

On Sakaguchi type functions

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Abstract

Two subclasses $\mathcal{S}(\alpha, t)$ and $\mathcal{T}(\alpha, t)$ are introduced concerning with Sakaguchi functions in the open unit disk $\mathbb{U}$. Further, by using the coefficient inequalities for the classes $\mathcal{S}(\alpha, t)$ and $\mathcal{T}(\alpha, t)$, two subclasses $\mathcal{S}_0(\alpha, t)$ and $\mathcal{T}_0(\alpha, t)$ are defined. The object of the present paper is to discuss some properties of functions belonging to the classes $\mathcal{S}_0(\alpha, t)$ and $\mathcal{T}_0(\alpha, t)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha, t)$ if it satisfies

$$\operatorname{Re} \left\{ \frac{(1-t)zf'(z)}{f(z) - f(tz)} \right\} > \alpha, \quad |t| \leq 1, t \neq 1 \quad (1.2)$$

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for some $\alpha (0 \leq \alpha < 1)$ and for all $z \in \mathbb{U}$. The class $\mathcal{S}(0, -1)$ was introduced by Sakaguchi [5]. Therefore, a function $f(z) \in \mathcal{S}(\alpha, -1)$ is called Sakaguchi function of order α . Incidentally the class of uniformly starlike functions introduced by Goodman [1] is following.

$$\mathcal{UST} = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \frac{(z - \zeta)f'(z)}{f(z) - f(\zeta)} > 0 \right\}, \quad (z, \zeta) \in \mathbb{U} \times \mathbb{U}.$$

As for $\mathcal{S}(\alpha, t)$ and $\mathcal{UST}$, Rønning [4] showed the following important result.

Remark 1.1 $f(z) \in \mathcal{UST}$ if and only if for every $z \in \mathbb{U}$, $|t| = 1$

$$\operatorname{Re} \frac{(1 - t)zf'(z)}{f(z) - f(tz)} > 0.$$

We also denote by $\mathcal{T}(\alpha, t)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in \mathcal{S}(\alpha, t)$. Recently Cho, Kwon and Owa [2], and , recently, Owa, Sekine and Yamakawa [3] have discussed some properties for functions $f(z)$ in $\mathcal{S}(\alpha, -1)$, $\mathcal{T}(\alpha, -1)$. Now we show some results for functions belonging to the classes $\mathcal{S}(\alpha, t)$ and $\mathcal{T}(\alpha, t)$.

2 $\mathcal{S}_0(\alpha, t)$ and $\mathcal{T}_0(\alpha, t)$

We first prove the following two theorems which are similar to the results of Cho, Kwon and Owa [2].

Theorem 2.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{ |n - u_n| + (1 - \alpha)|u_n| \} |a_n| \leq 1 - \alpha, \quad u_n = 1 + t + t^2 + \cdots + t^{n-1} \quad (2.1)$$

for some $\alpha (0 \leq \alpha < 1)$, then $f(z) \in \mathcal{S}(\alpha, t)$.

Proof For Theorem 1, we show that if $f(z)$ satisfies (2.1) then

$$\left| \frac{(1 - t)zf'(z)}{f(z) - f(tz)} - 1 \right| < 1 - \alpha.$$

Evidently, since

$$\frac{(1 - t)zf'(z)}{f(z) - f(tz)} - 1 = \frac{z + \sum_{n=2}^{\infty} na_n z^{n-1}}{z + \sum_{n=2}^{\infty} u_n a_n z^n} - 1 = \frac{\sum_{n=2}^{\infty} (n - u_n) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} u_n a_n z^{n-1}},$$

we see that

$$\left| \frac{(1 - t)zf'(z)}{f(z) - f(tz)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} |n - u_n| |a_n|}{1 - \sum_{n=2}^{\infty} |u_n| |a_n|}.$$

Therefore, if $f(z)$ satisfies (2.1), then we have

$$\left| \frac{(1 - t)zf'(z)}{f(z) - f(tz)} - 1 \right| < 1 - \alpha.$$

This completes the proof of Theorem 2.1.

