

**Study on Communication System  
From the Perspective of Improving  
Signal-to-Noise Ratio**

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# Chapter 1

## Communication Systems and Signal-to-Noise Ratio

### I OFDM and CDMA systems

Communication systems have been developed and they are necessary tools for our lives. Nowadays, as wireless communication systems, 4G/LTE systems are used. Further, 5G systems have been researched as future systems. In 4G/LTE systems, Orthogonal Frequency Division Multiplexing (OFDM) systems are used. In addition to these systems, Code Division Multiple Access (CDMA) systems are partially used. It is known that CDMA systems have been used in 3G systems.

To evaluate the efficiency of communication systems, the Shannon capacity is often used [1]. This capacity is defined as the maximum rate of a code which achieves zero error as the length of a code goes infinity [2]. In [1], the capacity has been only investigated under the assumption that each symbol is generated independently. In general situations, the capacity has been obtained in [3] [4]. From the definition of the capacity, it is clear that a large capacity means that data can be transmitted with high rate. Further, it is known that capacity increases as Signal-to-Noise Ratio (SNR) increases in Gaussian noise channels [5]. It is often the case that SNR is evaluated instead of a capacity.

From the perspective of the capacity, OFDM systems have some advantages. One of advantages in OFDM systems is that a large amount of data can be rapidly sent. Thus, OFDM systems can achieve the large capacity. In a practical situation, the capacity of OFDM systems has been investigated [6] [7]. In such a situation, it is known that SNR of OFDM systems depends on amplifiers. This advantage is obtained since OFDM systems have some carriers. Another advantage is that OFDM systems can deal with fading effects [8]. These effects are caused if there are some paths where signals go. Further, it is known that these effects decrease SNR. Since OFDM systems can avoid these effects with a zero-padding technique and a zero forcing technique, OFDM system can also achieve the large capacity in practical situations.

However, OFDM systems have some disadvantages. One of them is that OFDM signals have large Peak-to-Average Power Ratio (PAPR) [9]. Since large PAPR induces distortions and these distortions decrease SNR, large PAPR has

to be avoided. In the situation where distortions are caused, distortions are regarded as noise and Signal-to-Noise plus Distortion Ratio (SNDR) is defined. To avoid distortions, PAPR-reducing techniques have been proposed [10]. Further, properties and formulas about PAPR have been obtained [11]. Another one is that each OFDM system requires to synchronize to recover symbols [12]. In OFDM systems, each signal locates on the non-overlapped frequency band. Since symbols which other users send are regarded as interference noise, each OFDM systems have to synchronize to avoid the noise.

In contrast, asynchronous CDMA systems do not require synchronization with each user. Since each user has a different spreading sequence, each user can detect his/her desired signal. Further, all users share a common frequency band. However, due to this sharing, interference noise occurs in asynchronous CDMA systems. It is known that this interference noise depends on the spreading sequences of all the users [13]. Since interference noise appears in asynchronous CDMA systems, Signal to plus Interference Noise (SINR) is defined to evaluate systems. Therefore, one of obstacles in asynchronous CDMA systems is to increase SINR to achieve the large capacity.

This thesis shows how to improve SINR and SNDR in OFDM and CDMA systems. In OFDM systems, we propose methods to reduce PAPR since PAPR relates to SNDR (see Section IV). One of obstacles in reducing PAPR is that large calculation amount is required to reduce PAPR. To overcome this obstacle, we propose a randomization method and a unitary matrix method. In asynchronous CDMA systems, interference noise is the most significant quantity. Since interference noise depends on spreading sequences, we propose methods to design sequences to achieve large SINR.

In the remains of this chapter, we show the relations among SNR, SINR, Bit Error Rate (BER), capacity and PAPR.

## II Signal-to-Noise Ratio and Channel Capacity

In this section, we show the relation between Signal-to-Noise Ratio and the channel capacity. The channel capacity is an important quantity in communication systems since the channel capacity is the maximum data rate with zero error [14]. Thus, the channel capacity is one of limits in communication systems.

We only consider a discrete channel with finite alphabets. In continue variables and infinite alphabets, the channel capacity is defined with an information spectrum method [3] [4].

Figure 1.1 denotes the communication system model considered here. A sender communicates a message  $W \in \{1, 2, \dots, M\}$ , where  $M < \infty$ . To communicate, the sender encodes the message  $W$  into a codeword  $X^n$  with an encoder  $\varphi_n$  and sends  $X^n$  over the channel in  $n$  times instances. At the receiver side, the sequence  $Y^n$  is received. Then, the receiver decodes the sequence  $Y^n$  and obtain the estimated message  $\hat{W}$  with a decoder  $\psi_n$ .

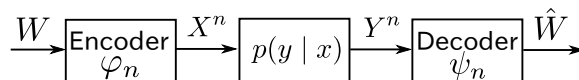


Figure 1.1: Communication system model

In this model, we consider a discrete memoryless channel (DMC). This channel consists of a finite input alphabet  $\mathcal{X}$ , a finite output  $\mathcal{Y}$ , and conditional probability math functions  $p(y | x)$  on  $\mathcal{Y}$  for every  $x \in \mathcal{X}$ . Further, the channel is stationary and memoryless, that is, the output at time  $i \in \{1, 2, \dots, n\}$ ,  $y_i$  is independent of the past inputs and outputs,  $x^i = (x_1, x_2, \dots, x_i)$  and  $y^{i-1} = (y_1, y_2, \dots, y_{i-1})$ , respectively. This property is written as

$$p(y_i | x^i, y^{i-1}) = p(y_i | x_i). \quad (1.1)$$

In the above situation, we define the following quantities.

**Definition.** A  $(M, n)$  code for the DMC  $(\mathcal{X}^n, p(y | x), \mathcal{Y}^n)$  consists of

1. an index (a message) set  $\{1, 2, \dots, M\}$
2. an encoding function (encoder)  $\varphi_n : \{1, 2, \dots, M\} \rightarrow \mathcal{X}^n$  which assigns a codeword  $x^n(1), x^n(2), \dots, x^n(M)$ . The set of codewords is called a codebook.
3. a decoding function (decoder)  $\psi_n : \mathcal{Y}^n \rightarrow \{1, 2, \dots, M\}$ .

**Definition.** The maximal probability of error for a  $(M, n)$  code is defined as

$$\lambda^{(n)} = \max_{i \in \{1, 2, \dots, M\}} \lambda_i, \quad (1.2)$$

where  $\lambda_i$  is the conditional probability of error defined as

$$\lambda_i = \Pr(\psi_n(Y^n) \neq i | X^n = x^n(i)) = \sum_{y^n} p(y^n | x^n(i)) I(\psi_n(y^n) \neq i) \quad (1.3)$$

and  $I(\cdot)$  is the indicator function. Note that the quantity  $\lambda_i$  is often called the conditional probability of error for a message  $i$ .

**Definition.** A rate  $R$  of a code  $(M, n)$  is defined as

$$R = \frac{\log M}{n}. \quad (1.4)$$

**Definition.** A rate  $R$  is called achievable if there exists a  $(\lceil 2^{nR} \rceil, n)$  code such that

$$\lim_{n \rightarrow \infty} \lambda^{(n)} = 0. \quad (1.5)$$

In what follows, a  $(2^{nR}, n)$  code denotes a  $(\lceil 2^{nR} \rceil, n)$  code.

**Definition.** The capacity  $C$  of DMC is the supremum over all achievable rates.

With the above definitions, Shannon has proven the following theorem.

**Theorem** (Shannon [1]). The capacity of DMC is given by the information capacity formula

$$C = \max_{p(x)} I(X; Y), \quad (1.6)$$

where  $I(X; Y)$  is mutual information written as

$$I(X; Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log \frac{p(x, y)}{p(x)p(y)}. \quad (1.7)$$



As seen in the above theorem, Shannon has proven the theorem only in DMC. In [4], it has been proven that the capacity is written in terms of mutual information in a general channel. Therefore, the information capacity formula is also established in continuous variables [15].

From the information capacity formula, capacity for the Gaussian channel is given as follows.

**Theorem 1.** Let  $\mathcal{N}(\mu, \sigma)$  be the Gaussian distribution whose mean and variance are  $\mu$  and  $\sigma$ , respectively. Consider the following channel

$$Y_i = X_i + Z_i, \quad (1.8)$$

where  $X_i$ ,  $Y_i$ , and  $Z_i$  are the input, the output, and the Gaussian noise which follows  $\mathcal{N}(0, N)$  at the time  $i$ , respectively. In particular, the average power of the input signal is restricted as

$$\frac{1}{n} \sum_{i=1}^n x_i^2 \leq P. \quad (1.9)$$

Then, capacity is written as

$$C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right). \quad (1.10)$$

The ratio  $P/N$  is called Signal-to-Noise Ratio (SNR).

A proof of the above theorem is obtained in [15]. From Eq. (1.10), capacity increases as SNR increases. In practical schemes, capacity has been obtained [16]. However, its capacity is not written in a closed form.

### III Signal-to-Noise Ratio and Bit Error Rate

In the previous section, we have shown the relation between SNR and capacity. In practical situations, Bit Error Rate (BER) is often evaluated since BER is regarded as the performance of communication systems. Further, as seen in the definition of the capacity, BER plays important role to define the capacity. In this section, we show the relation between SNR and BER.

