

TWISTED FIRST HOMOLOGY GROUP OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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Abstract: The automorphism group $\text{Aut } F_n$ and the outer automorphism group $\text{Out } F_n$ of a free group F_n of rank n act on the abelianized group H of F_n and the dual group H^* of H . The twisted first homology groups of $\text{Aut } F_n$ and $\text{Out } F_n$ with coefficients in H and H^* are calculated.

Keywords: automorphism group of a free group, mapping class group, Magnus representation

1. INTRODUCTION

Let F_n be a free group of rank n and $\text{Aut } F_n$ the automorphism group of F_n . There are remarkable results of the homology groups of $\text{Aut } F_n$ with trivial coefficients. For example, Gersten [2] showed $H_2(\text{Aut } F_n, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 5$ and Hatcher and Vogtmann [3] showed $H_i(\text{Aut } F_n, \mathbb{Q}) = 0$ for $n \geq 1$ and $1 \leq i \leq 6$, except for $H_4(\text{Aut } F_4, \mathbb{Q}) = \mathbb{Q}$. However, there are very few computations of twisted homology groups of $\text{Aut } F_n$.

Fix a free basis Y of F_n . Since the abelianized group H of F_n is isomorphic to $\mathbb{Z}^n$, abelianization induces a homomorphism $\varphi : \text{Aut } F_n \rightarrow \text{Aut } H = GL(n, \mathbb{Z})$. The map φ induces the action of $\text{Aut } F_n$ on H , and hence the dual group $H^* = \text{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$ of H . We denote by $\text{Out } F_n$ the outer automorphism group of F_n . Since φ induces a natural map $\bar{\varphi} : \text{Out } F_n \rightarrow GL(n, \mathbb{Z})$, $\text{Out } F_n$ also acts on H and H^* . In this paper, we calculate the twisted first homology groups of $\text{Aut } F_n$ and $\text{Out } F_n$ with coefficients in H and H^* . Let $\det : GL(n, \mathbb{Z}) \rightarrow \{\pm 1\}$ be the determinant map. The groups $\text{Aut}^+ F_n = \ker(\det \circ \varphi)$ and $\text{Out}^+ F_n = \ker(\det \circ \bar{\varphi})$ are called the special automorphism group and the special outer automorphism group of F_n respectively. The following theorem is our main result.

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Theorem 1. For $n \geq 2$, we have:

(1) If $\Gamma_n = \text{Aut } F_n$ or $\text{Aut}^+ F_n$,

$$H_1(\Gamma_n, H) = \begin{cases} 0 & \text{if } n \geq 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 3, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2 \text{ and } \Gamma_2 = \text{Aut } F_2, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2 \text{ and } \Gamma_2 = \text{Aut}^+ F_2, \end{cases}$$

$$H_1(\Gamma_n, H^*) = \begin{cases} \mathbb{Z} & \text{if } n \geq 4, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2, 3. \end{cases}$$

(2) If $\Omega_n = \text{Out } F_n$ or $\text{Out}^+ F_n$,

$$H_1(\Omega_n, H) = \begin{cases} 0 & \text{if } n \geq 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2, 3, \end{cases}$$

$$H_1(\Omega_n, H^*) = \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z} & \text{if } n \geq 4, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 3, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2. \end{cases}$$

In Section 2, we introduce Gersten's finite presentation for $\text{Aut}^+ F_n$. We simplify his presentation using Titz transformations. We use it to calculate the first cohomology group of $\text{Aut}^+ F_n$.

In Section 5, we give some consequences of our results. We show that the generator of $H^1(\text{Aut}^+ F_n, H) = \mathbb{Z}$ is induced by the Magnus representation of $\text{Aut}^+ F_n$. This shows that the natural map $M_{g,1} \hookrightarrow \text{Aut}^+ F_{2g}$ induces an isomorphism $H^1(\text{Aut}^+ F_{2g}, H) \rightarrow H^1(M_{g,1}, H)$ where $M_{g,1}$ is the mapping class group of a surface of genus g with one boundary component.

