

On the problem of Goldberg for the rational maps

Masayo FUJIMURA*

Department of Mathematics
National Defense Academy, Yokosuka 239-8686, JAPAN
E-mail: masayo@nda.ac.jp

Abstract

In this paper, we solve a problem of Goldberg that determine the number of equivalence classes of rational maps corresponding to each critical set, when the degree is small and ∞ is critical.

1 Introduction

In [3], Goldberg suggested a problem that determine the number of equivalence classes of rational maps corresponding to each critical set. This problem is based on her theorem (Theorem 1.3 in [3]), and it is known that the theorem deeply concern with B. and M. Shapiro conjecture (see [1]).

By using algebraic computation system, we solve a problem of Goldberg when the degree is small and ∞ is critical, and this gives a complete answer to this problem together with our results in [2]. This work is joint work with M. Karima and M. Taniguchi (Nara Women's Univ.).

A rational map of degree d is a map with the following form,

$$R(z) = \frac{P(z)}{Q(z)},$$

where P and Q are coprime polynomials with $\max\{\deg P, \deg Q\} = d$.

Definition 1. Two rational maps R_1 and R_2 are said to be *Möbius equivalent* if there is a Möbius transformation $M : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $R_2 = M \circ R_1$.

Let X_d be the set of all equivalence classes of rational maps of degree d , and $X_d^{(k)}$ be the subset of X_d consisting of all equivalence classes of rational maps with k -hold critical point at ∞ , where $k = 0$ means the rational maps that ∞ is non-critical.

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Remark 1. A rational map R of degree d has $2d - 2$ critical points counted including multiplicity. The set of critical points of R is invariant under taking a Möbius conjugate.

Every set of critical points of R is admissible, i.e., every critical point has multiplicity at most $d - 1$. Therefore, the space X_d is the disjoint union of $X_d^{(0)}, X_d^{(1)}, \dots$, and $X_d^{(d-1)}$.

Goldberg showed the following theorem.

Theorem (Goldberg [3]). A $(2d - 2)$ -tuple B is the critical set of at most $C(d)$ classes in X_d , where $C(d)$ means the d -th *Catalan number* $\frac{1}{d} \binom{2d-2}{d-1}$. The maximal is attained by a Zariski open subset of the space $\hat{\mathbb{C}}^{2d-2}$ of all B .

The map $\Phi_d : X_d \rightarrow \hat{\mathbb{C}}^{2d-2}$ is defined by sending a equivalence class to the set of critical points, and the restriction of Φ_d to $X_d^{(k)}$ is denoted by $\Phi_d^{(k)}$.

Then Goldberg's problem (see [3]) is written as follows.

Problem

- Describe in detail the ramification sets of the maps Φ_d .
- For every point $c \in \hat{\mathbb{C}}^{2d-2}$, determine the number of points in the preimage $\Phi_d^{-1}(c)$.

We give the complete answer to this problem for the case of $d = 3$ and 4.

2 The case that ∞ is non-critical

Theorem 2 (Fujimura, Karima and Taniguchi [2]).

For each class in $X_d^{(0)}$, there is a unique representative R of the form

$$R(z) = \frac{P(z)}{Q(z)} = z + \frac{a_{d-2}z^{d-2} + \dots + a_0}{z^{d-1} + b_{d-2}z^{d-2} + \dots + b_0}.$$

For each $R = \frac{P}{Q}$ in the above form, the critical points of R is obtained by the equation

$$P'(z)Q(z) - P(z)Q'(z) = z^{2d-2} + c_{2d-3}z^{2d-3} + \dots + c_0 = 0.$$

Then, the map $\Phi_d^{(0)}$ is defined as follows,

$$\begin{array}{ccc} \Phi_d^{(0)} : & \begin{array}{c} \mathbb{C}^{2d-2} \\ \cup \\ (a_{d-2}, \dots, a_0, b_{d-2}, \dots, b_0) \end{array} & \begin{array}{c} \rightarrow \\ \cup \\ \mapsto \end{array} \begin{array}{c} \mathbb{C}^{2d-2} \\ \cup \\ (c_{2d-3}, \dots, c_0) \end{array} \end{array}$$

The defining equation of the ramification locus of $\Phi_d^{(0)}$ gives the answer to a problem of Goldberg for the case that ∞ is non-critical. For the details, see [2].

Thereafter, we consider the case that ∞ is critical.

3 The case that ∞ is critical

3.1 The case of degree 3

Proposition 3.

1. For each class in $X_3^{(1)}$, there is a unique representative in $CB_3^{(1)}$, where

$$CB_3^{(1)} = \left\{ R(z) = z^2 + az + \frac{c}{z+b} \quad (c \neq 0) \right\}.$$

2. For each class in $X_3^{(2)}$, there is a unique representative in $CB_3^{(2)}$, where

$$CB_3^{(2)} = \{ R(z) = z^3 + az^2 + bz \}.$$

3.1.1 The case that ∞ is simple critical point

Let $R = \frac{P}{Q}$ be a rational map in $CB_3^{(1)}$, and $z^3 + c_2z^2 + c_1z + c_0 = 0$ be the equation defined by $P'(z)Q(z) - P(z)Q'(z) = 0$.