Similar to the previous section, we assume that noise follows Gaussian. Then, a transmission model is written as

$$r_k = s_k + n_k, \quad (1.11)$$

where  $r_k$ ,  $s_k$  and  $n_k$  are the received symbol, the transmitted symbol and additive white Gaussian noise at time  $k$ , respectively. Note that these quantities  $r_k$ ,  $s_k$  and  $n_k$  are either real or complex values. Further, we assume that  $E\{|s_k|^2\} = E_S$ , where  $E\{X\}$  is the average of  $X$  and that  $n_k$  follows a circularly symmetric complex Gaussian distribution [17], that is,  $\text{Re}\{n_k\}$  and  $\text{Im}\{n_k\}$  follow independent Gaussian distributions  $\mathcal{N}(0, N_0/2)$ , respectively (If  $n_k$  is a real random value, then  $n_k$  follows  $\mathcal{N}(0, N_0/2)$ ). Then, SNR is defined as

$$\text{SNR} = \frac{E_S}{N_0}. \quad (1.12)$$

In Eq. (1.11), the quantity  $s_k$  denotes the transmitted symbol. In practical schemes, the symbol  $s_k$  has  $\log M$  bits information. We note the example of how to map from  $\log M$  bits information to a real (complex) value. Let us consider the situation where only one bit is sent to the receiver. If the bit 1 is sent, then  $s_k$  is 1. On the other hand, if the bit 0 is sent, then  $s_k$  is 0. This scheme is called Binary Phase Shift Keying (BPSK). With the above procedures, we can construct the map from a bit to real value. One of more general maps is the Gray mapping [12]. In the Gray mapping, a  $M^2$ -QAM scheme is often used. In a  $M^2$ -QAM scheme, the symbol  $s_k$  takes a following value

$$s_k \in \{(2m_1 - 1) + j(2m_2 - 1) \mid m_1, m_2 \in \{-\sqrt{M}/2\} + 1, \dots, \sqrt{M}/2\}\}, \quad (1.13)$$

where  $j$  is the unit imaginary number. Note that if  $M = 1$  then the above scheme is equivalent to the binary case.

From the above definitions, the relation between SNR and BER is derived. To consider BER, we define the following quantity  $E_b$

$$E_b = \frac{E_S}{\log_2 M}. \quad (1.14)$$

The quantity  $E_b$  denotes the average power required to transmit each bit. Note that SNR is rewritten as

$$\text{SNR} = \log_2(M) \cdot \frac{E_b}{N_0}. \quad (1.15)$$

Then, BER in BPSK schemes is written as

$$P_b^{BPSK} = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right), \quad (1.16)$$

where

$$\text{erfc}(x) = \int_x^\infty \exp(-t^2) dt. \quad (1.17)$$

Similarly, BERs in  $M^2$ -QAM schemes with Gray mapping for  $M^2 = 4, 16$  and  $64$  are written as [12]

$$\begin{aligned} P_b^{4-QAM} &= \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right) \\ P_b^{16-QAM} &\approx \frac{3}{8} \text{erfc} \left( \sqrt{\frac{2}{5} \frac{E_b}{N_0}} \right) \\ P_b^{64-QAM} &\approx \frac{7}{24} \text{erfc} \left( \sqrt{\frac{1}{7} \frac{E_b}{N_0}} \right). \end{aligned} \quad (1.18)$$

Note that BERs in the 16-QAM scheme and the 64-QAM scheme are approximate values. As seen in Eq. (1.18), BER decreases as SNR increases.

## IV Signal-to-Noise plus Distortion Ratio and Peak-to-Average Power Ratio

In previous sections, we have shown the relations among SNR, capacity and BER. To derive them, there is no assumptions about amplifiers. Since amplifiers are not ideal in practical situations, distortions are generally caused. It is known that these distortions decrease SNR [6]. Thus, these distortions should be avoided. In this section, we discuss effects of an amplifier in multicarrier systems. In particular, we focus on OFDM systems. In OFDM systems, instead of SNR, Signal-to-Noise plus Distortion Ratio (SNDR) is defined. We derive SNDR and discuss the relation between SNDR and amplifiers.

To judge whether a given signal tends to be distorted or not, Peak-to-Average Power Ratio (PAPR) is often used. PAPR is defined as

$$\text{PAPR} = \max_t \frac{|\zeta(t)|^2}{P_{\text{av}}}, \quad (1.19)$$

where

$$\zeta(t) = \text{Re}\{s(t) \exp(2\pi j f_c t)\}, \quad (1.20)$$

$s(t)$  is a base-band signal,  $f_c$  is the carrier frequency and  $P_{\text{av}}$  is the average power of a base-band signal. Note that PAPR is defined with a Radio Frequency (RF) signal  $\zeta(t)$ . The ratio of a maximum amplitude of a base-band signal to the average power is called Peak-to-Mean Envelope Power Ratio (PMEPR). It has been shown that PMEPR is nearly equivalent to PAPR with sufficiently large  $f_c$  [18]. Thus, to simulate the effects of amplifiers, it is often the case that only base-band signals are considered. Let us consider effects by amplifiers. First, we rewritten the signal  $s(t)$  in polar coordinates as

$$s(t) = r(t) \exp(j\theta(t)). \quad (1.21)$$

Let  $f$  be an amplifier. Then, a output of a base-band signal with the amplifier  $f$  is modeled as [11] [19]

$$f(s(t)) = g(r(t)) \exp(j\theta(t) + j\psi(r(t))), \quad (1.22)$$

where  $g$  and  $\psi$  are maps  $g : [0, \infty) \rightarrow [0, \infty)$  and  $\psi : [0, \infty) \rightarrow (-\infty, \infty)$ . These function are called the AM/AM and AM/PM characteristics. To characterize amplifiers, many kinds of the sets  $(g, \psi)$  have been proposed.

- Soft limiter [6]

In the soft limiter, the set  $(g, \psi)$  is written as

$$g(\rho) = \begin{cases} \rho & \rho \leq A \\ A & \rho > A \end{cases} \quad (1.23)$$

$$\psi(\rho) = 0.$$

- Rapp model [20]

The Rapp model describes the conversion of the transformation of solid-state power amplifiers. The set  $(g, \psi)$  is written as

$$g(\rho) = \frac{\rho}{\left(1 + \left(\frac{\rho}{A}\right)^{2p}\right)^{\frac{1}{2p}}} \quad (1.24)$$

$$\psi(\rho) = 0,$$

where the parameter  $p$  denotes the smoothness of the transition from the linear region to the limiting saturation region. The parameter  $p$  is often chosen as  $p = 2$  and  $p = 3$  [21] [22] [23].

- Saleh model [24]

The set  $(g, \psi)$  of this model is written as

$$\begin{aligned} g(\rho) &= \frac{\alpha_1 \rho}{1 + \alpha_2 \rho} \\ \psi(\rho) &= \frac{\beta_1 \rho^2}{1 + \beta_2 \rho^2}, \end{aligned} \quad (1.25)$$

where  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  are parameters. In [24], these parameters are given by

$$\alpha_1 = 2.1587, \alpha_2 = 1.1517, \beta_1 = 4.0330, \beta_2 = 9.1040. \quad (1.26)$$

- Ghorbani model [25]

This model has more parameters than the Saleh model. The set  $(g, \psi)$  is written as

$$\begin{aligned} g(\rho) &= \frac{\alpha_1 \rho^{\alpha_2}}{1 + \alpha_3 \rho^{\alpha_2}} + \alpha_4 \rho \\ \psi(\rho) &= \frac{\beta_1 \rho^{\beta_2}}{1 + \beta_3 \rho^{\beta_2}} + \beta_4 \rho, \end{aligned} \quad (1.27)$$

where  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3$  and  $\beta_4$  are parameters. In [25], these parameters are given by

$$\begin{aligned} \alpha_1 &= 8, 1081, \alpha_2 = 1.5413, \alpha_3 = 6.5202, \alpha_4 = -0.0718 \\ \beta_1 &= 4.6645, \beta_2 = 2.0965, \beta_3 = 10.8800, \beta_4 = -0.0030. \end{aligned} \quad (1.28)$$

As seen in Eq. (1.20), since PAPR is the ratio of the squared maximum amplitude to the average power, large PAPR means that the signal has large amplitude. Thus, from the models shown above, it seems that a signal tends to be more distorted as PAPR gets larger. In particular, it is known that multi-carrier communication systems have large PAPR. In what follows, we consider a OFDM system, which is one of representative multicarrier communication systems.