2. A PRESENTATION FOR THE SPECIAL AUTOMORPHISM GROUP OF A FREE GROUP

In this section, we introduce Gersten's finite presentation for $\text{Aut}^+ F_n$. Let $Y = \{y_1, \dots, y_n\}$ be a free basis of F_n and let $Y^{\pm 1} = \{y \mid y \text{ or } y^{-1} \in Y\}$. For any $a, b \in Y^{\pm 1}$ with $a \neq b^{\pm 1}$, define the Nielsen automorphism E_{ab} by the rule $a \mapsto ab$, $c \mapsto c$ if $c \in Y^{\pm 1} \setminus \{a^{\pm 1}\}$ and let $w_{ab} = E_{ba}E_{a^{-1}b}E_{b^{-1}a^{-1}}$. The map w_{ab} induces a permutation σ of $Y^{\pm 1}$ $a \mapsto b^{-1}$, $b \mapsto a$, called the monomial map determined by w_{ab} . Gersten [2] showed that $\text{Aut}^+ F_n$ has a following presentation.

Theorem 2.1 (Gersten [2]). For $n \geq 3$, a presentation for $\text{Aut}^+ F_n$ is given by the generators E_{ab} and relations:

$$(R1): E_{ab}^{-1} = E_{ab^{-1}},$$

$$(R2): [E_{ab}, E_{cd}] = 1, \quad a \neq c, d^{\pm 1}, \quad b \neq c^{\pm 1},$$

$$(R3): [E_{ab}, E_{bc}] = E_{ac}, \quad a \neq c^{\pm 1},$$

$$(R4): w_{ab} = w_{a^{-1}b^{-1}}$$

$$(R5): w_{ab}^4 = 1.$$

Here $[,]$ denotes the commutator bracket: $[x, y] = xyx^{-1}y^{-1}$.

Remark 2.1. Gersten [2] also showed that if $n = 2$, the group $\text{Aut}^+ F_2$ has a presentation which is given by the generators E_{ab} subject to the relations (R1) – (R3), (R5) and

$$(R4)': w_{ab}^{-1} E_{cd} w_{ab} = E_{\sigma(c)\sigma(d)},$$

where σ is the monomial map determined by w_{ab} .

Using Titze transformations, we have the following presentation for $\text{Aut}^+ F_n$ for $n \geq 3$.

Theorem 2.2. For $n \geq 3$, a presentation for $\text{Aut}^+ F_n$ is given by the generators $E_{y_i y_j}$ and $E_{y_i^{-1} y_j}$ subject to the relations:

$$(R2-1): [E_{y_i y_j}, E_{y_i^{-1} y_j}] = 1,$$

$$(R2-2): [E_{y_i y_j}, E_{y_k y_j}] = 1,$$

$$(R2-3): [E_{y_i^{-1} y_j}, E_{y_k y_j}] = 1,$$

$$(R2-4): [E_{y_i^{-1} y_j}, E_{y_k^{-1} y_j}] = 1,$$

$$(R2-5): [E_{y_i y_j}, E_{y_i^{-1} y_k}] = 1,$$

$$(R2-6): [E_{y_i y_j}, E_{y_k y_l}] = 1,$$

$$(R2-7): [E_{y_i^{-1} y_j}, E_{y_k y_l}] = 1,$$

$$(R2-8): [E_{y_i^{-1} y_j}, E_{y_k^{-1} y_l}] = 1,$$

$$(R3-1): [E_{y_i y_k}, E_{y_k y_j}] = E_{y_i y_j},$$

$$(R3-2): [E_{y_i y_k^{-1}}, E_{y_k^{-1} y_j}] = E_{y_i y_j},$$

$$(R3-3): [E_{y_i^{-1} y_k}, E_{y_k y_j}] = E_{y_i^{-1} y_j},$$

$$(R3-4): [E_{y_i^{-1} y_k^{-1}}, E_{y_k^{-1} y_j}] = E_{y_i^{-1} y_j},$$

$$(R4-1): w_{y_i y_j} = w_{y_i^{-1} y_j^{-1}},$$

$$(R5-1): w_{y_i y_j}^4 = 1,$$

where $E_{y_i y_j^{-1}}$ is understood to be $E_{y_i y_j}^{-1}$.