Then, the map $\Phi_3^{(1)} : CB_3^{(1)} \rightarrow \mathbb{C}^3$ is defined by sending (a, b, c) to (c_0, c_1, c_2) .

Proposition 4. *The ramification locus of $\Phi_3^{(1)}$ is given by $a = 0$, $\Phi_3^{(1)}(CB_3^{(1)}) = \mathbb{C}^3 \setminus E^{(1)}(3)$ and $\Phi_3^{(1)}$ is 2-valent on the the set of the points in $\mathbb{C}^3 \setminus E^{(1)}(3)$ satisfying that*

$$4c_0c_2^3 - c_1^2c_2^2 - 4c_0c_1c_2 + c_1^3 + c_0^2 \neq 0 \quad \text{or} \quad 2c_2^3 - 2c_1c_2 + c_0 \neq 0.$$

Proof. The map $\Phi_3^{(1)}$ is defined by

$$(a, b, c) \mapsto (c_0, c_1, c_2) = \left(\frac{ab^2 - c}{2}, ab + b^2, b + \frac{a}{2} \right).$$

For $\mathbf{c} = (c_0, c_1, c_2) \in \mathbb{C}^3 \setminus E^{(1)}(3)$, every $(\Phi_3^{(1)})^{-1}(\mathbf{c})$ is given by

$$\begin{cases} B = b^2 - 2c_2b + c_1 = 0 \\ C = (4c_2^2 - 2c_1)b + c - 2c_1c_2 + 2c_0 = 0 \\ A = a + 2b - 2c_2 = 0, \end{cases} \quad (1)$$

which has exactly 2 solutions except for discriminant $_b(B) = c_2^2 - c_1 = 0$.

The map $\Phi_3^{(1)}$ is not defined on $\{(a, b, c) \mid c = 0\}$ where

$$\text{resultant}_z(\text{numerator}(R), \text{denominator}(R)) = c = 0.$$

From (1), for each (c_0, c_1, c_2) , the coefficient c is determined by

$$-c^2 + (-8c_2^3 + 8c_1c_2 - 4c_0)c - 16c_0c_2^3 + 4c_1^2c_2^2 + 16c_0c_1c_2 - 4c_1^3 - 4c_0^2 = 0. \quad (2)$$

Therefore, the exceptional set $E^{(1)}(3)$ corresponds to the condition that the equation (2) has 0 as a unique solution. Thus we have

$$E^{(1)}(3) = \{4c_0c_2^3 - c_1^2c_2^2 - 4c_0c_1c_2 + c_1^3 + c_0^2 = 0 \text{ and } 2c_2^3 - 2c_1c_2 + c_0 = 0\}.$$

□

3.1.2 The case that ∞ is double critical point

Let $R(z) = z^3 + az + b$ be a polynomial map in $CB_3^{(2)}$, and $z^2 + c_1z + c_0 = 0$ be the equation defined by $R'(z) = 0$.

Then, the map $\Phi_3^{(2)} : CB_3^{(2)} \rightarrow \mathbb{C}^2$ is defined by sending (a, b) to (c_0, c_1) .

Proposition 5. *The map $\Phi_3^{(2)}$ is bijective.*

Proof. Since the map $\Phi_3^{(2)}$ is given by $(a, b) \mapsto (c_0, c_1) = (\frac{2a}{3}, \frac{b}{3})$, the assertion follows. □

3.2 The case of degree 4

Proposition 6.

1. For each class in $X_4^{(1)}$, there is a unique representative in $CB_4^{(1)}$, where

$$CB_4^{(1)} = \left\{ R(z) = z^2 + cz + \frac{a_1z + a_0}{z^2 + b_1z + b_0} \quad (a_0a_1b_1 - b_0a_1^2 - a_0^2 \neq 0) \right\}.$$

2. For each class in $X_4^{(2)}$, there is a unique representative in $CB_4^{(2)}$, where

$$CB_4^{(2)} = \left\{ R(z) = z^3 + a_2z^2 + a_1z + \frac{c}{z + b} \quad (c \neq 0) \right\}.$$

3. For each class in $X_4^{(3)}$, there is a unique representative in $CB_4^{(3)}$, where

$$CB_4^{(3)} = \left\{ R(z) = z^4 + a_3z^3 + a_2z^2 + a_1z \right\}.$$

3.2.1 The case that ∞ is simple critical point

Let $R = \frac{P}{Q}$ be a rational map in $CB_4^{(1)}$, and $z^5 + c_4z^4 + \dots + c_0 = 0$ be the equation defined by $P'(z)Q(z) - P(z)Q'(z) = 0$.

