

On a problem in combinatorics.

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1. Introduction.

PROBLEM. There are n boys and n girls. They are sitting at table, the boys looking at the girls, and the girls the boys. Each member of each group chooses one of the other group. A boy and a girl that choose each other, we call them a couple. i) What is probability of that the number of couples is k ($0 \leq k \leq n$)? ii) How behave the probability distribution of the random variable k , for increasing n ?

This problem was proposed by Prof. Hisatsugu*) in statistics. More generally, we define the (fixed points) problem of $A_1 \times \cdots \times A_\lambda$ by the problem of counting the number

$$H_{A_1 \times \cdots \times A_\lambda}(k) = \#\{(f_1, \dots, f_\lambda) \in A_1 \times \cdots \times A_\lambda; \varphi(f_\lambda \circ \cdots \circ f_1) = k\},$$

where

$$A_1, \dots, A_\lambda \subset N^N = \{f; f \text{ is a map from } N \text{ into } N\},$$

$$\varphi(f) = \#\{x \in N; f(x) = x\} \quad (\text{for } f \in N^N)$$

and $N = \{1, \dots, n\}$. Let $H_\lambda(n, k) = H_{(N^N)^\lambda}(k)$. It is plain that

$$H_\lambda(n, n) = (n!)^{\lambda-1}, \quad \sum_{k=0}^n H_\lambda(n, k) = n^{\lambda n}.$$

Our probability asked at i) is $H_2(n, k)/n^{2n}$. We have explicit formula

$$(1.1) \quad H_\lambda(n, k) = \frac{n^{\lambda n}}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \left\{ \frac{(n)_{j+k}}{n_{j+k}} \right\}^\lambda,$$

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where $(x)_k = x(x-1)\dots(x-k+1)$, $(x)_0 = x^0 = 1$. (For some values of $H_\lambda(n, k)$, see the table in the appendix.) Especially in the case of $\lambda = 1$, we have

$$H_1(n, k) = \binom{n}{k} (n-1)^{n-k}.$$

We consider only the problem of $(N^N)^\lambda$ herein, however, we would think that is not vacant to compare with the problem of $(N!)^\lambda$, where $N!$ is symmetric group on N . We get following

TABLE I.

problem of	$(N^N)^\lambda$	$(N!)^\lambda$ *)	Poisson d. with pm. 1	def.
number	$H_\lambda(n, k)$	$(n!)^{\lambda-1} D(n, k)$		
probability	$\frac{H_\lambda(n, k)}{n^\lambda n} = \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \left\{ \frac{(n)_{j+k}}{n^{j+k}} \right\}^\lambda$	$\frac{D(n, k)}{n!} = \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}$	$\frac{1}{e \cdot k!}$	$p(k)$
j-th moment	$\sum_{m=0}^j S(j, m) \left\{ \frac{(n)_m}{n^m} \right\}^\lambda$	$\sum_{m=0}^j S(j, m) **)$	$\hat{B}_j$ †	$\{k^j\}_p$
characteristic function	$\sum_{m=0}^n \frac{(e^{it}-1)^m}{m!} \left\{ \frac{(n)_m}{n^m} \right\}^\lambda$	$\sum_{m=0}^n \frac{(e^{it}-1)^m}{m!}$	$e^{e^{it}-1}$	$\{e^{ikt}\}_p$

*) The problem of $(N!)^\lambda$ is reduced to the problem of $N!$, that is well known as 'problème des rencontres' (see [1]).

**) $S(j, m)$ is a Stirling number of the second kind, that is defined by $x^j = \sum_{m=0}^j S(j, m) (x)_m = \sum_{m=0}^{\infty} S(j, m) (x)_m$.

† $\hat{B}_j$ is the j-th Bell number, that is defined by $\hat{B}_j = \sum_{m=0}^j S(j, m)$.