3. THE AUTOMORPHISM GROUP OF A FREE GROUP

Until Section 4, we assume $n \geq 3$. For any integer $q \geq 2$, let $A_q = H \otimes_{\mathbb{Z}} (\mathbb{Z}/q\mathbb{Z})$ and $A_q^* = H^* \otimes_{\mathbb{Z}} (\mathbb{Z}/q\mathbb{Z})$. Let $M = H, H^*, A_q$ or A_q^* . Using the presentation for $\text{Aut}^+ F_n$ obtained by Theorem 2.2, we can calculate the twisted first cohomology groups of $\text{Aut}^+ F_n$ as follows:

Proposition 3.1. *Let $q \geq 2$ and $e \geq 1$ be positive integers. For $n \geq 3$, we have*

$$H^1(\text{Aut}^+ F_n, H) = \mathbb{Z},$$

$$H^1(\text{Aut}^+ F_n, A_q) = \begin{cases} \mathbb{Z}/q\mathbb{Z} & \text{if } (q, 2) = 1, \\ \mathbb{Z}/q\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 3 \text{ and } q = 2^e. \end{cases}$$

Proposition 3.2. *Let $q \geq 2$ and $e \geq 1$ be positive integers. For $n \geq 3$, we have*

$$H^1(\text{Aut}^+ F_n, H^*) = 0,$$

$$H^1(\text{Aut}^+ F_n, A_q^*) = \begin{cases} 0 & \text{if } (q, 2) = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 3 \text{ and } q = 2^e. \end{cases}$$

Observing the spectral sequence of the group extension

$$1 \rightarrow \text{Aut}^+ F_n \rightarrow \text{Aut } F_n \rightarrow \{\pm 1\} \rightarrow 1,$$

we see that $H^1(\text{Aut } F_n, M) \simeq H^1(\text{Aut}^+ F_n, M)$ For $M = H, H^*, A_q$ or A_q^* . Then, using the universal coefficient theorem, we obtain the twisted first homology groups of $\text{Aut } F_n$.

4. THE OUTER AUTOMORPHISM GROUP OF A FREE GROUP

Let $\text{Inn } F_n$ be the group of inner automorphisms of F_n . Observing the spectral sequence of the group extension

$$1 \rightarrow \text{Inn } F_n \rightarrow \text{Aut}^+ F_n \rightarrow \text{Out}^+ F_n \rightarrow 1,$$

we calculate the twisted first cohomology groups of $\text{Out}^+ F_n$ as follows:

Proposition 4.1. *Let $q \geq 2$ and $e \geq 1$ be positive integers. For $n \geq 3$, we have*

$$H^1(\text{Out}^+ F_n, H) = 0, \quad H^1(\text{Out}^+ F_n, H^*) = 0.$$

Proposition 4.2. *Let $q \geq 2$ and $e \geq 1$ be positive integers. For $n \geq 3$, we have*

(1) *If $n = 3$,*

$$H^1(\text{Out}^+ F_3, A_q) = \begin{cases} 0 & \text{if } (q, 2) = 1, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } q = 2^e, \end{cases}$$

$$H^1(\text{Out}^+ F_3, A_q^*) = \begin{cases} 0 & \text{if } (q, 2) = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q = 2^e. \end{cases}$$