Then, the map $\Phi_4^{(1)} : CB_4^{(1)} \rightarrow \mathbb{C}^5$ is defined by sending (a_0, a_1, b_0, b_1, c) to $(c_0, \dots, c_4)$.

Proposition 7. *The ramification locus of the map $\Phi_4^{(1)}$ is given by*

$$(b_1^2 - 4b_0)c^2 + (-2b_1^3 + 8b_0b_1 - a_1)c + 4b_0b_1^2 + 2a_1b_1 - 2a_0 - 16b_0^2 = 0,$$

$\Phi_4^{(1)}(CB_4^{(1)}) = \mathbb{C}^5 \setminus E^{(1)}(4)$, and $\Phi_4^{(1)}$ is 5-valent on the set of points in $\mathbb{C}^5 \setminus E^{(1)}(4)$, where defining equation of $E^{(1)}(4)$ is given in the proof.

Proof. The five critical points of R is given as the solution of the following equation,

$$2z^5 + (c + 4b_1)z^4 + (2b_1c + 2b_1^2 + 4b_0)z^3 + ((b_1^2 + 2b_0)c + 4b_0b_1 - a_1)z^2 + (2b_0b_1c - 2a_0 + 2b_0^2)z + b_0^2c - a_0b_1 + b_0a_1 = 0. \quad (3)$$

Therefore, the map $\Phi_4^{(1)}$ is defined by $(a_0, a_1, b_0, b_1, c) \mapsto (c_0, \dots, c_4)$, where

$$\begin{aligned} c_0 &= (b_0^2c - a_0b_1 + b_0a_1)/2, \\ c_1 &= (2b_0b_1c - 2a_0 + 2b_0^2)/2, \\ c_2 &= ((b_1^2 + 2b_0)c + 4b_0b_1 - a_1)/2, \\ c_3 &= (2b_1c + 2b_1^2 + 4b_0)/2, \\ c_4 &= (c + 4b_1)/2. \end{aligned} \quad (4)$$

The ramification locus is obtained from the Jacobian of the map $\Phi_4^{(1)}$,

$$(b_1^2 - 4b_0)c^2 + (-2b_1^3 + 8b_0b_1 - a_1)c + 4b_0b_1^2 + 2a_1b_1 - 2a_0 - 16b_0^2 = 0.$$

For $c \in \mathbb{C}^5 \setminus E^{(1)}(4)$, every $(\Phi_4^{(1)})^{-1}(c)$ is given by,

$$\left\{ \begin{array}{l} B_1 = 81b_1^5 - 162c_4b_1^4 + (108c_4^2 + 54c_3)b_1^3 + (-24c_4^3 - 72c_3c_4 + 12c_2)b_1^2 \\ \quad + (24c_3c_4^2 - 8c_2c_4 + 9c_3^2 - 4c_1)b_1 - 6c_3^2c_4 + 4c_2c_3 + 8c_0 \\ B_0 = -3b_1^2 + 2c_4b_1 + 2b_0 - c_3 \\ A_1 = -10b_1^3 + 12c_4b_1^2 + (-4c_4^2 - 2c_3)b_1 - a_1 + 2c_3c_4 - 2c_2 \\ A_0 = 15b_1^4 - 16c_4b_1^3 + (4c_4^2 + 2c_3)b_1^2 + 4a_0 - c_3^2 + 4c_1 \\ C = c + 4b_1 - 2c_4, \end{array} \right.$$