Our purpose is to get an asymptotic expression of $H_\lambda(n, k)$ for increasing n . We firstly obtain an infinite sum form

$$(1.2) \quad H_{\lambda}(n, k) = \frac{n^{\lambda n}}{e \cdot k!} \sum_{m=0}^{\infty} c_m(\lambda, n) d_m(k),$$

where

$$c_m(\lambda, n) = \frac{(-1)^m}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} \left\{ \frac{(n)_j}{n^j} \right\}^{\lambda},$$

$$d_m(k) = \sum_{t=0}^m (-1)^{m-t} \binom{m}{t} (k)_t.$$

When we set

$$(1.3) \quad c_m(\lambda, n) = \sum_{\ell=0}^{\lambda m} \alpha_{\lambda}(\ell, m) n^{-\ell},$$

we have

$$(1.4) \quad \alpha_{\lambda}(0, 0) = 1,$$

$$(1.5) \quad c_m(\lambda, n) = \sum_{\ell=\lfloor \frac{m+1}{2} \rfloor}^{\lambda(m-1)} \alpha_{\lambda}(\ell, m) n^{-\ell} \quad (m > 0)$$

(for some values of $\alpha_{\lambda}(\ell, m)$, see the appendix). Observing (1.2) and (1.5), we may conjecture that

$$(1.6) \quad \frac{H_{\lambda}(n, k)}{n^{\lambda n}} = \frac{1}{e \cdot k!} \sum_{\ell=0}^{L-1} n^{-\ell} \sum_{\frac{\ell}{\lambda} \leq m \leq 2\ell} \alpha_{\lambda}(\ell, m) d_m(k) + O_{\lambda, L}(n^{-L}).$$

We get following

THEOREM. For $\lambda = 1, 2$, (1.6) is correct.

COROLLARY. If $\lambda = 1$ or 2 , $H_{\lambda}(n, k)/n^{\lambda n}$ converges to $1/e \cdot k!$, uniformly for $k = 0, 1, 2, \dots$, as n tends to infinity.

EXAMPLE. By the theorem, we have the asymptotic expansion

$$H_2(n, 0) \approx \frac{n^{2n}}{e} \left(1 - \frac{1}{n} - \frac{2}{3n^2} - \frac{1}{2n^3} + \dots \right).$$

If $n = 10$, we have

$$H_2(10, 0) = 32845 \ 29863 \ 68548 \ 28800,$$

$$\frac{n^{2n}}{e} \left(1 - \frac{1}{n} - \frac{2}{3n^2} - \frac{1}{2n^3} \right) \Big|_{n=10} = 0.32845 \ 50277 \times 10^{20}.$$

2. Proof of (1.1).

$$\begin{aligned}
\sum_{k=j}^n \binom{k}{j} H_\lambda(n, k) &= \sum_{k=j}^n \binom{k}{j} \sum_{\substack{(f_1, \dots, f_\lambda) \in (N^N)^\lambda \\ \varphi(f_\lambda \circ \dots \circ f_1) = k}} 1 \\
&= \sum_{k=j}^n \sum_{\substack{(f_1, \dots, f_\lambda) \in (N^N)^\lambda \\ \varphi(f_\lambda \circ \dots \circ f_1) = k}} \sum_{\substack{A_1 \subset \{x \in N; f_\lambda \circ \dots \circ f_1(x) = x\} \\ \# A_1 = j}} 1 \\
&= \sum_{\substack{A_1 \subset N \\ \# A_1 = j}} \sum_{\substack{(f_1, \dots, f_\lambda) \in (N^N)^\lambda \\ \{x \in N; f_\lambda \circ \dots \circ f_1(x) = x\} \supset A_1}} 1 \\
&= \sum_{\substack{A_1 \subset N \\ \# A_1 = j}} \dots \sum_{\substack{A_\lambda \subset N \\ \# A_\lambda = j}} \sum_{\substack{(f_1, \dots, f_\lambda) \in (N^N)^\lambda \\ f_i(A_i) = A_{i+1} \ (i=1, \dots, \lambda-1) \\ (f_\lambda|A_\lambda) \circ \dots \circ (f_1|A_1) = I_{A_1}}} 1 \\
&= \left\{ \binom{n}{j} \right\}^\lambda (j!)^{\lambda-1} (n^{n-j})^\lambda \\
(1.2) \quad &= \frac{n^{\lambda n}}{j!} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda,
\end{aligned}$$

where $f_i|A_i$ is a restriction of f_i to A_i and I_{A_1} is the identity map on A_1 . Thus we have

$$\begin{pmatrix} \binom{0}{0} & \dots & \binom{n}{0} \\ & \ddots & \vdots \\ 0 & & \binom{n}{n} \end{pmatrix} \begin{pmatrix} H_\lambda(n, 0) \\ \vdots \\ H_\lambda(n, n) \end{pmatrix} = \begin{pmatrix} \frac{n^{\lambda n}}{0!} \left\{ \frac{(n)_0}{n^0} \right\}^\lambda \\ \vdots \\ \frac{n^{\lambda n}}{n!} \left\{ \frac{(n)_n}{n^n} \right\}^\lambda \end{pmatrix}.$$