Let us discuss the relation between SNDR and PAPR. Their relation has been originally discussed in [6]. To discuss the relation between amplifier and SNDR in OFDM systems, we define signals of OFDM systems. A continuous base-band OFDM signal is defined as

$$s(t) = \frac{1}{\sqrt{K}} \sum_{k=1}^K A_k \exp \left( 2\pi j \frac{k-1}{T} t \right), \quad (1.29)$$

where  $A_k$  is the transmitted symbol with the  $k$ -th carrier,  $j$  is the unit imaginary number and  $T$  is the duration of symbols. On the other hand, a discrete base-band OFDM signal is defined as

$$s_n = s(n\Delta t) = \frac{1}{\sqrt{K}} \sum_{k=1}^K A_k \exp \left( 2\pi j \frac{k-1}{JK} n \right), \quad (1.30)$$

where  $n = 0, 1, \dots, JK - 1$  and  $J$  is the parameter called an oversampling parameter. To generate signals and to recover symbols, Inverse Discrete Fourier Transformation (IDFT) and Discrete Fourier Transformation (DFT) are used in OFDM systems, respectively. Thus, we consider a discrete signal. To analyze Eq. (1.30), we write the FFT and IFFT transformations as follows

$$\begin{aligned} \text{DFT}(L, \mathbf{X})_n &= \frac{1}{\sqrt{L}} \sum_{l=1}^L X_l \exp \left( -2\pi j \frac{l-1}{L} n \right) \\ \text{IDFT}(L, \mathbf{X})_n &= \frac{1}{\sqrt{L}} \sum_{l=1}^L X_l \exp \left( 2\pi j \frac{l-1}{L} n \right), \end{aligned} \quad (1.31)$$

where  $\mathbf{X} = \{X_1, X_2, \dots, X_L\}$  is a complex-valued input vector. With the definition of IDFT, Eq. (1.30) is rewritten as

$$s_n = \sqrt{J} \text{IDFT}(JK, \mathbf{A}'), \quad (1.32)$$

where

$$\mathbf{A}' = \left\{ A_1, A_2, \dots, A_K, \underbrace{0, 0, \dots, 0}_{(J-1)K \text{ } 0's} \right\}. \quad (1.33)$$

Thus, the vector  $\mathbf{A}'$  is obtained from the symbols  $A_k$  with a zero-padding technique.

At the receiver side, the signal  $r_n$  is received for  $n = 0, 1, \dots, JK - 1$  and the estimated symbols  $\hat{\mathbf{A}}'$  are obtained as

$$\hat{A}'_k = \text{DFT}(JK, \{r_n\})_k. \quad (1.34)$$

In the above equation,  $\hat{A}'_k$  is equivalent to  $A_k$  for  $k = 1, 2, \dots, K$  if  $r_n = s_n$ . In general, the received signal  $r_n$  is not equivalent to  $s_n$  since there is a channel noise. Then, the symbols  $\hat{\mathbf{A}}'$  are decomposed as

$$\hat{\mathbf{A}}' = \{ \underbrace{\hat{A}'_1, \hat{A}'_2, \dots, \hat{A}'_K}_{\text{estimated symbols}}, \underbrace{\hat{A}'_{K+1}, \hat{A}'_{K+2}, \dots, \hat{A}'_{JK}}_{\text{out of bands}} \}. \quad (1.35)$$

Thus, the first  $K$  symbols of  $\hat{\mathbf{A}}'$  are chosen as estimated symbols and the other symbols are ignored. From the above discussions, the estimated symbols  $\hat{\mathbf{A}}$  are written as

$$\hat{\mathbf{A}} = \{ \hat{A}'_1, \hat{A}'_2, \dots, \hat{A}'_K \}. \quad (1.36)$$

With the above definitions, we derive SNDR formula in OFDM systems. In what follows, we make the following assumptions.

- There is no fading.
- Each transmitted symbol is independently chosen from the same set.

In fading channels, SNDR has been discussed in [6]. Here, only the effects of amplifier is discussed here.

From assumption 2 and Central Limit Theorem, the base-band signal  $s(t)$  can be regarded as a Gaussian process. Note that the independence of transmitted symbols is not necessary for regarding  $s(t)$  as a Gaussian process. In general,

it has been known that the base-band signal can be regarded as a Gaussian process if transmitted symbols are uncorrelated and satisfy some conditions [26]. Further, it is known that most of good error correction codes give uncorrelated symbols [26]. Therefore, following discussions can be applied to most cases. Since the base-band signal can be regarded as a Gaussian process, each sample of the signal,  $s(nT/(JK))$ , follows Gaussian. Then, from the Bussgang's theorem, which is the special case of Price's theorem [27], output signal with the amplifier  $f$  is written as

$$f(s(nT/(JK))) = \alpha s(nT/(JK)) + d_n, \quad (1.37)$$

where the distortion term  $d_n$  is uncorrelated with  $s(nT/(JK))$  and  $\alpha$  is a coefficient which is written as

$$\alpha = \frac{\mathbb{E}\{f(s(nT/(JK)))s(nT/(JK))^*\}}{\mathbb{E}\{|s(nT/(JK))|^2\}}. \quad (1.38)$$

Note that the denominator of the right hand side of Eq. (1.38) is written as  $\mathbb{E}\{|s(nT/(JK))|^2\} = 1$  since the envelope of the signal  $s(nT/(JK))$  follows the Rayleigh distribution. Thus, Eq. (1.38) is rewritten as

$$\alpha = \mathbb{E}\{f(s(nT/(JK)))s(nT/(JK))^*\}. \quad (1.39)$$

If the amplifier  $f$  is a soft limiter, which has been defined in Eq. (1.23),  $\alpha$  is calculated as

$$\alpha = 1 - \exp(-\rho^2) + \frac{\sqrt{\pi}\rho}{2} \operatorname{erfc}(\rho), \quad (1.40)$$

where  $\rho$  is the parameter defined in Eq. (1.23) and  $\operatorname{erfc}(\cdot)$  is the error function, which is written as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2) dt. \quad (1.41)$$

As seen in the above discussions, the effect of amplifier can be calculated when the base-band signal  $s(t)$  is regarded as a Gaussian process.

Let  $r_n$  be the received signal which is written as

$$r_n = f(s(nT/(JK))) + w_n, \quad (1.42)$$

where  $w_n$  denotes Gaussian noise whose mean and variance are 0 and  $N_0$ , respectively. From Eqs. (1.34) (1.38), the  $k$ -th estimated symbol is written as

$$\begin{aligned} \hat{A}'_k &= \text{DFT}(JK, \{r_n\})_k \\ &= \alpha \underbrace{\frac{1}{\sqrt{JK}} \sum_{n=0}^{JK-1} s_n \exp\left(-2\pi j \frac{k-1}{JK} n\right)}_{A_k} \\ &\quad + \underbrace{\frac{1}{\sqrt{JK}} \sum_{n=0}^{JK-1} d_n \exp\left(-2\pi j \frac{k-1}{JK} n\right)}_{D_k} \\ &\quad + \underbrace{\frac{1}{\sqrt{JK}} \sum_{n=0}^{JK-1} w_n \exp\left(-2\pi j \frac{k-1}{JK} n\right)}_{W_k}. \end{aligned} \quad (1.43)$$

Thus, SNDR with the  $k$ -th carrier is written as

$$\text{SNDR}_k = \frac{\alpha^2 \text{E}\{|A_k|^2\}}{\text{E}\{|D_k|^2\} + \text{E}\{|W_k|^2\}}. \quad (1.44)$$

In the above equation, we have used the assumption that  $D_n$  is independent of  $W_n$ . Note that the terms relating to the amplifier  $f$  are  $\alpha$  and  $D_n$ . It is obvious that SNDR is  $\text{E}\{|A_k|^2\} / \text{E}\{|W_n|^2\}$  if the amplifier is ideal.

## Chapter 2

# Correlation, Eigenvalue and Eigenvector

### I Introduction

In this chapter, we derive an expression of correlation in quadratic forms. In communication systems, correlation is used in synchronization [12]. In CDMA systems, correlation is used to recover symbols. Further, in OFDM systems, correlation terms appear in analysis of PAPR [28]. Therefore, correlation plays important roles in analyzing communication systems.

In wireless communication systems, there are two kinds of correlations, a periodic correlation and an odd periodic correlation. These two are written in terms of aperiodic correlation terms. It is known that SNR in asynchronous CDMA systems can be written in terms of aperiodic correlation terms [13]. Thus, to analyze communication systems, it is necessary to analyze correlation terms.

In this chapter, we give a technique to analyze correlation, which is based on the works [29] [30]. As seen in Chapter 4, with this technique, the terms of Signal to Noise Ratio in asynchronous CDMA systems can be written in terms of quadratic forms. From such a expression, we can consider problems in communication systems from another perspective.

### II Types of Correlations

In this section, we define three kinds of correlations, an aperiodic correlation, a periodic correlation, and an odd periodic correlation. These correlations appear not only in analyzing of asynchronous CDMA systems but also in one of OFDM systems.

To define these three kinds of correlations, we define the vectors  $\mathbf{x}_i, \mathbf{x}_k \in \mathbb{C}^n$ . These vectors  $\mathbf{x}_i$  and  $\mathbf{x}_k$  are written as

$$\begin{aligned}\mathbf{x}_i &= (x_{i,1}, x_{i,2}, \dots, x_{i,n})^\top, \\ \mathbf{x}_k &= (x_{k,1}, x_{k,2}, \dots, x_{k,n})^\top,\end{aligned}\tag{2.1}$$



where  $\mathbf{z}^\top$  is the transpose of  $\mathbf{z}$ . With this definition, an aperiodic correlation is defined as

$$C_{i,k}(l) = \begin{cases} \sum_{m=1}^{n-l} \overline{x_{i,m+l}} x_{k,m} & 0 \leq l \leq n-1, \\ \sum_{m=1}^{n+l} \overline{x_{i,m}} x_{k,m-l} & 1-n \leq l < 0, \\ 0 & |l| \geq n \end{cases} \quad (2.2)$$

where  $\bar{z}$  is the conjugate of  $z$ . With the aperiodic correlation, a periodic correlation  $\theta_{i,k}(l)$  and an odd periodic correlation  $\hat{\theta}_{i,k}(l)$  are defined as

$$\begin{aligned} \theta_{i,k}(l) &= C_{i,k}(l) + C_{i,k}(l-N), \\ \hat{\theta}_{i,k}(l) &= C_{i,k}(l) - C_{i,k}(l-N). \end{aligned} \quad (2.3)$$

The above two kinds of correlations, a periodic correlation and an odd periodic correlation appear explicitly in analysis of CDMA systems [13].