which has exactly 5 solutions except for discriminant $_{b_1}(B_1) = 0$,

$$\begin{aligned}
& 1296c_0c_1^2c_4^7 + ((-1296c_0c_1c_2 - 324c_1^3)c_3 + 384c_0c_2^3 + 108c_1^2c_2^2 - 7776c_0^2c_1)c_4^6 + \\
& (324c_0c_1c_3^3 + (-108c_0c_2^2 + 324c_1^2c_2)c_3^2 + (-204c_1c_2^2 + 3888c_0^2c_2 - 7452c_0c_1^2)c_3 + \\
& 32c_2^5 - 936c_0c_1c_2^2 + 108c_1^3c_2 + 11664c_0^3)c_4^5 + (-81c_1^2c_3^4 + (54c_1c_2^2 - 972c_0^2)c_3^3 + \\
& (-9c_2^4 + 8316c_0c_1c_2 + 2106c_1^3)c_3^2 + (-2412c_0c_2^3 - 738c_1^2c_2^2 + 49572c_0^2c_1)c_3 + 8c_1c_2^4 + \\
& 108c_0^2c_2^2 + 4284c_0c_1^2c_2 + 27c_1^4)c_4^4 + (-1944c_0c_1c_3^4 + (648c_0c_2^2 - 2052c_1^2c_2)c_3^3 + \\
& (1296c_1c_2^3 - 24624c_0^2c_2 + 9288c_0c_1^2)c_3^2 + (-204c_2^5 + 1512c_0c_1c_2^2 - 1800c_1^3c_2 - \\
& 72900c_0^3)c_3 + 1320c_0c_2^4 + 368c_1^2c_2^2 - 26460c_0^2c_1c_2 + 3396c_0c_1^3)c_4^3 + (486c_1^2c_3^5 + \\
& (-324c_1c_2^2 + 5832c_0^2)c_3^4 + (54c_2^4 - 13608c_0c_1c_2 - 3834c_1^3)c_3^3 + (3672c_0c_2^3 + 2592c_1^2c_2^2 - \\
& 86670c_0^2c_1)c_3^2 + (-738c_1c_2^2 + 12690c_0^2c_2^2 - 13284c_0c_1^2c_2 - 984c_1^4)c_3 + 108c_2^6 - \\
& 2124c_0c_1c_3^3 + 634c_1^3c_2^2 + 40500c_0^3c_2 - 49950c_0^2c_1^2)c_4^2 + (2916c_0c_1c_3^5 + (-972c_0c_2^2 + \\
& 3240c_1^2c_2)c_3^4 + (-2052c_1c_3^3 + 38880c_0^2c_2 + 6156c_0c_1^2)c_3^3 + (324c_2^5 + 3024c_0c_1c_2^2 + \\
& 5544c_1^3c_2 + 121500c_0^3)c_3^2 + (-3888c_0c_2^4 - 1800c_1^2c_2^3 + 118800c_0^2c_1c_2 + 12240c_0c_1^3)c_3 + \\
& 108c_1c_2^5 - 8100c_0^2c_2^2 - 5220c_0c_1^2c_2^2 + 352c_1^4c_2 + 202500c_0^3c_1)c_4 - 729c_1^2c_3^6 + (486c_1c_2^2 - \\
& 8748c_0^2)c_3^5 + (-81c_2^4 + 972c_0c_1c_2 + 972c_1^3)c_3^4 + (324c_0c_2^3 - 3834c_1^2c_2^2 + 12150c_0^2c_1)c_3^3 + \\
& (2106c_1c_2^4 - 36450c_0^2c_2^2 - 18360c_0c_1^2c_2 - 432c_1^4)c_3^2 + (-324c_2^6 + 14580c_0c_1c_2^2 - \\
& 984c_1^3c_2^2 - 202500c_0^3c_2 - 27000c_0^2c_1^2)c_3 - 648c_0c_2^5 + 27c_1^2c_2^4 + 20250c_0^2c_1c_2^2 - \\
& 2400c_0c_1^3c_2 + 64c_1^5 - 253125c_0^4 = 0.
\end{aligned}$$

The map $\Phi_4^{(1)}$ is not defined on

$$r := \text{resultant}_z(\text{numerator}(R), \text{denominator}(R)) = -a_0a_1b_1 + b_0a_1^2 + a_0^2 = 0.$$

From (4), for each $(c_0, \dots, c_4)$, r is determined by the equation of the form,

$$\begin{aligned}
& 8503056r^5 + P_4r^4 + P_3r^3 + P_2r^2 + P_1r + P_0 = 0 \\
& (P_k \in \mathbb{C}[c_0, c_1, c_2, c_3, c_4], \quad k = 0, 1, 2, 3, 4). \quad (5)
\end{aligned}$$

Therefore, the exceptional set $E^{(1)}(4)$ corresponds to the condition that this equation has 0 as a unique solution. Thus we have

$$E^{(1)}(4) = \{P_0 = P_1 = P_2 = P_3 = P_4 = 0\},$$

where

$$\begin{aligned}
P_0 = & -256(256c_0^3c_4^5 + (-192c_0^2c_1c_3 - 128c_0^2c_2^2 + 144c_0c_1^2c_2 - 27c_1^4)c_4^4 + ((144c_0^2c_2 - \\
& 6c_0c_1^2)c_3^2 + (-80c_0c_1c_2^2 + 18c_1^3c_2 - 1600c_0^3)c_3 + 16c_0c_2^4 - 4c_1^2c_2^3 + 160c_0^2c_1c_2 - \\
& 36c_0c_1^3)c_4^3 + (-27c_0^2c_3^4 + (18c_0c_1c_2 - 4c_1^3)c_3^3 + (-4c_0c_2^3 + c_1^2c_2^2 + 1020c_0^2c_1)c_3^2 + \\
& (560c_0^2c_2^2 - 746c_0c_1^2c_2 + 144c_1^4)c_3 + 24c_0c_1c_2^3 - 6c_1^3c_2^2 + 2000c_0^3c_2 - 50c_0^2c_1^2)c_4^2 + \\
& ((-630c_0^2c_2 + 24c_0c_1^2)c_3^3 + (356c_0c_1c_2^2 - 80c_1^3c_2 + 2250c_0^3)c_3^2 + (-72c_0c_2^4 + 18c_1^2c_2^3 - \\
& 2050c_0^2c_1c_2 + 160c_0c_1^3)c_3 - 900c_0^2c_2^3 + 1020c_0c_1^2c_2^2 - 192c_1^4c_2 - 2500c_0^3c_1)c_4 + 108c_0^2c_3^5 + \\
& (-72c_0c_1c_2 + 16c_1^3)c_3^4 + (16c_0c_2^3 - 4c_1^2c_2^2 - 900c_0^2c_1)c_3^3 + (825c_0^2c_2^2 + 560c_0c_1^2c_2 - \\
& 128c_1^4)c_3^2 + (-630c_0c_1c_2^3 + 144c_1^3c_2^2 - 3750c_0^3c_2 + 2000c_0^2c_1^2)c_3 + 108c_0c_2^5 - 27c_1^2c_2^4 + \\
& 2250c_0^2c_1c_2^2 - 1600c_0c_1^3c_2 + 256c_1^5 + 3125c_0^4)^2.
\end{aligned}$$