Since

$$\begin{pmatrix} \binom{0}{0} & \cdots & \binom{n}{0} \\ & \ddots & \vdots \\ 0 & & \binom{n}{n} \end{pmatrix}^{-1} = \begin{pmatrix} \binom{0}{0} & -\binom{1}{0} & \binom{2}{1} & \cdots & (-1)^{n-0} \binom{n}{0} \\ & \binom{1}{1} & -\binom{2}{1} & & \vdots \\ & & \binom{2}{2} & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & (-1)^{n-n} \binom{n}{n} \end{pmatrix},$$

we have

$$\begin{aligned} H_\lambda(n, k) &= \sum_{j=k}^n (-1)^{j-k} \binom{j}{k} \frac{n^{\lambda n}}{j!} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda \\ &= \frac{n^{\lambda n}}{k!} \sum_{j=k}^n \frac{(-1)^{j-k}}{(j-k)!} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda = \frac{n^{\lambda n}}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \left\{ \frac{(n)_{j+k}}{n^{j+k}} \right\}^\lambda. \end{aligned}$$

q.e.d.

3. Moments.

By (2.1), we have

$$\sum_{k=j}^n (k)_j \frac{H_\lambda(n, k)}{n^{\lambda n}} = \left\{ \frac{(n)_j}{n^j} \right\}^\lambda \quad (0 \leq j \leq n).$$

Since $(k)_j = 0$ ($k < j$), the right hand side may be replaced by $\sum_{k=0}^n$ instead of $\sum_{k=j}^n$. Let it be so. Then the both sides vanish if $j > n$.

Hence we have, for all non-negative integer j ,

$$\sum_{k=0}^n (k)_j \frac{H_\lambda(n, k)}{n^{\lambda n}} = \left\{ \frac{(n)_j}{n^j} \right\}^\lambda$$

Thus, for $j = 0, 1, 2, \dots$, we have the j -th moment

$$\begin{aligned} (3.1) \quad \sum_{k=0}^n k^j \frac{H_\lambda(n, k)}{n^{\lambda n}} &= \sum_{k=0}^n \frac{H_\lambda(n, k)}{n^{\lambda n}} \sum_{m=0}^j S(j, m) (k)_m \\ &= \sum_{m=0}^j S(j, m) \sum_{k=0}^n (k)_m \frac{H_\lambda(n, k)}{n^{\lambda n}} = \sum_{m=0}^j S(j, m) \left\{ \frac{(n)_m}{n^m} \right\}^\lambda. \end{aligned}$$

4. Characteristic function.

We consider the function $\varphi_{\lambda,n}(t)$ that is called characteristic function of the probability distribution

$$\Phi_{\lambda,n}(I) = \sum_{k \in I} \frac{H_{\lambda}(n,k)}{n^{\lambda n}} \quad (I \subset N).$$

We have, by (2.1),

$$\begin{aligned} (4.1) \quad \varphi_{\lambda,n}(t) &\stackrel{\text{def.}}{=} \sum_{k=0}^n e^{ikt} \frac{H_{\lambda}(n,k)}{n^{\lambda n}} = \sum_{k=0}^n \frac{H_{\lambda}(n,k)}{n^{\lambda n}} \sum_{m=0}^k \binom{k}{m} (e^{it}-1)^m \\ &= \sum_{m=0}^n (e^{it}-1)^m \sum_{k=m}^n \binom{k}{m} \frac{H_{\lambda}(n,k)}{n^{\lambda n}} = \sum_{m=0}^n \frac{(e^{it}-1)^m}{m!} \left\{ \frac{(n)_m}{n^m} \right\}^{\lambda}. \end{aligned}$$

(3.1) could be deduced also from (4.1), by using the formula

$$\frac{(e^x-1)^m}{m!} = \sum_{j=m}^{\infty} S(j,m) \frac{x^j}{j!} \quad (\text{see [1]}).$$