### III Eigenvalue Decomposition of Correlation Matrix

In this section, the expressions of the correlations which have been discussed in the previous section are derived. These expressions derived in this section are used in Chapters 4, 5, and 6.

#### III.I Periodic Correlation

First, we derive the expression of a periodic correlation. From Eq. (2.3), a periodic correlation  $\theta_{i,k}(l)$  is rewritten as

$$\theta_{i,k}(l) = \mathbf{z}^* B(l) \mathbf{x}, \quad (2.4)$$

where  $\mathbf{z}^*$  is the conjugate transpose of  $\mathbf{z}$ ,  $B(l)$  is a matrix written as

$$B(l) = \begin{pmatrix} O & E_l \\ E_{n-l} & O \end{pmatrix} \quad (2.5)$$

and  $E_l$  is the identity matrix whose size is  $l$ . Since the matrix  $B(l)$  is unitary,  $B(l)$  can be decomposed with an eigenvector decomposition. This decomposition is written as

$$B(l) \mathbf{v}_k^{(l)} = \lambda_k^{(l)} \mathbf{v}_k^{(l)}, \quad (2.6)$$

where  $\lambda_k^{(l)}$  and  $\mathbf{v}_k^{(l)}$  are the  $k$ -th eigenvalue and the corresponding eigenvector of the matrix  $B(l)$ , respectively.

For the decomposition, let us investigate the property of the matrix  $B(l)$ . From Eq. (2.5), the matrix  $B(l)$  satisfies the following relation

$$B(l) = B(1)B(l-1). \quad (2.7)$$

Thus, from Eq. (2.6), the eigenvalues  $\lambda_k^{(l)}$  and the eigenvectors  $\mathbf{v}_k^{(l)}$  satisfy the following relations

$$\begin{aligned}\lambda_k^{(l)} \mathbf{v}_k^{(l)} &= B(l) \mathbf{v}_k^{(l)} \\ &= B(1) B(l-1) \mathbf{v}_k^{(l)} \\ &= B(1)^l \mathbf{v}_k^{(l)} \\ &= (\lambda_k^{(1)})^l \mathbf{v}_k^{(l)}.\end{aligned}\tag{2.8}$$

Therefore, we obtain  $\lambda_k^{(l)} = (\lambda_k^{(1)})^l$ . We merely write  $\lambda_k = \lambda_k^{(1)}$  since the eigenvalue  $\lambda_k^{(l)}$  is written as the product of  $\lambda_k^{(1)}$ . Further, from Eq. (2.8), the matrices  $B(l)$  and  $B(1)$  have the same eigenvectors. From this discussion, we also merely write  $\mathbf{v}_k = \mathbf{v}_k^{(1)}$ .

Since the matrix  $B(n)$  is identity, the eigenvalue  $\lambda_k$  is written as

$$\lambda_k = \exp\left(-2\pi j \frac{k}{n}\right),\tag{2.9}$$

where  $j$  is the unit imaginary number. Thus, the eigenvector  $\mathbf{v}_k$  is written as

$$\mathbf{v}_k = \frac{1}{\sqrt{n}} \left(1, \exp\left(2\pi j \frac{k}{n}\right), \dots, \exp\left(2\pi j \frac{k(n-1)}{n}\right)\right)^\top.\tag{2.10}$$

From the above results, the eigenvalues and eigenvectors of the matrix  $(B(l))$  are written as

$$\begin{aligned}\lambda_k^{(l)} &= \exp\left(-2\pi j \frac{kl}{n}\right) \\ \mathbf{v}_k &= \frac{1}{\sqrt{n}} \left(1, \exp\left(2\pi j \frac{k}{n}\right), \dots, \exp\left(2\pi j (n-1) \frac{k}{n}\right)\right)^\top.\end{aligned}\tag{2.11}$$

With the above results, the matrix  $B(l)$  is decomposed as

$$B(l) = V \Lambda^{(l)} V^*,\tag{2.12}$$

where  $\Lambda^{(l)} = \text{diag}(\lambda_1^{(l)}, \lambda_2^{(l)}, \dots, \lambda_n^{(l)})$ ,  $\text{diag}(\mathbf{z})$  is the diagonal matrix whose diagonal components are  $\mathbf{z}$  and  $V = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ .

### III.II Odd Periodic Correlation

Finally, we derive the expression of a odd periodic correlation. From Eq. (2.3), a odd periodic correlation  $\hat{\theta}_{i,k}(l)$  is rewritten as

$$\hat{\theta}_{i,k}(l) = \mathbf{x}^* \hat{B}(l) \mathbf{x},\tag{2.13}$$

where

$$\hat{B}(l) = \begin{pmatrix} O & -E_l \\ E_{n-l} & O \end{pmatrix}.\tag{2.14}$$

Similar to the periodic correlation, the matrices  $\hat{B}(l)$  satisfy

$$\hat{B}(l) = \hat{B}(1) \hat{B}(l-1).\tag{2.15}$$

Let  $\hat{\lambda}_k^{(l)}$  and  $\hat{\mathbf{v}}_k^{(l)}$  be the  $k$ -th eigenvalue and the corresponding eigenvector, respectively. These eigenvalues and eigenvectors satisfy

$$\hat{B}(l)\hat{\mathbf{v}}_k^{(l)} = \hat{\lambda}_k^{(l)}\hat{\mathbf{v}}_k^{(l)}, \quad (2.16)$$

Similar to Eq. (2.8), we obtain the following relation

$$\begin{aligned} \hat{\lambda}_k^{(l)}\mathbf{v}_k^{(l)} &= \hat{B}(l)\hat{\mathbf{v}}_k^{(l)} \\ &= \left(\hat{\lambda}_k^{(1)}\right)^l \hat{\mathbf{v}}_k^{(l)}. \end{aligned} \quad (2.17)$$

Since the matrices  $B(l)$  and  $B(1)$  have the same eigenvectors and the eigenvalue  $\hat{\lambda}_k^{(l)}$  is written as  $\left(\hat{\lambda}_k^{(1)}\right)^l$ , we merely write  $\hat{\lambda}_k = \hat{\lambda}_k^{(1)}$  and  $\mathbf{v}_k = \mathbf{v}_k^{(1)}$ .

Since the matrix  $B(n) = -E_n$ , the eigenvalue  $\lambda_k$  is written as

$$\hat{\lambda}_k = \exp\left(-2\pi j\left(\frac{k}{n} + \frac{1}{2n}\right)\right), \quad (2.18)$$

Thus, the eigenvalue  $\lambda_k^{(l)}$  and the eigenvector  $\hat{\mathbf{v}}_k$  are written as

$$\begin{aligned} \hat{\lambda}_k^{(l)} &= \exp\left(-2\pi j l \left(\frac{k}{n} + \frac{1}{2n}\right)\right) \\ \hat{\mathbf{v}}_k &= \frac{1}{\sqrt{n}} \left(1, \exp\left(2\pi j \left(\frac{k}{n} + \frac{1}{2n}\right)\right), \dots, \exp\left(2\pi j(n-1) \left(\frac{k}{n} + \frac{1}{2n}\right)\right)\right)^\top. \end{aligned} \quad (2.19)$$

From the above results, the matrix  $\hat{B}(l)$  is decomposed as

$$\hat{B}(l) = \hat{V}\hat{\Lambda}^{(l)}\hat{V}^*, \quad (2.20)$$

where  $\hat{\Lambda}^{(l)} = \text{diag}(\hat{\lambda}_1^{(l)}, \hat{\lambda}_2^{(l)}, \dots, \hat{\lambda}_n^{(l)})$  and  $\hat{V} = (\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \dots, \hat{\mathbf{v}}_n)$ .

## IV Correlation Formula with Eigenvalues

In the previous section, we have seen that matrices which appear in a periodic correlation and an odd periodic correlation can be decomposed with an eigenvalue decomposition technique. In this section, we derive formulas of correlations with eigenvalues.