$$\begin{aligned}
P_1 = & 256((144c_0c_2 - 54c_1^2)c_4^4 + (-54c_0c_3^2 + 18c_1c_2c_3 - 4c_2^3 - 36c_0c_1)c_4^3 + ((-702c_0c_2 + 279c_1^2)c_3 - 6c_1c_2^2 - 1350c_0^2)c_4^2 + (243c_0c_3^3 - 81c_1c_2c_3^2 + (18c_2^3 + 810c_0c_1)c_3 + 1440c_0c_2^2 - 624c_1^2c_2)c_4 + (-405c_0c_2 - 216c_1^2)c_3^2 + (279c_1c_2^2 + 3375c_0^2)c_3 - 54c_2^4 - 3600c_0c_1c_2 + 1120c_1^3, 1], [256c_0^3c_4^5 + (-192c_0^2c_1c_3 - 128c_0^2c_2^2 + 144c_0c_1^2c_2 - 27c_1^4)c_4^4 + ((144c_0^2c_2 - 6c_0c_1^2)c_3^2 + (-80c_0c_1c_2^2 + 18c_1^3c_2 - 1600c_0^3)c_3 + 16c_0c_2^4 - 4c_1^2c_2^3 + 160c_0^2c_1c_2 - 36c_0c_1^3)c_4^3 + (-27c_0^2c_3^4 + (18c_0c_1c_2 - 4c_1^3)c_3^3 + (-4c_0c_2^3 + c_1^2c_2^2 + 1020c_0^2c_1)c_3^2 + (560c_0^2c_2^2 - 746c_0c_1^2c_2 + 144c_1^4)c_3 + 24c_0c_1c_2^3 - 6c_1^3c_2^2 + 2000c_0^3c_2 - 50c_0^2c_1^2)c_4^2 + ((-630c_0^2c_2 + 24c_0c_1^2)c_3^3 + (356c_0c_1c_2^2 - 80c_1^3c_2 + 2250c_0^3)c_3^2 + (-72c_0c_2^4 + 18c_1^2c_2^3 - 2050c_0^2c_1c_2 + 160c_0c_1^3)c_3 - 900c_0^2c_3^2 + 1020c_0c_1^2c_2^2 - 192c_1^4c_2 - 2500c_0^3c_1)c_4 + 108c_0^2c_3^5 + (-72c_0c_1c_2 + 16c_1^3)c_3^4 + (16c_0c_2^3 - 4c_1^2c_2^2 - 900c_0^2c_1)c_3^3 + (825c_0^2c_2^2 + 560c_0c_1^2c_2 - 128c_1^4)c_3^2 + (-630c_0c_1c_2^2 + 144c_1^3c_2^2 - 3750c_0^3c_2 + 2000c_0^2c_1^2)c_3 + 108c_0c_2^5 - 27c_1^2c_2^4 + 2250c_0^2c_1c_2^2 - 1600c_0c_1^3c_2 + 256c_1^5 + 3125c_0^4),
\end{aligned}$$