Theorem 2.2 If $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} n\{|n - u_n| + (1 - \alpha)|u_n|\}|a_n| \leq 1 - \alpha \quad (2.2)$$

for some α ($0 \leq \alpha < 1$), then $f(z) \in \mathcal{T}(\alpha, t)$.

Proof Noting that $f(z) \in \mathcal{T}(\alpha, t)$ if and only if $zf'(z) \in \mathcal{S}(\alpha, t)$, we can prove Theorem 2.

We now define

$$\mathcal{S}_0(\alpha, t) = \{f(z) \in \mathcal{A} : f(z) \text{ satisfies (2.1)}\}$$

and

$$\mathcal{T}_0(\alpha, t) = \{f(z) \in \mathcal{A} : f(z) \text{ satisfies (2.2)}\}.$$

In view of the above theorems, we see

Example 2.1 Let us consider a function $f(z)$ given by

$$f(z) = z + (1 - \alpha) \left(\frac{\lambda \delta_2}{2(2 - \alpha)} z^2 + \frac{(1 - \lambda) \delta_3}{7 - 3\alpha} z^3 \right), \quad 0 \leq \lambda \leq 1, |\delta_2| = |\delta_3| = 1 \quad (2.3)$$

Then for any t ($|t| \leq 1, t \neq 1$), $f(z) \in \mathcal{S}_0(\alpha, t) \subset \mathcal{S}(\alpha, t)$.

Example 2.2 Let us consider a function $f(z)$ given by

$$f(z) = z + (1 - \alpha) \left(\frac{\lambda \delta_2}{4(2 - \alpha)} z^2 + \frac{(1 - \lambda) \delta_3}{3(7 - 3\alpha)} z^3 \right), \quad 0 \leq \lambda \leq 1, |\delta_2| = |\delta_3| = 1 \quad (2.4)$$

Then for any t ($|t| \leq 1, t \neq 1$), $f(z) \in \mathcal{T}_0(\alpha, t) \subset \mathcal{T}(\alpha, t)$.

3 Coefficient inequalities

Next applying Carathéodry function $p(z)$ defined by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (3.1)$$

in $\mathbb{U}$, we discuss the coefficient inequalities for functions $f(z)$ in $\mathcal{S}(\alpha, t)$ and $\mathcal{T}(\alpha, t)$.

Theorem 3.1 If $f(z) \in \mathcal{S}(\alpha, t)$, then

$$\begin{aligned} |a_n| \leq \frac{\beta}{|v_n|} \left\{ 1 + \beta \sum_{j=2}^{n-1} \frac{|u_j|}{|v_j|} + \beta^2 \sum_{j_2 > j_1}^{n-1} \sum_{j_1=2}^{n-2} \frac{|u_{j_1} u_{j_2}|}{|v_{j_1} v_{j_2}|} + \beta^3 \sum_{j_3 > j_2}^{n-1} \sum_{j_2 > j_1}^{n-2} \sum_{j_1=2}^{n-3} \frac{|u_{j_1} u_{j_2} u_{j_3}|}{|v_{j_1} v_{j_2} v_{j_3}|} \right. \\ \left. + \cdots + \beta^{n-2} \prod_{j=2}^{n-1} \frac{|u_j|}{|v_j|} \right\}, \end{aligned} \quad (3.1)$$

where

$$\beta = 2(1 - \alpha), \quad v_n = n - u_n. \quad (3.2)$$

Proof We define the function $p(z)$ by

$$p(z) = \frac{1}{1-\alpha} \left(\frac{(1-t)zf'(z)}{f(z)-f(tz)} - \alpha \right) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (3.3)$$

for $f(z) \in \mathcal{S}(\alpha, t)$. Then $p(z)$ is a Carathéodory function and satisfies

$$|p_n| \leq 2 \quad (n \geq 1). \quad (3.4)$$

Since

$$(1-t)zf'(z) = (f(z) - f(tz))(\alpha + (1-\alpha)p(z)),$$

we have

$$z + \sum_{n=2}^{\infty} na_n z^n = \left(z + \sum_{n=2}^{\infty} u_n a_n z^n \right) \left(1 + (1-\alpha) \sum_{n=2}^{\infty} p_n z^n \right)$$

where

$$u_n = 1 + t + t^2 + \cdots + t^{n-1}.$$

So we get

$$a_n = \frac{1-\alpha}{n-u_n} (p_1 u_{n-1} a_{n-1} + p_2 u_{n-2} a_{n-2} + \cdots + p_{n-2} u_2 a_2 + p_{n-1}). \quad (3.5)$$