Let us define the vectors  $\boldsymbol{\alpha}_k = V^*\mathbf{x}_k$  and  $\boldsymbol{\beta}_k = \hat{V}^*\mathbf{x}_k$ , where the vector  $\mathbf{x}_k$  has been defined in Section I. With the above expressions, we obtain the expressions of a periodic correlation and an odd periodic correlation as

$$\begin{aligned} \theta_{i,k}(l) &= \sum_{m=1}^n \exp\left(2\pi j \frac{ml}{n}\right) \bar{\alpha}_{i,m} \alpha_{k,m} \\ \hat{\theta}_{i,k}(l) &= \sum_{m=1}^n \exp\left(2\pi j m \left(\frac{l}{n} + \frac{1}{2n}\right)\right) \bar{\beta}_{i,m} \beta_{k,m}, \end{aligned} \quad (2.21)$$

where  $\alpha_{k,m}$  and  $\beta_{k,m}$  are the  $m$ -th component of  $\boldsymbol{\alpha}_k$  and  $\boldsymbol{\beta}_k$ , respectively. From Eq. (2.21), a periodic correlation and an odd periodic correlation are written as the sum of the product of two terms. One is the term which depends on  $l$  and the other is the term which is independent of  $l$ .

## V Summary

In this chapter, we have derived the formulas of a periodic correlation and an odd periodic correlation. These formulas play an important role in analysis of CDMA and OFDM systems. In asynchronous CDMA systems, these formula appear in SINR formula. In OFDM systems, these formulas appear in the bound of Peak-to-Power Ratio (PAPR) (see Chapter 5 and 6)[28].

## Chapter 3

# Sequence in Weyl Class to Reduce Interference Noise

### I Introduction

As seen in Chapter 1, SNR plays important roles in communication systems. In CDMA systems, instead of SNR, Signal to Interference plus Noise Ratio (SINR) is an important index for wireless communication systems since there is a interference term in CDMA systems. The difference between SNR and SINR is that SINR has interference noise term and channel noise term and that SNR has only channel noise term. Similar to the Gaussian channel, which is discussed Chapter 1, SINR is also the most significant parameter in achieving high capacity in CDMA systems [1]. In general, it is necessary and sufficient for achieving high capacity to increase SINR under the condition that the width of the frequency band is constant when the inter-symbol interference is approximated as Additive White Gaussian Noise (AWGN) [31]. Similarly, the performance of wireless communication is evaluated in Bit Error Rate (BER). However, these two are not independent, and it is known that BER decreases as SINR increases since inter-symbol interference is the most important factor relating to the system performance in CDMA systems.

In this chapter, we focus on a code division multiple access (CDMA) system [32], in particular, an asynchronous CDMA system. It is one of the multiple access systems with which many people can communicate each other at the same time [33]. In CDMA systems, spreading sequences are utilized as codes to multiplex. Each user is assigned a different code and uses it to modulate and demodulate his signal.

To increase SINR, many methods have been proposed. One such method is based on the blind multi-user detection [34]. On the other hand, improving the receiver [35] with the application of digital implementation of ICA [36] and Maximum Likelihood (ML) estimation [37] is also efficient. However, in particular, ML estimation method needs a large amount of computation.

In uplink of W-CDMA systems, the current spreading sequence is the Gold code [38]. It is known [29] that the Gold code is optimal in all the binary spreading sequences as well as the Kasami sequence [39] in a sense of the maximum value among all periodic autocorrelation and periodic cross-correlation. To ex-

plore a better sequence for asynchronous CDMA systems, in [40] and [41], the use of chaotic spreading sequences has been proposed. These chaotic spreading sequences are multivalued sequences, not binary ones, and are obtained from chaotic maps. Examples of such spreading sequences have been given in [41] [42] [43] [44] [45]. In [46], the approach to obtain the capacity of spreading sequences has been proposed.

In [47], Sarwate has shown two kinds of characterized sequences on his limitation. One kind is a set of sequences whose periodic cross-correlation is always zero. We call them Sarwate sequences. The other kind is a set of sequences whose periodic autocorrelation is always zero except for only one point, that is, Frank-Zadoff-Chu (FZC) sequences [48] [49]. In [50], the extended set of the FZC sequences, the Oppermann sequences are proposed. They have three parameters and their SINR, autocorrelation and cross-correlation have been investigated.

In this chapter, we define the Weyl sequence class, which is a set of sequences generated with the Weyl transformation [51]. This class is similar to the set of Oppermann sequences and includes the Sarwate sequences. Sequence in the Weyl sequence class have a desired property that the order of cross-correlation is low. We evaluate the upper bound of cross-correlation and construct the optimization problem: minimize the upper bound of cross-correlation. From the problem, we derive optimal spreading sequences in the Weyl sequence class. We show their SINR in a special case and compare them with other sequences in a sense of BER.

## II Weyl Sequence Class

In this section, we define the Weyl sequence class and show their properties. Let  $N$  be the length of spreading sequences. We define the Weyl sequence  $(x_n)$  as the following formula [51]

$$x_n = (n\rho + \Delta) \bmod 1 \quad (n = 1, 2, \dots, N), \quad (3.1)$$

where  $\rho$  and  $\Delta$  are real parameters. From the above definition, we can assume that the parameters  $\rho$  and  $\Delta$  satisfy  $0 \leq \Delta < 1$  and  $0 \leq \rho < 1$ . The sequences whose  $\rho$  is an irrational number are used in a Quasi-Monte Carlo method [52]. We apply this sequence to a spreading sequence. Then, the Weyl spreading sequence  $(w_{k,n})$  is defined as [53]

$$\begin{aligned} x_{k,n} &= (n\rho_k + \Delta_k) \bmod 1, \\ w_{k,n} &= \exp(2\pi j x_{k,n}) \quad (n = 1, 2, \dots, N), \end{aligned} \quad (3.2)$$

where  $k$  is the number of the user,  $j$  is the unit imaginary number, and  $\rho_k$  is a real-valued initial point assigned to the user  $k$ . In CDMA systems, the value of  $\Delta_k$  has no effects to Signal to Interference plus Noise Ratio (SINR) since  $\exp(2\pi j \Delta_k)$  is united to the phase term of the signal. Thus, we set  $\Delta_k = 0$ . We call the class which consists of Weyl spreading sequences the Weyl sequence class. Note that this class is similar to the set of Oppermann sequences [50]. The  $n$ -th element of the Oppermann sequences is defined as

$$u_{k,n} = (-1)^{nM_k} \exp\left(\frac{j\pi(M_k^p n^q + n^r)}{N}\right) \quad (n = 1, 2, \dots, N), \quad (3.3)$$

where  $M_k$  is an integer that is relatively prime to  $N$  such that  $1 \leq M_k < N$  and  $p, q$  and  $r$  are any real numbers. The triple  $\{p, q, r\}$  specifies the set of sequences. When the triple  $\{p, q, r\}$  is  $\{2, 1, -\infty\}$ , we obtain the element of the FZC sequence [48] [49]

$$u_{k,n} = (-1)^{nM_k} \exp\left(\frac{j\pi(M_k^2 n)}{N}\right). \quad (3.4)$$

The Weyl sequence class is obtained when the triple is  $\{1, 1, -\infty\}$  and  $M_k = \rho_k \cdot 2N/(N+1)$ . The number  $-1$  is written as  $\exp(j\pi)$ . Substituting  $\{p, q, r\} = \{1, 1, -\infty\}$  and  $M_k = \rho_k \cdot 2N/(N+1)$  into Eq. (3.3), we have

$$\begin{aligned} u_{k,n} &= \exp(j\pi)^{nM_k} \exp\left(\frac{j\pi(M_k^p n^q + n^r)}{N}\right) \\ &= \exp\left(2\pi j n \frac{N}{N+1} \rho_k\right) \exp\left(2\pi j n \frac{1}{N+1} \rho_k\right) \\ &= \exp(2\pi j n \rho_k). \end{aligned} \quad (3.5)$$

Note that  $M_k$  is not always an integer. Thus, the Weyl sequence class is similar to the set of Oppermann sequences.

The element of the Weyl sequence class,  $(w_{k,n})$  has a desired property that cross-correlation is low. We define the periodic correlation function  $\theta_{i,k}(l)$  and odd periodic correlation function  $\hat{\theta}_{i,k}(l)$  as

$$\theta_{i,k}(l) = C_{i,k}(l) + C_{i,k}(l-N), \quad (3.6)$$

$$\hat{\theta}_{i,k}(l) = C_{i,k}(l) - C_{i,k}(l-N), \quad (3.7)$$

where

$$C_{i,k}(l) = \begin{cases} \sum_{n=1}^{N-l} \overline{w_{i,n+l}} w_{k,n} & 0 \leq l \leq N-1, \\ \sum_{n=1}^{N+l} \overline{w_{i,n}} w_{k,n-l} & 1-N \leq l < 0, \\ 0 & |l| \geq N \end{cases} \quad (3.8)$$

and  $\bar{z}$  is the conjugate of  $z$ . The correlation functions  $\theta_{i,k}(l)$  and  $\hat{\theta}_{i,k}(l)$  have been studied in [54] [47] [29]. When  $i \neq k$ ,  $\theta_{i,k}(l)$  and  $\hat{\theta}_{i,k}(l)$  are the periodic function and the odd periodic cross-correlation function, respectively. It is necessary for achieving high SINR to keep the value of the cross-correlation functions,  $|\theta_{i,k}(l)|$  and  $|\hat{\theta}_{i,k}(l)|$  low for all  $0 \leq l < N$ . The absolute values of cross-correlation functions  $|\theta_{i,k}(l)|$  and  $|\hat{\theta}_{i,k}(l)|$  have the common upper bound that

$$|\theta_{i,k}(l)| \leq |C_{i,k}(l)| + |C_{i,k}(l-N)|, \quad (3.9)$$

$$|\hat{\theta}_{i,k}(l)| \leq |C_{i,k}(l)| + |C_{i,k}(l-N)|. \quad (3.10)$$

With the sequences in the Weyl sequence class, we evaluate the absolute value

of  $C_{i,k}(l)$  as

$$\begin{aligned}
|C_{i,k}(l)| &= \left| \frac{1 - \exp(2\pi j(N-l)(\rho_k - \rho_i))}{1 - \exp(2\pi j(\rho_k - \rho_i))} \right| \\
&= \sqrt{\frac{1 - \cos(2\pi(N-l)(\rho_k - \rho_i))}{1 - \cos(2\pi(\rho_k - \rho_i))}} \\
&= \left| \frac{\sin(\pi(N-l)(\rho_k - \rho_i))}{\sin(\pi(\rho_k - \rho_i))} \right| \\
&\leq \frac{1}{|\sin(\pi(\rho_k - \rho_i))|} \\
&= \frac{1}{|\sin(\pi(\rho_i - \rho_k))|}.
\end{aligned} \tag{3.11}$$