$$\begin{aligned}
P_2 = & 864(2048c_0^3c_4^9 + (-1536c_0^2c_1c_3 - 1024c_0^2c_2^2 + 1152c_0c_1^2c_2 - 216c_1^4)c_4^8 + ((1152c_0^2c_2 - 48c_0c_1^2)c_3^2 + (-640c_0c_1c_2^2 + 144c_1^3c_2 - 23040c_0^3)c_3 + 128c_0c_2^4 - 32c_1^2c_3^2 + 1280c_0^2c_1c_2 - 288c_0c_1^3)c_4^7 + (-216c_0^2c_3^3 + (144c_0c_1c_2 - 32c_1^3)c_3^2 + (-32c_0c_2^3 + 8c_1^2c_2^2 + 15840c_0^2c_1)c_3 + (9600c_0^2c_2^2 - 11728c_0c_1^2c_2 + 2232c_1^4)c_3 + 192c_0c_1c_2^3 - 48c_1^3c_2^2 + 85632c_0^3c_2 - 21136c_0^2c_1^2)c_4^6 + ((-10800c_0^2c_2 + 432c_0c_1^2)c_3^3 + (6048c_0c_1c_2^2 - 1360c_1^3c_2 + 68688c_0^3)c_3^2 + (-1216c_0c_2^4 + 304c_1^2c_2^3 - 54288c_0^2c_1c_2 + 13088c_0c_1^3)c_3 - 34336c_0^2c_2^3 + 33504c_0c_1^2c_2^2 - 6936c_1^4c_2 - 53280c_0^3c_1)c_4^5 + (1944c_0^2c_3^3 + (-1296c_0c_1c_2 + 288c_1^3)c_3^4 + (288c_0c_2^3 - 72c_1^2c_2^2 - 43200c_0^2c_1)c_3^3 + (16200c_0^2c_2^2 + 22608c_0c_1^2c_2 - 7405c_1^4)c_3^2 + (-17776c_0c_1c_2^3 + 5856c_1^3c_2^2 - 615600c_0^3c_2 + 198480c_0^2c_1^2)c_3 + 3424c_0c_2^5 - 1160c_1^2c_2^4 + 124240c_0^2c_1c_2^2 - 86240c_0c_1^3c_2 + 13820c_1^5 - 135000c_0^4)c_4^4 + ((16200c_0^2c_2 + 486c_0c_1^2)c_3^4 + (-9504c_0c_1c_2^2 + 2958c_1^3c_2 + 10800c_0^3)c_3^3 + (1824c_0c_2^4 - 1380c_1^2c_2^3 + 208080c_0^2c_1c_2 - 58180c_0c_1^3)c_3^2 + (304c_1c_2^5 + 152480c_0^2c_2^3 - 142160c_0c_1^2c_2^2 + 30360c_1^4c_2 + 468000c_0^3c_1)c_3 - 32c_2^7 - 1760c_0c_1c_2^4 + 256c_1^3c_2^3 + 672000c_0^3c_2^2 - 304400c_0^2c_1^2c_2 + 29520c_0c_1^4)c_4^3 + (-3645c_0^2c_3^3 + (2430c_0c_1c_2 - 756c_1^3)c_3^2 + (-540c_0c_2^3 + 351c_1^2c_2^2 + 12960c_0^2c_1)c_3^4 + (-72c_1c_2^4 - 176040c_0^2c_2^2 + 27810c_0c_1^2c_2 + 9600c_1^4)c_3^3 + (8c_2^6 + 80040c_0c_1c_2^3 - 27518c_1^3c_2^2 + 918000c_0^3c_2 - 557550c_0^2c_1^2)c_3^2 + (-16240c_0c_2^5 + 5856c_1^2c_2^4 - 820800c_0^2c_1c_2^2 + 524360c_0c_1^3c_2 - 76480c_1^5 + 675000c_0^4)c_3 - 48c_1c_2^6 - 209200c_0^2c_2^4 + 188640c_0c_1^2c_2^3 - 39240c_1^4c_2^2 - 1260000c_0^3c_1c_2 + 233000c_0^2c_1^3)c_4^2 + ((39366c_0^2c_2 - 5832c_0c_1^2)c_3^5 + (-21060c_0c_1c_2^2 + 1008c_1^3c_2 - 182250c_0^3)c_3^4 + (4680c_0c_2^4 + 2958c_1^2c_2^3 + 106650c_0^2c_1c_2 + 34560c_0c_1^3)c_3^3 + (-1360c_1c_2^5 + 233100c_0^2c_2^3 - 133740c_0c_1^2c_2^2 + 2560c_1^4c_2 - 742500c_0^3c_1)c_3^2 + (144c_2^7 - 77440c_0c_1c_2^4 + 30360c_1^3c_2^3 - 1890000c_0^3c_2^2 + 1737000c_0^2c_1^2c_2 - 227200c_0c_1^4)c_3 + 19296c_0c_2^6 - 6936c_1^2c_2^5 + 1112000c_0^2c_1c_2^3 - 881200c_0c_1^3c_2^2 + (157952c_1^5 + 225000c_0^4)c_2 + 450000c_0^3c_1^2)c_4 - 2916c_0^2c_3^7 + (1944c_0c_1c_2 + 432c_1^3)c_3^6 + (-432c_0c_2^3 - 756c_1^2c_2^2 + 4860c_0^2c_1)c_3^5 + (288c_1c_2^4 - 34425c_0^2c_2^2 + 6480c_0c_1^2c_2 - 8640c_1^4)c_3^4 + (-32c_2^6 + 16470c_0c_1c_2^3 + 9600c_1^3c_2^2 + 303750c_0^3c_2 + 216000c_0^2c_1^2)c_3^3 + (-3756c_0c_2^5 - 7405c_1^2c_2^4 - 384750c_0^2c_1c_2^2 - 201600c_0c_1^3c_2 + 64768c_1^5 - 928125c_0^4)c_3^2 + (2232c_1c_2^6 - 33000c_0^2c_2^4 + 294200c_0c_1^2c_2^3 - 76480c_1^4c_2^2 + 2475000c_0^3c_1c_2 - 920000c_0^2c_1^3)c_3 - 216c_2^8 - 48240c_0c_1c_2^5 + 13820c_1^3c_2^4 + 500000c_0^3c_2^3 - 1765000c_0^2c_1^2c_2^2 + 992000c_0c_1^4c_2 - 148480c_1^6 - 562500c_0^4c_1),
\end{aligned}$$