Since $\varphi_{\lambda,n}(t)/\exp(e^{it}-1)$ is an entire function of $e^{it}-1$, we may put

$$(4.2) \quad e^{-z} \varphi_{\lambda,n}(t) = \sum_{n=0}^{\infty} c_m(\lambda, n) z^m \quad (z = e^{it}-1).$$

Then, by (4.1), we have

$$\begin{aligned} (4.3) \quad c_m(\lambda, n) &= \sum_{j=0}^{\min(m,n)} \frac{1}{j!} \left\{ \frac{(n)_j}{n^j} \right\}^{\lambda} (-1)^{m-j} \frac{1}{(m-j)!} \\ &= \frac{(-1)^m}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} \left\{ \frac{(n)_j}{n^j} \right\}^{\lambda}, \end{aligned}$$

$$(4.4) \quad |c_m(\lambda, n)| \leq \frac{2^m}{m!}.$$

5. Proof of (1.2).

In this section, we shall show (1.2) with error term, that is the following

LEMMA 1.

$$\frac{H_\lambda(n, k)}{n^{\lambda n}} = \frac{1}{e k!} \sum_{m=0}^{M-1} c_m(\lambda, n) d_m(k) + E_M \quad (|E_M| \leq \frac{e^4 4^M}{M!}),$$

where

$$(4.5) \quad d_m(k) = \sum_{t=0}^m (-1)^{m-t} \binom{m}{t} (k)_t.$$

COROLLARY. (1.2) is valid.

PROOF. By Cauchy's theorem with (4.1), (4.2),

$$\begin{aligned} \frac{H_\lambda(n, k)}{n^{\lambda n}} &= \operatorname{Res}_{z=-1} \frac{\mathcal{P}_{\lambda, n}(t)}{(z+1)^{k+1}} = \sum_{m=0}^{\infty} c_m(\lambda, n) \operatorname{Res}_{z=-1} \frac{e^z z^m}{(z+1)^{k+1}} \\ &= \frac{1}{e} \sum_{m=0}^{M-1} c_m(\lambda, n) \operatorname{Res}_{z=0} \frac{e^z (z-1)^m}{z^{k+1}} + \frac{1}{2\pi i} \int_{|z|=p} \frac{e^{z-1}}{z^{k+1}} \sum_{m=M}^{\infty} c_m(\lambda, n) (z-1)^m dz. \end{aligned}$$

We have

$$\operatorname{Res}_{z=0} \frac{e^z (z-1)^m}{z^{k+1}} = \frac{d_m(k)}{k!},$$

where $d_m(k)$ is defined by (4.5). On the other hand, by Taylor's theorem with (4.4),

$$\left| \sum_{m=M}^{\infty} c_m(n) x^m \right| \leq \sum_{m=M}^{\infty} \frac{2^m}{m!} |x|^m \leq \frac{|2x|^M}{M!} e^{2|x|}.$$

Therefore,

$$\begin{aligned} |E_M| &\leq \frac{1}{2\pi e} \int_{|z|=p} \frac{e^{\operatorname{Re} z}}{\rho^{k+1}} e^{2|z-1|} \frac{(2|z-1|)^M}{M!} |dz| \\ &\leq \frac{2^M}{M!} \rho^{-k} e^{3\rho+1} (\rho+1)^M. \end{aligned}$$

If we take $\rho = 1$, we have $|E_M| \leq e^4 \cdot 4^M / M!$. q.e.d.

6. On $s_\lambda(j, \ell)$.

Let

$$(6.1) \quad \{(x)_j\}^\lambda = \sum_{\ell=0}^{\lambda j} \Delta_\lambda(j, \ell) x^\ell = \sum_{\ell=-\infty}^{\infty} \Delta_\lambda(j, \ell) x^\ell$$