The equality is attained if and only if

$$(N-l)(\rho_k - \rho_i) = \frac{1}{2} + m, \tag{3.12}$$

where  $m \in \mathbb{Z}$ . From the above result,  $|C_{i,k}(l)|$  obeys

$$|C_{i,k}(l)| = O(1), \tag{3.13}$$

with  $O$  being an order function. Similarly,  $|C_{i,k}(l-N)|$  obeys  $O(1)$ . Thus, the common upper bound of  $|\theta_{i,k}(l)|$  and  $|\hat{\theta}_{i,k}(l)|$  is independent of  $N$ . For general spreading sequences, due to the central limit theorem (CLT), the cross-correlations  $|\theta_{i,k}(l)|$  and  $|\hat{\theta}_{i,k}(l)|$  become large as  $N$  becomes large. For this reason, compared to general spreading sequences, the Weyl spreading sequence is expected to have low cross-correlation.

### III Optimal Spreading Sequence in Weyl Sequence Class

In this section, we consider an asynchronous binary phase shift keying (BPSK) CDMA system. Our goal is to derive the spreading sequences whose inter-symbol interference is the smallest in the Weyl sequence class. Let  $K$ ,  $T$  and  $T_c$  be the number of users, the duration of the symbol and each chip, respectively. In this situation, the user  $i$  despreads the spreading sequences  $(w_{k,n})$  with the spreading sequence of the user  $i$ ,  $(w_{i,n})$ . The symbols  $b_{k,-1}, b_{k,0} \in \{-1, 1\}$  denote bits which the user  $k$  send. The transmitted signal of the user  $k$  has time delay  $\tau_k$ . From [13], we assume that time delay  $\tau_k$  is distributed in  $[0, T)$  and satisfies  $l_k T_c \leq \tau_k \leq (l_k + 1)T_c$ , where  $l_k \in \{0, 1, \dots, N-1\}$  is an integer. Then, the inter-symbol interference between the user  $i$  and the user  $k$ ,  $I_{i,k}(\tau_k)$  is obtained as

$$\begin{aligned}
I_{i,k}(\tau_k) &= \exp(j\phi_k) [( \tau_k - l_k T_c ) ( b_{k,-1} C_{i,k}(l_k) \\
&\quad + b_{k,0} C_{i,k}(l_k - N) ) + ((l_k + 1)T_c - \tau_k) \\
&\quad \cdot \{ b_{k,-1} C_{i,k}(l_k + 1) + b_{k,0} C_{i,k}(l_k + 1 - N) \} ],
\end{aligned} \tag{3.14}$$



where  $\phi_k \in [0, 2\pi)$  is the phase of user  $k$ 's carrier. With Eq. (3.11), the absolute value of the inter-symbol interference  $I_{i,k}(\tau_k)$  is evaluated as

$$\begin{aligned} |I_{i,k}(\tau_k)| &\leq (\tau_k - l_k T_c) \{|C_{i,k}(l_k)| + |C_{i,k}(l_k - N)|\} \\ &\quad + ((l_k + 1)T_c - \tau_k) \cdot \{|C_{i,k}(l_k + 1)| + |C_{i,k}(l_k + 1 - N)|\} \\ &\leq \frac{2T_c}{|\sin(\pi(\rho_i - \rho_k))|}. \end{aligned} \quad (3.15)$$

Thus, we have shown that the upper bound of inter-symbol interference between two sequences is inversely proportional to  $|\sin(\pi d(\rho_i - \rho_k))|$ . To reduce the inter-symbol interference  $I_{i,k}(\tau_k)$ , it is necessary to reduce  $2T_c/|\sin(\pi(\rho_i - \rho_k))|$ . To eliminate the absolute value function, we introduce the distance between the phases  $\rho_i$  and  $\rho_k$ . The distance  $d(\rho_i, \rho_k)$  we propose here is given by

$$d(\rho_i, \rho_k) = \min\{|\rho_i - \rho_k|, 1 - |\rho_i - \rho_k|\}. \quad (3.16)$$

Note that this  $d$  satisfies the axiom of distance, and

$$|\sin(\pi(\rho_i - \rho_k))| = \sin(\pi d(\rho_i, \rho_k)), \quad (3.17)$$

$$0 \leq d(\rho_i, \rho_k) \leq \frac{1}{2} \quad (3.18)$$

if we regard  $\rho = 1$  in the same light as  $\rho = 0$ . From Eq. (3.16), we rewrite Eq. (3.15) without any absolute value as

$$|I_{i,k}(\tau_k)| \leq \frac{2T_c}{\sin(\pi d(\rho_i, \rho_k))}. \quad (3.19)$$

We should take into account the whole inter-symbol interference in the users. The whole inter-symbol interference  $I$  is written as

$$I = \sum_{i=1}^K \sum_{\substack{k=1 \\ k \neq i}}^K I_{i,k}. \quad (3.20)$$

With Eq. (3.19), it is clear that  $|I|$  has the upper bound:

$$|I| \leq \sum_{i=1}^K \sum_{\substack{k=1 \\ k \neq i}}^K \frac{2T_c}{\sin(\pi d(\rho_i, \rho_k))}. \quad (3.21)$$

Thus, we minimize Eq. (3.21) and obtain the problem  $(\tilde{P})$

$$\begin{aligned} (\tilde{P}) \quad &\min \sum_{i=1}^K \sum_{\substack{k=1 \\ k \neq i}}^K \frac{1}{\sin(\pi d(\rho_i, \rho_k))} \\ &\text{subject to } \rho_k \in [0, 1) \quad (1 \leq k \leq K). \end{aligned} \quad (3.22)$$

This problem is equivalent to that we minimize the sum of the upper bound of  $C_{i,k}(l)$ . Thus, the cross-correlation among all the users is expected to be always low when we solve this problem. From Eq. (3.17), it is clear that

$d(\rho_i, \rho_k) = d(\rho_k, \rho_i)$ . Then, in the problem  $(\tilde{P})$ , we count two times the same distance. Thus, we obtain the equivalent problem  $(\tilde{P}')$

$$\begin{aligned} (\tilde{P}') \quad & \min \sum_{i < k} \frac{1}{\sin(\pi d(\rho_i, \rho_k))} \\ \text{subject to} \quad & \rho_k \in [0, 1) \quad (1 \leq k \leq K). \end{aligned} \quad (3.23)$$

It is not clear whether the objective function of the problem  $(\tilde{P}')$  is convex or not since the form of function  $d$  is complicated. To eliminate the function  $d$ , we introduce slack variables  $t_{i,k}$  for  $(P)$ . Then, the problem  $(\tilde{P}')$  is rewritten as

$$\begin{aligned} (\tilde{P}'') \quad & \min \sum_{i < k} \frac{1}{\sin(\pi t_{i,k})}, \\ \text{subject to} \quad & \rho_k \in [0, 1) \quad (1 \leq k \leq K), \\ & |\rho_i - \rho_k| \geq t_{i,k} \quad (i < k), \\ & 1 - |\rho_i - \rho_k| \geq t_{i,k} \quad (i < k), \\ & t_{i,k} \geq 0 \quad (i < k). \end{aligned} \quad (3.24)$$

Without loss of generality, we assume  $\rho_k \leq \rho_{k+1}$ . Then, the problem  $(\tilde{P}'')$  can be rewritten as

$$\begin{aligned} (P) \quad & \min \sum_{i < k} \frac{1}{\sin(\pi t_{i,k})}, \\ \text{subject to} \quad & \rho_k - \rho_i \geq t_{i,k} \quad (i < k), \\ & 1 - \rho_k + \rho_i \geq t_{i,k} \quad (i < k), \\ & \rho_{i+1} \geq \rho_i \quad (1 \leq i \leq K-1), \\ & \rho_1 \geq 0, \rho_K \leq 1, \\ & t_{i,k} \geq 0 \quad (i < k). \end{aligned} \quad (3.25)$$