$$\begin{aligned}
P_3 = & 11664(768c_0^2c_4^6 + (-384c_0c_1c_3 - 256c_0c_2^2 + 144c_1^2c_2)c_4^5 + ((288c_0c_2 - 6c_1^2)c_3^2 + \\
& (-80c_1c_2^2 - 5760c_0^2)c_3 + 16c_2^4 + 1184c_0c_1c_2 - 360c_1^3)c_4^4 + (-54c_0c_3^4 + 18c_1c_2c_3^3 + \\
& (-4c_2^3 + 2196c_0c_1)c_3^2 + (1008c_0c_2^2 - 656c_1^2c_2)c_3 + 13200c_0^2c_2 - 2800c_0c_1^2)c_4^3 + \\
& ((-1350c_0c_2 - 9c_1^2)c_3^3 + (402c_1c_2^2 + 9450c_0^2)c_3^2 + (-80c_2^4 - 9720c_0c_1c_2 + 2500c_1^3)c_3 - \\
& 3040c_0c_2^3 + 1416c_1^2c_2^2 - 9000c_0^2c_1)c_4^2 + (243c_0c_3^5 - 81c_1c_2c_3^4 + (18c_2^3 - 1890c_0c_1)c_3^3 + \\
& (3960c_0c_2^2 - 492c_1^2c_2)c_3^2 + (-656c_1c_2^3 - 45000c_0^2c_2 + 15000c_0c_1^2)c_3 + 144c_2^5 + \\
& 20000c_0c_1c_2^2 - 7360c_1^3c_2)c_4 + (-405c_0c_2 + 216c_1^2)c_3^4 + (-9c_1c_2^2 + 3375c_0^2)c_3^3 + \\
& (-6c_2^4 + 900c_0c_1c_2 - 2240c_1^3)c_3^2 + (-2400c_0c_2^3 + 2500c_1^2c_2^2 + 22500c_0^2c_1)c_3 - 360c_1c_2^4 + \\
& 30000c_0^2c_2^2 - 40000c_0c_1^2c_2 + 9600c_1^4), \\
\end{aligned}$$

$$\begin{aligned}
P_4 = & 19683(768c_0c_4^3 + (-192c_1c_3 - 128c_2^2)c_4^2 + (144c_2c_3^2 - 2880c_0c_3 + 1024c_1c_2)c_4 - \\
& 27c_3^4 + 216c_1c_3^2 - 192c_2^2c_3 + 4800c_0c_2 - 2480c_1^2).
\end{aligned}$$

□

3.2.2 The case that ∞ is double critical point

Let $R = \frac{P}{Q}$ be a rational map in $CB_4^{(2)}$, and $z^4 + c_3z^3 + \cdots + c_0 = 0$ be the equation defined by $P'(z)Q(z) - P(z)Q'(z) = 0$.

Then, the map $\Phi_4^{(2)} : CB_4^{(2)} \rightarrow \mathbb{C}^4$ is defined by sending (a_1, a_2, b, c) to $(c_0, \dots, c_3)$.

Proposition 8. *The ramification locus of $\Phi_4^{(2)}$ is given by $3b^2 - 2a_2b + a_1 = 0$, $\Phi_4^{(2)}(CB_4^{(2)}) = \mathbb{C}^4 \setminus E^{(2)}(4)$, and $\Phi_4^{(2)}$ is 3-valent on the set of the points in $\mathbb{C}^4 \setminus E^{(2)}(4)$, where the defining equation of $E^{(2)}(4)$ is given in the proof.*

Proof. The four critical points of R in $\mathbb{C}$ is given as the solution of

$$3z^4 + (6b + 2a_2)z^3 + (3b^2 + 4a_2b + a_1)z^2 + (2a_2b^2 + 2a_1b)z + a_1b^2 - c = 0.$$

Therefore, the map $\Phi_4^{(2)}$ is defined by $(a_1, a_2, b, c) \mapsto (c_0, \dots, c_3)$, where

$$\begin{aligned}
c_0 &= (a_1b^2 - c)/3, \\
c_1 &= (2a_2b^2 + 2a_1b)/3, \\
c_2 &= (3b^2 + 4a_2b + a_1)/3, \\
c_3 &= (6b + 2a_2)/3.
\end{aligned} \tag{6}$$