From the equation (3.5), we easily have that

$$|a_2| = \left| \frac{1-\alpha}{2-u_2} p_1 \right| \leq \frac{2(1-\alpha)}{|2-u_2|},$$

$$|a_3| \leq \frac{2(1-\alpha)}{|3-u_3|} (|u_2 a_2| + 1) \leq \frac{2(1-\alpha)}{|3-u_3|} \left(1 + 2(1-\alpha) \frac{|u_2|}{|2-u_2|} \right),$$

and

$$|a_4| \leq \frac{2(1-\alpha)}{|4-u_4|} \left\{ 1 + 2(1-\alpha) \left(\frac{|u_2|}{|2-u_2|} + \frac{|u_3|}{|3-u_3|} \right) + 2^2(1-\alpha)^2 \frac{|u_2 u_3|}{|2-u_2||3-u_3|} \right\}.$$

Thus, using the mathematical induction, we obtain the inequality (3.1).

Remark 3.1 Equalities in Theorem 3.1 are attended for $f(z)$ given by

$$\frac{zf'(z)}{f(z)-f(tz)} = \frac{1+(1-2\alpha)z}{1-z}.$$

Remark 3.2 If we put $\alpha = 0, t = 0$ in Theorem 3.2, then we have well known result

$$f(z) \in S^* \implies |a_n| \leq n$$

where S^* is usual starlike class. And if we put $\alpha = 0, t = -1$, then we have the result due to Sakaguchi [5]

$$f(z) \in STS \implies |a_n| \leq 1,$$

where $\mathcal{STS}$ is Sakaguchi function class.

For functions $\mathcal{T}(\alpha, t)$, similarly we have

Theorem 3.2 *If $f(z) \in \mathcal{T}(\alpha, t)$, then*

$$|a_n| \leq \frac{\beta}{n|v_n|} \left\{ 1 + \beta \sum_{j=2}^{n-1} \frac{|u_j|}{|v_j|} + \beta^2 \sum_{j_2 > j_1}^{n-1} \sum_{j_1=2}^{n-2} \frac{|u_{j_1} u_{j_2}|}{|v_{j_1} v_{j_2}|} + \beta^3 \sum_{j_3 > j_2}^{n-1} \sum_{j_2 > j_1}^{n-2} \sum_{j_1=2}^{n-3} \frac{|u_{j_1} u_{j_2} u_{j_3}|}{|v_{j_1} v_{j_2} v_{j_3}|} \right. \\ \left. + \cdots + \beta^{n-2} \prod_{j=2}^{n-1} \frac{|u_j|}{|v_j|} \right\}, \quad (3.6)$$

where

$$\beta = 2(1 - \alpha), \quad v_n = n - u_n.$$

4 Distortion inequalities

For functions $f(z)$ in the classes $\mathcal{S}_0(\alpha, t)$ and $\mathcal{T}_0(\alpha, t)$, we derive

Theorem 4.1 *If $f(z) \in \mathcal{S}_0(\alpha, t)$, then*

$$|z| - \sum_{n=2}^j |a_n| |z|^n - A_j |z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^j |a_n| |z|^n + A_j |z|^{j+1} \quad (4.1)$$

where

$$A_j = \frac{1 - \alpha - \sum_{n=2}^j \{|n - u_n| + (1 - \alpha)|u_n|\} |a_n|}{j + 1 - \alpha|u_{j+1}|} \quad (j \geq 2). \quad (4.2)$$