Notice that the objective function and the inequality constraints of the problem  $(P)$  are convex. It has been known that convex programming can be solved with the KKT conditions [55]. To write such conditions, we define the variable vector  $\mathbf{x}$  as

$$\begin{aligned} \boldsymbol{\rho} &= (\rho_1, \rho_2, \dots, \rho_K)^T, \\ \mathbf{t} &= (t_{1,2}, t_{1,3}, \dots, t_{1,K}, t_{2,3}, \dots, t_{K-1,K})^T, \\ \mathbf{x} &= \begin{pmatrix} \boldsymbol{\rho} \\ \mathbf{t} \end{pmatrix}, \end{aligned} \quad (3.26)$$

where  $\boldsymbol{\rho} \in \mathbb{R}^K$ ,  $\mathbf{t} \in \mathbb{R}^{K(K-1)/2}$ ,  $\mathbf{x} \in \mathbb{R}^{K(K+1)/2}$  and  $\mathbf{z}^T$  is the transpose of  $\mathbf{z}$ . From the KKT conditions, the solution  $\mathbf{x}^*$  is a global solution of  $(P)$  if  $\mathbf{x}^*$  satisfies the following equation:

$$\begin{aligned} & \nabla f(\mathbf{x}^*) + \sum_{i < k} \lambda_{i,k} \nabla c_{i,k}(\mathbf{x}^*) + \sum_{i < k} \mu_{i,k} \nabla d_{i,k}(\mathbf{x}^*) \\ & + \sum_{i=1}^{K-1} \nu_i \nabla e_i(\mathbf{x}^*) + \xi_1 \nabla g_1(\mathbf{x}^*) \\ & + \xi_K \nabla g_K(\mathbf{x}^*) + \sum_{i < k} o_{i,k} \nabla h_{i,k}(\mathbf{x}^*) = \mathbf{0}, \end{aligned} \quad (3.27)$$

where

$$\begin{aligned}
f(\mathbf{x}) &= \sum_{i < k} \frac{1}{\sin(\pi t_{i,k})}, \\
c_{i,k}(\mathbf{x}) &= t_{i,k} + \rho_i - \rho_k, \\
d_{i,k}(\mathbf{x}) &= t_{i,k} - 1 - \rho_i + \rho_k, \\
e_i(\mathbf{x}) &= \rho_i - \rho_{i+1}, \\
g_1(\mathbf{x}) &= -x_1, \\
g_K(\mathbf{x}) &= x_K - 1, \\
h_{i,k}(\mathbf{x}) &= -t_{i,k},
\end{aligned} \tag{3.28}$$

and the Lagrange multipliers  $\lambda_{i,k}, \mu_{i,k}, \nu_i, \xi_1, \xi_K$  and  $o_{i,k}$  are non-negative real numbers. They have to satisfy the following conditions:

$$\begin{aligned}
c_{i,k}(\mathbf{x}) < 0 &\Rightarrow \lambda_{i,k} = 0, \\
d_{i,k}(\mathbf{x}) < 0 &\Rightarrow \mu_{i,k} = 0, \\
e_i(\mathbf{x}) < 0 &\Rightarrow \nu_i = 0, \\
g_1(\mathbf{x}) < 0 &\Rightarrow \xi_1 = 0, \\
g_K(\mathbf{x}) < 0 &\Rightarrow \xi_K = 0, \\
h_{i,k}(\mathbf{x}) < 0 &\Rightarrow o_{i,k} = 0.
\end{aligned} \tag{3.29}$$

In the appendix A, we prove that the global optimal solutions  $\rho_i^*$  and  $t_{i,k}^*$  are given by

$$\begin{aligned}
\rho_i^* &= \gamma + \frac{i-1}{K} \quad (i = 1, 2, \dots, K), \\
t_{i,k}^* &= \min \left\{ \frac{|k-i|}{K}, 1 - \frac{|k-i|}{K} \right\},
\end{aligned} \tag{3.30}$$

where  $\gamma$  is a real number. From the above result, the optimal spreading sequence of the user  $k$ ,  $(\tilde{w}_{k,n})$  is given by

$$\tilde{w}_{k,n} = \exp \left( 2\pi j n \left( \gamma + \frac{k-1}{K} \right) \right) \tag{3.31}$$

for a real number  $\gamma$ . This is equivalent to the following spreading sequences:

$$\tilde{w}_{k,n} = \exp \left( 2\pi j n \left( \gamma + \frac{\sigma_k}{K} \right) \right), \tag{3.32}$$

where  $\sigma_k \in \{0, 1, 2, \dots, K-1\}$  which satisfies  $\sigma_k \neq \sigma_i$  when  $k \neq i$ . This sequence belongs to the Weyl sequence class. Therefore, similar to Eq. (3.5), this sequence is obtained from the triple  $\{1, 1, -\infty\}$  and  $M_k = 2N(\gamma + \sigma_k/K)/(N+1)$  in the Oppermann sequences. If  $N = 2(L+1)$ ,  $K = L+1$  and  $\gamma = \frac{1}{2}$ , where  $L$  is an even number, then our sequences are equivalent to the Song-Park (SP) sequences ( $\sigma_k = L+1$  is not used) [56]. If  $K = N$  and  $\gamma = 0$ , our sequences are equivalent to the Sarwate sequences [47].

## IV SINR with the Optimal Spreading Sequence

In the previous section, we fix the number of users  $K$  and derive optimal spreading sequences in the Weyl spreading sequence class. This spreading sequence

is expected to be useful when the number of the users  $K$  is fixed. However, the number of users in a channel changes as time passes. Thus, in this section, we fix the maximum number of users in a channel,  $K_{\max}$  and assign the  $\sigma_k \in \{0, 1, 2, \dots, K_{\max} - 1\}$  to  $K$  users.

The spreading sequences assigned to the user  $k$  is expressed as

$$\tilde{w}_{k,n}(K_{\max}, \gamma) = \exp \left( 2\pi j n \left( \gamma + \frac{\sigma_k}{K_{\max}} \right) \right) \quad (n = 1, 2, \dots, N). \quad (3.33)$$

Note that the optimal spreading sequence of the problem (P) is expressed as

$$\tilde{w}_{k,n}(K, \gamma) = \exp \left( 2\pi j n \left( \gamma + \frac{\sigma_k}{K} \right) \right) \quad (n = 1, 2, \dots, N). \quad (3.34)$$

In particular, when  $K_{\max} = N$  and  $\gamma = 0$ , this spreading sequence is equivalent to the Sarwate's sequence [47]. The Sarwate's sequence has the feature that the periodic cross-correlation function is always 0, that is,

$$\theta_{i,k}(l) = C_{i,k}(l) + C_{i,k}(l - N) = 0 \quad (3.35)$$

for all  $l$  and  $k \neq i$ .

When we set  $K_{\max} = N$ ,  $\theta_{i,k}(0) = 0$  for all the  $\gamma$  and  $k \neq i$ . Thus, the sequence  $\tilde{w}_{k,n}(N, \gamma)$  is the WBE sequence since the orthogonal condition is satisfied. From the above reason, we define  $K_{\max}$  as  $N$ , that is, we consider the following sequences

$$\tilde{w}_{k,n}(N, \gamma) = \exp \left( 2\pi j n \left( \gamma + \frac{\sigma_k}{N} \right) \right) \quad (n = 1, 2, \dots, N). \quad (3.36)$$

In this section, we assume that  $\sigma_k$  is a random variable and is uniformly distributed in  $\{0, 1, 2, \dots, N - 1\}$ . In the next section, we consider how to assign  $\sigma_k$  in a systematic approach.

The expression of SINR of the user  $i$  is obtained in [13] [45] as

$$\text{SINR}_i = \left\{ (6N^3)^{-1} \sum_{\substack{k=1 \\ k \neq i}}^K r_{i,k} + \frac{N_0}{2E} \right\}^{-1/2}, \quad (3.37)$$

where

$$\begin{aligned} r_{i,k} = & \sum_{l=0}^{N-1} \{ |C_{i,k}(l - N)|^2 + \text{Re}[C_{i,k}(l - N) \overline{C_{i,k}(l - N - 1)}] \\ & + |C_{i,k}(l - N + 1)|^2 + |C_{i,k}(l)|^2 \\ & + \text{Re}[C_{i,k}(l) \overline{C_{i,k}(l + 1)}] + |C_{i,k}(l + 1)|^2 \}, \end{aligned} \quad (3.38)$$

$E$  is the energy per data bit and  $N_0$  is the power of Gaussian noise. SINR is the ratio between the variance of a desired signal and the one of a noise signal. In the appendix B, with the spreading sequence  $\{\tilde{w}_{k,n}(N, \gamma)\}$ , we prove that SINR of the user  $i$  is given by

$$\text{SINR}_i = \left\{ R_i + \frac{N_0}{2E} \right\}^{-1/2}, \quad (3.39)$$

where

$$R_i = \frac{(K-1)}{18N^2} \left\{ 2(N+1) + (N-2) \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right) \right\}. \quad (3.40)$$

Equation (3.39) is obtained when the ratio  $K/N$  is close to 1, that is, the number of users  $K$  is sufficiently large. From Eqs. (3.39) and (3.40), the spreading sequence  $\{\tilde{w}_{i,n}(N, \gamma)\}$  has different SINR in  $\sigma_i$ . Thus, some users have high SINR and other users have low SINR. The lower bound of  $\text{SINR}_i$  is

$$\underline{\text{SINR}}_i = \left\{ \frac{K-1}{6N} + \frac{N_0}{2E} \right\}^{-\frac{1}{2}}. \quad (3.41)$$

## V How to Assign $\sigma_k$

In this section, we consider how to assign  $\sigma_k$  to the each users. Let us consider the spreading sequences

$$\tilde{w}_{k,n}(N, \gamma) = \exp \left( 2\pi j n \left( \gamma + \frac{\sigma_k}{N} \right) \right),$$

where  $\sigma_k \in \{0, 1, \dots, N-1\}$ . From Eq. (3.30), it is demanded that we assign  $\sigma_k$  at regular interval. However, these sequences cannot be used if the number of users changes. Thus, we have to make the rule to assign  $\sigma_k$  when the number of users changes.

