup generated by the vectors $\mathbf{c}^0 = (\tilde{\lambda}_1, 0, \dots, 0), \mathbf{c}^1 = (\mathbf{a}^1, \tilde{\lambda}_2, 0, \dots, 0), \dots, \mathbf{c}^\ell = (\mathbf{a}^\ell, \tilde{\lambda}_{\ell+1}, 0, \dots, 0), \dots, \mathbf{c}^{n-1} = (\mathbf{a}^{n-1}, \tilde{\lambda}_n)$.

5. REDUCTIONS OF EXTENSIONS

The special fibres of the group schemes of type $(\lambda_1, \lambda_2, \dots, \lambda_n)$ can be decided as follows.

Theorem 5.1. *Let*

$$\mathcal{W}_n = \text{Spec } A[X_0, X_1, \dots, X_{n-1}, \frac{1}{1 + \lambda_1 X_0}, \frac{1}{D_1(X_0) + \lambda_2 X_1}, \dots, \frac{1}{D_{n-1}(\mathbb{X}) + \lambda_n X_{n-1}}]$$

be the group scheme of type $(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ defined by

$$D_1(X_0) = E_p(\mathbf{a}_1^1, \lambda_1; X_0)$$

and for $1 \leq k \leq n-2$,

$$D_{k+1}(\mathbb{X}) = E_p(\mathbf{a}^{k+1}, (\lambda_\ell)_{1 \leq \ell \leq k+1}; X_0, X_1, \dots, X_k),$$

and

$$\bar{\mathbf{a}}^k \in \text{Ker}(U^k : \widehat{W}(A/\lambda_{k+1})^k \rightarrow \widehat{W}(A/\lambda_k)^k).$$

Here

$$\mathbf{b}^i = {}^t(b_1^i, b_2^i, \dots, b_{i-1}^i) = \frac{1}{\lambda_i} U^{i-1} \mathbf{a}^{i-1} \quad (i = 2, \dots, n).$$

If $\mathbf{b}_\ell^k \equiv \mathbf{0} \pmod{\mathfrak{m}}$ for $3 \leq k \leq n$, $1 \leq \ell \leq k-2$, and $\mathbf{b}_{k-1}^k \equiv (1, 0, \dots) \pmod{\mathfrak{m}}$, then we have

$$\mathcal{W}_{n,k} = \mathcal{W}_n \otimes_A k = \mathcal{W}_{n,k}.$$

6. CONDITIONS FOR KRSW GROUP SCHEMES

Let

$$\mathcal{W}_n = \text{Spec } A[X_0, \dots, X_{n-1}, \frac{1}{1 + \lambda_{(1)} X_0}, \frac{1}{D_1(X_0) + \lambda_{(1)} X_1}, \dots, \frac{1}{D_{n-1}(X_0, \dots, X_{n-2}) + \lambda_{(1)} X_{n-1}}]$$

be a KASW group scheme over a DVR $(A, \mathfrak{m})$, and λ be an element of $\mathfrak{m} \setminus \{0\}$. Here D_i 's are given by

$$D_i(\mathbb{X}) = E_p(\mathbf{a}^i, (\lambda_{(1)})^n; \mathbb{X})$$

with

$$\bar{\mathbf{a}}^i \in \text{Ker}(U^i : \widehat{W}(A/\lambda_{(1)}) \rightarrow \widehat{W}(A/\lambda_{(1)})).$$

We look at the exact sequence

$$0 \rightarrow (\mathbb{Z}/p^n)_A \xrightarrow{i_n} \mathcal{W}_n \xrightarrow{\psi_n} \mathcal{W}_n/(\mathbb{Z}/p^n)_A \rightarrow 0.$$

Then we have

$$\begin{array}{ccc}
 i_n^* : \text{Ext}_A^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}) & \rightarrow & \text{Ext}_A^1((\mathbb{Z}/p^n)_A, \mathcal{G}^{(\lambda)}) \\
 \parallel \wr & & \parallel \wr \\
 \frac{\text{Ker}(\widehat{W}(A/\lambda)^n \xrightarrow{U^n} \widehat{W}(A/\lambda))}{\langle c^0, c^1, \dots, c^{n-1} \rangle} & \rightarrow & (1 + \lambda A) / (1 + \lambda A)^{p^n} \\
 \mathbf{a}^n & \mapsto & \prod_{r>0} \left(E_p(\mathbf{a}_{1,r}^n, \lambda_{(1)}^{p^r}; 1)^{p^n} \prod_{i=2}^n E_p(\mathbf{a}_{i,r}^n, \lambda_{(1)}^{p^r}; \left(\frac{c_{i-1}}{D_{i-1}(i_n(1))} \right)^{p^r})^{p^n} \right)
 \end{array}$$

Under these notations, we have the following.

Theorem 6.1. *Let $\mathcal{W}_{n+1} \in \text{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)})$ be the extension corresponding to a vector $\mathbf{a}^n = (\mathbf{a}_i^n)_{1 \leq i \leq n}$ by the isomorphism*

$$\frac{\text{Ker}(U^n : \widehat{W}(A/\lambda)^n \rightarrow \widehat{W}(A/\lambda)^n)}{(c^1, c^2, \dots, c^{n-1})} \simeq \text{Ext}^1(\mathcal{W}_n, \mathcal{G}^{(\lambda)}).$$

Then there exists an inclusion $(\mathbb{Z}/p^{n+1})_A \subset \mathcal{W}_{n+1}$ fitting into a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (\mathbb{Z}/p)_A & \longrightarrow & (\mathbb{Z}/p^{n+1})_A & \longrightarrow & (\mathbb{Z}/p^n)_A \longrightarrow 0 \\
 & & i_1 \downarrow & & i_{n+1} \downarrow & & i_n \downarrow \\
 0 & \longrightarrow & \mathcal{G}^{(\lambda)} & \longrightarrow & \mathcal{W}_{n+1} & \longrightarrow & \mathcal{W}_n \longrightarrow 0,
 \end{array}$$

if and only if

$$E_p(\mathbf{a}^n, (\lambda, \dots, \lambda); i_n(1))^{p^n} = \zeta_1.$$

Using these results, we construct explicitly the KASW group schemes.

Theorem 6.2 (Main Theorem). *For each positive integer n , we construct explicitly a standard KASW group scheme $\mathcal{W}_n$ over $\mathbb{Z}_{(p)}[\mu_{p^n}]$.*

Finally, we remark that for a KASW group scheme $\mathcal{W}_n$, the quotient $\mathcal{V}_n := \mathcal{W}_n/(\mathbb{Z}/p^n)$ is a group scheme of type $(\lambda_{(1)}^p)^n$, and which is given explicitly.

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