Proof From the inequality (2.1), we know that

$$\sum_{n=j+1}^{\infty} \{|n - u_n| + (1 - \alpha)|u_n|\} |a_n| \leq 1 - \alpha - \sum_{n=2}^j \{|n - u_n| + (1 - \alpha)|u_n|\} |a_n|.$$

On the other hand

$$|n - u_n| + (1 - \alpha)|u_n| \geq n - \alpha|u_n|,$$

and hence $n - \alpha|u_n|$ is monotonically increasing with respect to n . Thus we deduce

$$(j + 1 - \alpha|u_{j+1}|) \sum_{n=j+1}^{\infty} |a_n| \leq 1 - \alpha - \sum_{n=2}^j \{|n - u_n| + (1 - \alpha)|u_n|\} |a_n|,$$

which implies that Therefore

$$\sum_{n=j+1}^{\infty} |a_n| \leq A_j. \quad (4.3)$$

Therefore we have that

$$|f(z)| \leq |z| + \sum_{n=2}^j |a_n| |z|^n + A_j |z|^{j+1}$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^j |a_n| |z|^n - A_j |z|^{j+1}.$$

This completes the proof of the theorem.

Similarly we have

Theorem 4.2 *If $f(z) \in \mathcal{T}_0(\alpha, t)$, then*

$$|z| - \sum_{n=2}^j |a_n| |z|^n - B_j |z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^j |a_n| |z|^n + B_j |z|^{j+1} \quad (4.4)$$

and

$$1 - \sum_{n=2}^j n |a_n| |z|^{n-1} - C_j |z|^{j-1} \leq |f'(z)| \leq 1 + \sum_{n=2}^j n |a_n| |z|^{n-1} + C_j |z|^{j-1} \quad (4.5)$$

where

$$B_j = \frac{1 - \alpha - \sum_{n=2}^j n \{|n - u_n| + (1 - \alpha)|u_n|\} |a_n|}{(j+1)\{j+1 - \alpha|u_{j+1}|\}} \quad (j \geq 2). \quad (4.6)$$

and

$$C_j = \frac{1 - \alpha - \sum_{n=2}^j n \{|n - u_n| + (1 - \alpha)|u_n|\} |a_n|}{j+1 - \alpha|u_{j+1}|} \quad (j \geq 2). \quad (4.7)$$

5 Relation between the classes

By the definitions for the classes $\mathcal{S}_0(\alpha, t)$, and $\mathcal{T}_0(\alpha, t)$, evidently we have

$$\mathcal{S}_0(\alpha, t) \subset \mathcal{S}_0(\beta, t) \quad (0 \leq \beta \leq \alpha < 1)$$

and

$$\mathcal{T}_0(\alpha, 1) \subset \mathcal{T}_0(\beta, t) \quad (0 \leq \beta \leq \alpha < 1).$$

Let us consider a relation between $\mathcal{S}_0(\beta, t)$ and $\mathcal{T}_0(\alpha, t)$.

Theorem 5.1 *If $f(z) \in \mathcal{T}_0(\alpha, t)$, then $f(z) \in \mathcal{S}_0\left(\frac{1+\alpha}{2}, t\right)$.*

Proof Let $f(z) \in \mathcal{T}_0(\alpha, t)$. Then, if β satisfies

$$\frac{|n - u_n| + (1 - \beta)|u_n|}{1 - \beta} \leq n \frac{|n - u_n| + (1 - \alpha)|u_n|}{1 - \alpha} \quad (5.1)$$

for all $n \geq 2$, then we have that $f(z) \in \mathcal{S}_0(\beta, t)$. From (5.1), we have

$$\beta \leq 1 - \frac{(1 - \alpha)|n - u_n|}{n|n - u_n| + (1 - \alpha)(n - 1)|u_n|}. \quad (5.2)$$

Furthermore, since for all $n \geq 2$

$$\frac{|n - u_n|}{n|n - u_n| + (1 - \alpha)(n - 1)|u_n|} \leq \frac{1}{n} \leq \frac{1}{2}, \quad (5.3)$$

we obtain

$$f(z) \in S_0\left(\frac{1 + \alpha}{2}, t\right).$$

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