Especially, $s(j, \ell) = s_1(j, \ell)$ is a Stirling number of the first kind. By the definition (6.1) of $s_\lambda(j, \ell)$, we have recurrence formulae

$$(6.2) \quad \begin{aligned} \Delta_\lambda(j+1, \ell) &= \sum_{\ell=0}^{\lambda} (-1)^{\lambda-\ell} \binom{\lambda}{\ell} j^{\lambda-\ell} \Delta_\lambda(j, \ell-\ell) \quad (j \geq 0), \\ \Delta_\lambda(j+1, \lambda(j+1)-\ell) &= \sum_{\ell=0}^{\lambda} (-1)^{\ell} \binom{\lambda}{\ell} j^{\ell} \Delta_\lambda(j, \lambda j - (\ell-\ell)) \quad (j \geq 0). \end{aligned}$$

The latter implies that $s_\lambda(j, \lambda j - \ell)$ is a polynomial of degree 2ℓ in $\mathbb{Q}[j]$, for $\ell \geq 0$. So we may define $\alpha_\lambda(\ell, m)$ by

$$(6.3) \quad \Delta_\lambda(j, \lambda j - \ell) = \sum_{m=0}^{2\ell} \alpha_\lambda(\ell, m) (j)_m$$

We define $\alpha_\lambda(\ell, m) = 0$ outside $\ell \geq 0, 0 \leq m \leq 2\ell$. (6.2) leads us to

$$(6.4) \quad (m+1)\alpha_\lambda(\ell, m+1) = \sum_{\ell=1}^{\lambda} (-1)^{\ell} \binom{\lambda}{\ell} \sum_{u=0}^{\ell} \alpha_\lambda(\ell-\ell, m-u) \eta_{\ell, u}(m-u),$$

where

$$\eta_{\ell, u}(m) = \sum_{t=u}^{\ell} \binom{\ell}{t} m^{\ell-t} S(t, u) \in \mathbb{Z}[m].$$

We now show the following

LEMMA 2. i) $\alpha_\lambda(0, 0) = 1$. ii) If $\ell > 0$, then $\alpha_\lambda(\ell, 0) = 0$. iii) For $m > 0$, if $\ell > \lambda(m-1)$ then $\alpha_\lambda(\ell, m) = 0$.

PROOF. i) It is clear from (6.1) and (6.3). ii) If $\ell > 0$, by the definition (6.1) of $s_\lambda(j, \ell)$, we have $\alpha_\lambda(0, -\ell) = 0$. When we substitute $j = 0$ in the both hand sides of (6.3), we obtain $\alpha_\lambda(\ell, 0) = 0$. iii) Induction on m . If $m = 1$ and $\ell > 0$, we have, by (6.4),

$$\alpha_\lambda(\ell, 1) = \sum_{g=1}^{\lambda} (-1)^g \binom{\lambda}{g} \alpha_\lambda(\ell-g, 0) \eta_{g,0}(0).$$

Since $\eta_{q,0}(0) = S(q, 0) = 0$ ($q > 0$), we have $\alpha_\lambda(\ell, 1) = 0$. Suppose $\alpha_\lambda(\ell, \mu) = 0$, for $\ell > \lambda(\mu - 1)$ and $\mu = 1, \dots, m$. Let $\ell > \lambda m$. If $1 \leq q \leq \lambda$ and $0 \leq u \leq q$, we have

$$\ell - q > \lambda m - q \geq \lambda(m - u) - q \geq \lambda(m - u) - \lambda$$

and $\alpha_\lambda(\ell - q, m - u) = 0$. By (6.4) we get $\alpha_\lambda(\ell, m+1) = 0$. q.e.d.

COROLLARY. If $\ell > 0$, then

$$(6.5) \quad \Delta_\lambda(j, \lambda j - \ell) = \sum_{\frac{\ell}{\lambda} + 1 \leq m \leq 2\ell} \alpha_\lambda(\ell, m) (j)_m.$$

7. On $c_m(\lambda, n)$.

We have

$$\left\{ \frac{(n)_j}{n^j} \right\}^\lambda = \frac{1}{n^{\lambda j}} \sum_{j=0}^m \Delta_\lambda(j, \ell) n^\ell = \sum_{\ell=0}^{\lambda j} \Delta_\lambda(j, \lambda j - \ell) n^{-\ell}.$$

By (4.3),

$$\begin{aligned} c_m(\lambda, n) &= \frac{(-1)^m}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} \sum_{\ell=0}^{\lambda j} \Delta_\lambda(j, \lambda j - \ell) n^{-\ell} \\ &= \frac{(-1)^m}{m!} \sum_{\ell=0}^{\lambda m} n^{-\ell} \sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} \Delta_\lambda(j, \lambda j - \ell). \end{aligned}$$