The ramification locus is obtained from the Jacobian of the map $\Phi_4^{(2)}$,

$$3b^2 - 2a_2b + a_1 = 0.$$

For $c \in \mathbb{C}^4 \setminus E^{(2)}(4)$, every $(\Phi_4^{(2)})^{-1}(c)$ is given by,

$$\begin{cases} B = 4b^3 - 3c_3b^2 + 2c_2b - c_1 \\ A_1 = -9b^2 + 6c_3b + a_1 - 3c_2 \\ A_2 = 6b + 2a_2 - 3c_3 \\ C = (9c_3^2 - 24c_2)b^2 + (-6c_2c_3 + 36c_1)b - 16c + 3c_1c_3 - 48c_0, \end{cases}$$

which has exactly 2 solutions except for $\text{discriminant}_b(B) = 0$.

The map $\Phi_4^{(2)}$ is not defined on

$$\text{resultant}_z(\text{numerator}(R), \text{denominator}(R)) = c = 0.$$

From (6), for each $(c_0, \dots, c_3)$, c is determined by the equation,

$$\begin{aligned} & 256c^3 - 3(27c_3^4 - 144c_2c_3^2 + 192c_1c_3 + 128c_2^2 - 768c_0)c^2 \\ & - 18(27c_0c_3^4 - 9c_1c_2c_3^3 + (2c_2^3 - 144c_0c_2 + 3c_1^2)c_3^2 + (40c_1c_2^2 + 192c_0c_1)c_3 \\ & - 8c_2^4 + 128c_0c_2^2 - 72c_1^2c_2 - 384c_0^2)c \\ & - 27(27c_0^2c_3^4 + (-18c_0c_1c_2 + 4c_1^3)c_3^3 + (4c_0c_2^3 - c_1^2c_2^2 - 144c_0^2c_2 + 6c_0c_1^2)c_3^2 \\ & + (80c_0c_1c_2^2 - 18c_1^3c_2 + 192c_0^2c_1)c_3 - 16c_0c_2^4 + 4c_1^2c_2^3 + 128c_0^2c_2^2 - 144c_0c_1^2c_2 \\ & + 27c_1^4 - 256c_0^3) = 0. \end{aligned}$$

Therefore, the exceptional set $E^{(2)}(4)$ corresponds to the condition that this equation has 0 as a unique solution.

Hence, the defining equation of $E^{(2)}(4)$ is

$$P_0 = P_1 = P_2 = 0,$$

where

$$\begin{aligned} P_0 &= -729c_0^2c_3^4 + (486c_0c_1c_2 - 108c_1^3)c_3^3 \\ &+ (-108c_0c_2^3 + 27c_1^2c_2^2 + 3888c_0^2c_2 - 162c_0c_1^2)c_3^2 \\ &+ (-2160c_0c_1c_2^2 + 486c_1^3c_2 - 5184c_0^2c_1)c_3 + 432c_0c_2^4 - 108c_1^2c_2^3 \\ &- 3456c_0^2c_2^2 + 3888c_0c_1^2c_2 - 729c_1^4 + 6912c_0^3, \\ P_1 &= -486c_0c_3^4 + 162c_1c_2c_3^3 + (-36c_2^3 + 2592c_0c_2 - 54c_1^2)c_3^2 \\ &+ (-720c_1c_2^2 - 3456c_0c_1)c_3 + 144c_2^4 - 2304c_0c_2^2 + 1296c_1^2c_2 + 6912c_0^2, \\ P_2 &= -81c_3^4 + 432c_2c_3^2 - 576c_1c_3 - 384c_2^2 + 2304c_0. \end{aligned}$$

□

3.2.3 The case that ∞ is triple critical point

Let R be a polynomial map in $CB_4^{(3)}$, $z^3 + c_2z^2 + c_1z + c_0 = 0$ be the equation defined by $R'(z) = 0$.

Then, the map $\Phi_4^{(3)} : CB_4^{(3)} \rightarrow \mathbb{C}^3$ is defined by sending (a_1, a_2, a_3) to (c_0, c_1, c_2) .

Proposition 9. *The map $\Phi_4^{(3)}$ is bijective.*

Proof. The three critical points of R in $\mathbb{C}$ is given as the solution of the following equation

$$4z^3 + 3a_3z^2 + 2a_2z + a_1 = 0.$$

Therefore, the map $\Phi_4^{(3)}$ is defined by

$$(a_1, a_2, a_3) \mapsto (c_0, c_1, c_2) = \left(\frac{a_1}{4}, \frac{2a_2}{4}, \frac{3a_3}{4}\right),$$

and the assertion follows. □

For $d = 3, 4$, the complete answer for the problem of Goldberg is obtained.

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