(2) If $n \geq 4$,

$$H^1(\text{Out}^+ F_n, A_q) = \begin{cases} 0 & \text{if } (q, n-1) = 1, \\ \mathbf{Z}/q\mathbf{Z} & \text{if } q \mid (n-1), \\ \mathbf{Z}/(n-1)\mathbf{Z} & \text{if } (n-1) \mid q, \end{cases}$$

$$H^1(\text{Out}^+ F_n, A_q^*) = 0.$$

Then, using the universal coefficient theorem, we obtain the twisted first homology groups of $\text{Out}^+ F_n$. Furthermore, observing the spectral sequence of the group extension

$$1 \rightarrow \text{Out}^+ F_n \rightarrow \text{Out } F_n \rightarrow \{\pm 1\} \rightarrow 1,$$

we see that $H^1(\text{Out } F_n, M) \simeq H^1(\text{Out}^+ F_n, M)$ For $M = H, H^*, A_q$ or A_q^* . Then, using the universal coefficient theorem, we obtain the twisted first homology groups of $\text{Out } F_n$.

5. SOME CONSEQUENCES

we show that the generator of $H^1(\text{Aut}^+ F_n, H) = \mathbf{Z}$ is induced by the Magnus representation of $\text{Aut}^+ F_n$. For any generator y_j ($1 \leq j \leq n$) of F_n , let

$$\frac{\partial}{\partial y_j} : \mathbf{Z}[F_n] \longrightarrow \mathbf{Z}[F_n]$$

be the Fox free derivatives. (See [1].) Let $\bar{\cdot} : \mathbf{Z}[F_n] \rightarrow \mathbf{Z}[F_n]$ be the antiautomorphism induced from the map $F_n \ni y \mapsto y^{-1} \in F_n$. Then the Magnus representation $r : \text{Aut}^+ F_n \rightarrow GL(n, \mathbf{Z}[F_n])$ of $\text{Aut}^+ F_n$ is defined to be

$$r(\sigma) = \left(\frac{\partial \sigma(y_j)}{\partial y_i} \right)_{(i,j)}.$$

Let $\sigma_* : \mathbf{Z}[F_n] \rightarrow \mathbf{Z}[F_n]$ be the automorphism of $\mathbf{Z}[F_n]$ induced from σ . The map r satisfies

$$(1) \quad r(\sigma\tau) = r(\sigma) \cdot r(\tau)^\sigma.$$

Here $r(\tau)^\sigma$ denotes the matrix obtained from $r(\tau)$ by applying σ_* on each entry. (See [5].) Let $a' : GL(n, \mathbf{Z}[F_n]) \rightarrow GL(n, \mathbf{Z}[H])$ be the homomorphism induced from the abelianizer $a : F_n \rightarrow H$ and $\det : GL(n, \mathbf{Z}[H]) \rightarrow \mathbf{Z}[H]$ the determinant homomorphism. Then we put

$$f_M = \det \circ a' \circ r : \text{Aut}^+ F_n \longrightarrow \mathbf{Z}[H].$$

Observing our results obtained in Section 3, we have

Lemma 5.1. *The map f_M is a crossed homomorphism from $\text{Aut}^+ F_n$ to H and generates $H^1(\text{Aut}^+ F_n, H)$.*

Remark 5.1. *We should remark that the same argument does not hold in the case $H^1(\text{Aut } F_n, H)$. In this case, the image of the crossed homomorphism $f_M : \text{Aut } F_n \rightarrow \mathbb{Z}[H]$ is not included in H .*

Morita [4] calculated $H^1(M_{g,1}, H_1(\Sigma_{g,1}, \mathbb{Z})) = \mathbb{Z}$ and show that the generator of $H^1(M_{g,1}, H_1(\Sigma_{g,1}, \mathbb{Z}))$ is also given by the Magnus representation of $M_{g,1}$. (See [5].) Hence we have

Corollary 5.1. *The natural map $M_{g,1} \hookrightarrow \text{Aut}^+ F_{2g}$ induces an isomorphism*

$$\text{res} : H^1(\text{Aut}^+ F_{2g}, H) \rightarrow H^1(M_{g,1}, H_1(\Sigma_{g,1}, \mathbb{Z})).$$

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