If $\ell = 0$, we have

$$\sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} \Delta_\lambda(j, \lambda j - \ell) = \sum_{j=0}^m (-1)^j \binom{m}{j} = \delta_{m,0}.$$

If $\ell > 0$,

$$\begin{aligned} \sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} \Delta_\lambda(j, \lambda j - \ell) &= \sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} \sum_{\frac{\ell}{\lambda} + 1 \leq t \leq 2\ell} \alpha_\lambda(\ell, t) (j)_t \\ &= \sum_{\frac{\ell}{\lambda} + 1 \leq t \leq 2\ell} \alpha_\lambda(\ell, t) \sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} (j)_t, \end{aligned}$$

since $(j)_t = 0$ (if $m < t$, $j \leq m$),

$$= \sum_{\substack{\frac{\ell}{\lambda}+1 \leq t \leq 2\ell \\ t \leq m}} \alpha_\lambda(\ell, t) \sum_{\substack{\frac{\ell}{\lambda} \leq j \leq m}} (-1)^j \binom{m}{j} (j)_t,$$

here the parameter t runs in the range $t > \ell/\lambda$, so that,

$$\begin{aligned} &= \sum_{\substack{\frac{\ell}{\lambda}+1 \leq t \leq 2\ell \\ t \leq m}} \alpha_\lambda(\ell, t) t! \sum_{j=t}^m (-1)^j \binom{m}{j} \binom{j}{t} \\ &= \sum_{\substack{\frac{\ell}{\lambda}+1 \leq t \leq 2\ell \\ t \leq m}} \alpha_\lambda(\ell, t) t! (-1)^t \delta_{m,t} \\ &= (-1)^m m! \alpha_\lambda(\ell, m). \end{aligned}$$

We have

$$c_m(\lambda, n) = \sum_{\ell=0}^{\lambda m} n^{-\ell} \alpha_\lambda(\ell, m).$$

Thus (1.3)-(1.5) are consistent with (6.1)-(6.5).

8. Inequalities.

We provide several inequalities for proving the theorem.

LEMMA 3.

$$(8.1) \quad |d_m(k)| \leq 2^m k!,$$

$$(8.2) \quad |c_m(1, n)| \leq n^{-[\frac{m+1}{2}]},$$

$$(8.3) \quad |c_m(2, n)| \leq n^{-[\frac{m+1}{2}]} (m+1)!.$$

PROOF. (8.1) is clear from (4.5). To prove (8.2) and (8.3), we introduce functions

$$f_{\lambda, m}(x) = \sum_{\ell=0}^{\lambda m} \alpha_\lambda(\ell, m) x^\ell$$

of x . Then we have $c_m(\lambda, n) = f_{\lambda, m}(1/n)$. Since

$$|\alpha_\lambda(\ell, m)| = (-1)^\ell \alpha_\lambda(\ell, m) \quad (\text{see (6.4)}),$$

if $m > 0$, we have

$$|c_m(\lambda, n)| \leq \sum_{\ell=\lfloor \frac{m+1}{2} \rfloor}^{\lambda(m-1)} |\alpha_\lambda(\ell, m)| n^{-\ell} \leq n^{-\lfloor \frac{m+1}{2} \rfloor} f_{\lambda, m}(-1).$$

Let $\beta_{\lambda, m} = f_{\lambda, m}(-1)$. By (6.4), we have

$$(m+1) f_{\lambda, m+1}(x) = \sum_{q=1}^{\lambda} (-x)^q \binom{\lambda}{q} \sum_{u=0}^q \eta_{q, u}(m-u) f_{\lambda, m-u}(x),$$

$$(8.4) \quad m \beta_{1, m} = \beta_{1, m-2} + (m-1) \beta_{1, m-1},$$

$$(8.5) \quad m \beta_{2, m} = \beta_{2, m-3} + (2m-1) \beta_{2, m-2} + (m^2-1) \beta_{2, m-1}.$$

It is enough to show $\beta_{1, m} \leq 1$ and $\beta_{2, m} \leq (m+1)!$. It follows from (8.4) that

$$\beta_{1, m} = \sum_{j=0}^m \frac{(-1)^j}{m!} \leq 1.$$

We now prove $\beta_{2, m} \leq (m+1)!$ by induction. $\beta_{2, 0} = 1 \leq 1!$, $\beta_{2, 1} = 0 \leq 2!$, $\beta_{2, 2} = 3/2 \leq 3!$. Suppose $m \geq 3$ and $\beta_{2, \mu} \leq (\mu+1)!$ for $\mu < m$. Thus, by

(8.5), we have

$$\begin{aligned} \beta_{2, m} &\leq \frac{1}{m} \left\{ (m-2)! + (2m-1)(m-1)! + (m-1)(m+1)! \right\} \\ &= (m+1)! \left\{ \frac{1}{(m+1)m^2(m-1)} + \frac{2-\frac{1}{m}}{(m+1)m} + \frac{m-1}{m} \right\} \\ &\leq (m+1)! \left\{ \frac{1}{4 \cdot 3 \cdot 2 m} + \frac{2}{m^2} + 1 - \frac{1}{m} \right\} \\ &= (m+1)! \left\{ 1 - \frac{1}{m} \left(\frac{23}{24} - \frac{2}{m} \right) \right\} \leq (m+1)!. \end{aligned}$$

q.e.d.

9. Proof of the theorem.

We divide the series (1.2) into three parts as

$$\frac{H_\lambda(n, k)}{n^{\lambda n}} = \frac{1}{e k!} \sum_{m=0}^{2\lambda L} c_m(\lambda, n) d_m(k) + \frac{1}{e k!} \sum_{m=2\lambda L+1}^{[x]} c_m(\lambda, n) d_m(k) + E_{[x]+1}.$$

By Lemma 1, we have

$$(9.1) \quad |E_{[x]+1}| \leq e^4 \frac{4^{x+1}}{\Gamma(x+1)}.$$

By (1.4), (1.5) and (8.1), we obtain

$$(9.2) \quad \frac{1}{e k!} \sum_{m=0}^{2\lambda L} c_m(\lambda, n) d_m(k) = \frac{1}{e k!} \sum_{\ell=0}^{L-1} n^{-\ell} \sum_{\substack{\ell \\ \lambda \leq m \leq 2\ell}} \alpha_\lambda(\ell, m) d_m(k) + O_{\lambda, L}(n^{-L}).$$

By Lemma 3, for $\lambda = 1, 2$, we have

$$(9.3) \quad \frac{1}{e k!} \left| \sum_{m=2\lambda L+1}^{[x]} c_m(\lambda, n) d_m(k) \right| \leq e^{-1} n^{-(\lambda L+1)} \sum_{m=0}^{[x]} 2^m A_\lambda(m),$$

where $A_\lambda(m) = 1$ if $\lambda = 1$, $= (m+1)!$ if $\lambda = 2$.

Suppose $\lambda = 1$. Then

$$(9.4) \quad e^{-1} n^{-(L+1)} \sum_{m=0}^{[x]} 2^m \leq e^{-1} 2^{x+1} n^{-(L+1)}.$$

For a sufficiently large n , there exists $x = x(n) > 0$ such that

$$(9.5) \quad n^{-\mu} = \frac{2^x}{\Gamma(x+1)},$$

where $\mu = L+1$. If we set $x = x(n)$ as (9.5), by Stirling's formula,

$$n^\mu = 2^{-x} \Gamma(x+1) = 2^{-x} \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} (1+o(1)) \quad (n \rightarrow \infty),$$

$$\mu \log n = \frac{1}{2} \log 2\pi + (x+\frac{1}{2}) \log x - x(1+\log 2) + o(1),$$

$$\mu \log n \sim x \log x,$$

$$(9.6) \quad \log \mu + \log \log n = \log x + \log \log x + o(1).$$

Hence we get

$$(9.7) \quad \log \log n \sim \log x.$$

Let $x = y(n) \log n$. Then by (9.6),

$$\log y = \log x - \log \log n = \log \mu - \log \log x + o(1),$$

$$y \sim \frac{\mu}{\log x}.$$

By (9.7), we have

$$(9.8) \quad x \sim \mu \frac{\log n}{\log \log n}.$$

(9.3)-(9.5) and (9.8) lead us to

$$\begin{aligned} \frac{1}{e k!} \sum_{m=2L+1}^{[x]} c_m(1, n) d_m(k) &= O(n^{-\mu} 2^x) \\ &= O(n^{-\mu} 2^{\mu \frac{\log n}{\log \log n} (1+o(1))}) = O(n^{-\mu+o(1)}) = O_L(n^{-L}). \end{aligned}$$

Similarly by (9.1), (9.5), (9.8), we have

$$E_{[x]+1} = O(n^{-\mu} 2^x) = O_L(n^{-L}).$$

In the case of $\lambda = 1$, we get the theorem.

Suppose $\lambda = 2$. We have

$$\sum_{m=0}^{[x]} 2^m (m+1)! \leq 2^{[x]} [x+1]! \sum_{m=0}^{[x]} 2^{-m} \leq 2^{x+1} \Gamma(x+2).$$

Let set

$$n^{-\mu} = \frac{2^x}{(x+1)\Gamma^2(x+1)} \quad (\mu = 2L+1),$$

for large n . On a similar way, we shall get

$$x \sim \frac{\mu}{2} \frac{\log n}{\log \log n}.$$

Thus we obtain

$$\begin{aligned}
& \frac{1}{e^{k!}} \sum_{m=4L+1}^{\lfloor x \rfloor} c_m(2, n) dm(k) = O(n^{-\mu} 2^x \Gamma(x+2)) \\
& = O(n^{-\frac{\mu}{2}} 2^{\frac{3}{2}x} \sqrt{x}) = O_{\mu}(n^{-\mu/2 + o(1)} \sqrt{\frac{\log n}{\log \log n}}) = O_L(n^{-L}), \\
& E_{\lfloor x \rfloor + 1} = O\left(\frac{4^x}{\Gamma(x+1)}\right) = O(n^{-\mu/2} 2^{\frac{3}{2}x} \sqrt{x}) = O_L(n^{-L}).
\end{aligned}$$

q.e.d.

Reference

- [1] Riordan, An introduction to combinatorial analysis, Wiley, 1958.
- [2] R.Kaneiwa, On the multiplicative partition function, Tsukuba J. of Math.
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APPENDIX

Table of $H_\lambda(n,k)$. $H_1(n,k)$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	2	1						
3	8	12	6	1					
4	81	108	54	12	1				
5	1024	1280	640	160	20	1			
6	15625	18750	9375	2500	375	30	1		
7	279936	326592	163296	45360	7560	756	42	1	
8	5764801	6588344	3294172	941192	168070	19208	1372	56	1

 $H_2(n,k)$

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	2	12	2				
3	156	423	144	6			
4	16920	33185	13968	1440	24		
5	2764880	4581225	2088800	316200	14400	120	
6	650696400	973830816	460350000	85521600	6231600	151200	720

 $H_3(n,k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	4	56	4			
3	2880	13959	2808	36		
4	3392064	10139392	3100032	145152	576	
5	7258985600	16544150125	6178856000	526644000	8928000	14400

Table of $\alpha_\lambda(\ell, m)$.
$$\alpha_1(\ell, m)$$

$\ell \backslash m$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	0	$-\frac{1}{2}$								
2	0	0	0	$\frac{1}{3}$	$\frac{1}{8}$						
3	0	0	0	0	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{48}$				
4	0	0	0	0	0	$\frac{1}{5}$	$\frac{13}{72}$	$\frac{1}{24}$	$\frac{1}{384}$		
5	0	0	0	0	0	0	$-\frac{1}{6}$	$-\frac{11}{60}$	$-\frac{17}{288}$	$-\frac{1}{144}$	$-\frac{1}{3840}$

$$\alpha_2(\ell, m)$$

$\ell \backslash m$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	0	-1								
2	0	0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{1}{2}$						
3	0	0	0	-2	-4	$-\frac{5}{3}$	$-\frac{1}{6}$				
4	0	0	0	$\frac{2}{3}$	$\frac{59}{8}$	$\frac{337}{30}$	$\frac{185}{36}$	$\frac{5}{6}$	$\frac{1}{24}$		
5	0	0	0	0	$-\frac{11}{2}$	$-\frac{83}{3}$	$-\frac{827}{24}$	$-\frac{241}{15}$	$-\frac{29}{9}$	$-\frac{5}{18}$	$-\frac{1}{120}$