We give the rule to assign  $\sigma_k$  in the situation that the number of users monotonic increases. From the demand of the problem (P), it is desirable that we assign  $\sigma_k$  to each users at regular interval. Thus, it is appropriate to assign them at nearly regular interval in every number of users. We apply the Van der Corput sequence [57] to the method to assign  $\sigma_k$  since the sequence is a regular interval sequence in some situations. For example, the Van der Corput sequence  $(v_n)$  is obtained as

$$(v_n) = \left\{ 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \dots \right\}. \quad (3.42)$$

In particular, when we take the first eight elements out from  $(v_n)$  and sort them, we obtain the sequence

$$\left\{ 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8} \right\}. \quad (3.43)$$

This sequence is a regular interval sequence. We can consider  $(v_n)$  as a nearly regular interval sequence.

When the length of spreading sequences  $N$  equals  $2^m$ , where  $m > 1$  is an integer,  $(v_n)$  is rewritten in terms of  $1/N$ . For example, when  $N = 16$ , the sequence  $(v_n)$  is obtained as

$$(v_n) = \left\{ 0, \frac{8}{16}, \frac{4}{16}, \frac{12}{16}, \frac{2}{16}, \frac{10}{16}, \frac{6}{16}, \frac{14}{16}, \frac{1}{16}, \dots \right\}. \quad (3.44)$$

Thus, we propose that we use the  $k$ -th element of  $(v_n)$  as  $\sigma_k/N$ , that is, the spreading sequences are expressed as

$$\tilde{w}_{k,n}(N) = \exp (2\pi j n (\gamma + v_k)), \quad (3.45)$$

where  $v_k$  is the  $k$ -th element of  $(v_n)$ .

## VI Simulation Results

In this section, we simulate an asynchronous CDMA system and discuss the performance of the spreading sequences obtained by Eqs. (3.32) (3.36) (3.45). We use two parameters  $\gamma = 1/(2N)$  and  $\gamma = 1/(2K)$ . Choosing the parameter  $\gamma = 0$ , we have a spreading sequence whose elements are 1. This sequence is trivial. Therefore, to avoid such a sequence, we use the two cases for  $\gamma$ ,  $\gamma = \frac{1}{2N}$  and  $\gamma = \frac{1}{2K}$ . We focus on BPSK and QPSK systems with AWGN channel and no fading signals. In this simulation, we make the following assumptions about the receivers and the channel

1. the receiver has the perfect synchronization with the desired signal and no knowledge about the time delay of the other signals.
2. there are no fading effects.
3. the time delay  $\tau_k$ , the symbols  $b_{k,-1}$  and  $b_{k,0}$ , and the phase  $\theta_k$  are normally distributed in  $[0, T)$ ,  $\{-1, 1\}$ , and  $[0, 2\pi)$ , where  $T$  is the duration of each symbol.
4. the spreading sequences are uniformly and randomly chosen. With the Weyl spreading sequences, the parameter  $\sigma_k$  is uniformly and randomly chosen.
5. the matched filter is used in the correlation receiver.

The detail of transmitters and receivers are described in [42] [45] [13].

We measure the averaged BER

$$\text{BER} = \frac{1}{K} \frac{1}{U} \sum_{k=1}^K \sum_{u=1}^U \text{BER}_{k,u}, \quad (3.46)$$

where  $U$  is the trial numbers,  $u$  is the  $u$ -th trial number and  $\text{BER}_{k,u}$  is the BER of the user  $k$  at the  $u$ -th trial. This section consists of three subsections. In the first subsection, we compare the spreading sequences obtained by Eq. (3.32) and (3.36) with other sequences, the Gold codes [38], the optimal Chebyshev spreading sequences [42] and the Oppermann sequences [50]. In particular, we choose the triple  $\{p, q, r\} = \{1.0, 1.0, 1.275\}$  as the parameters of the Oppermann sequences. This triple is shown in [50] as the optimal parameters when  $N = 31$  with  $N$  being the length. In the second subsection, we compare the spreading sequences obtained by Eq. (3.32) and (3.36) with one obtained by Eq. (3.45). We compare the random assigning approach with the systematic approach. In the final subsection, we compare the spreading sequences obtained by Eq. (3.32) and (3.36) with the SP sequences [56].

### VI.I Comparison with Other Sequences

We consider the following spreading sequences:

$$\begin{aligned} \tilde{w}_{k,n}(N, \gamma) &= \exp \left( 2\pi j n \left( \gamma + \frac{\sigma_k}{N} \right) \right) \quad (n = 1, 2, \dots, N), \\ \tilde{w}_{k,n}(K, \gamma) &= \exp \left( 2\pi j n \left( \gamma + \frac{\sigma_k}{K} \right) \right) \quad (n = 1, 2, \dots, N). \end{aligned}$$

The former sequence is obtained from Eq. (3.36) and the latter sequence is obtained from Eq. (3.32). In this section, the “Weyl” spreading sequence is  $\{\tilde{w}_{k,n}(N, \gamma)\}$  and the “Optimal” one is  $\{\tilde{w}_{k,n}(K, \gamma)\}$ . Note that the optimal spreading sequences are different in the number of users  $K$ . The “Upper Bound” is obtained from Eq. (3.41).

Figure 3.1 shows the relation between the number of users and BER when  $N = 31$  and  $E_b/N_0 = 25$ [dB], where  $E_b$  is the energy per data bit. In this figure, we set  $\gamma = \frac{1}{2N}$ . The BER of the Weyl spreading sequences is lower than the one of the Gold codes and the optimal Chebyshev spreading sequences. However, it is higher than one of the Oppermann sequences. The upper bound is established when the number of the users  $K$  is larger than 20. On the other hands, the BER of the global solution of the problem (P), the Optimal Weyl sequences  $\{\tilde{w}_{k,n}(K, \gamma)\}$  have the lowest BER. These sequences are dramatically efficient when the number of the users  $K$  is fixed.

Figure 3.2 shows the relation between the  $E_b/N_0$ [dB] and BER in  $K = 7$ . In this figure, the BER of the Weyl spreading sequences  $\{\tilde{w}_{k,n}(N, \frac{1}{2K})\}$  is lower than one of the Oppermann sequences. This result shows that BER of the Weyl spreading sequences is changed when the value of  $\gamma$  is varied.

As a modulation technique, a Quadrature Phase-Shift Keying (QPSK) modulation is often used. In AWGN channels, the relation of BER between BPSK and QPSK systems is known with a given Signal to Noise Ratio (SNR) [12]. Figure 3.3 shows the relation between the  $E_b/N_0$ [dB] and BER in QPSK systems and  $K = 7$ . Similar to BPSK systems, the optimal sequences have the lowest BER in QPSK systems and the BER of the optimal sequences is independent of the parameter  $\gamma$ . Further, the BER of Weyl spreading sequences does not depend on the parameter  $\gamma$  while BER depends on  $\gamma$  in BPSK systems. This result implies that the distribution of the inter-symbol interference of the Weyl spreading sequences is not Gaussian. The reason for this is stated as follows. In BPSK and QPSK systems, the BER is known when the SINR is given and inter-symbol interference obeys Gaussian. As seen in Fig. 3.2, with the Weyl spreading sequences, the BER curves depend on  $\gamma$  in BPSK systems. Therefore, if the distribution of the inter-symbol interference of the Weyl spreading sequences is Gaussian, then the BER curves should depend on the parameter  $\gamma$  in QPSK systems. However, as seen in Fig 3.3, the BER curves are independent of the parameter  $\gamma$  in QPSK systems. Thus, we conclude that the distribution of inter-symbol interference is not Gaussian.

In Section III, we have shown that the optimal sequences have an arbitrary parameter  $\gamma$ . It is unknown whether the BER of the optimal sequences is independent of  $\gamma$  or not since we minimize the upper bound of inter-symbol interference. We numerically verify whether the BER of the optimal sequences is independent of  $\gamma$  or not. We choose the parameters  $N = 31$  and  $K = 7$  and a system is a BPSK system. Figure 3.4 shows the BER curves in various parameters  $\gamma = \{0, 1/(8K), \dots, 7/(8K)\}$ . As seen in Fig 3.4, each BER of the optimal sequences is the same. Therefore, we conclude that the BER of the optimal sequences is independent of the parameter  $\gamma$ .

## VI.II Comparison with Systematic Approaches

In Section V, we have discussed how to assign the element  $\sigma_k$  to the user  $k$  and proposed the method to assign. We set the length  $N = 32$  and the parameter

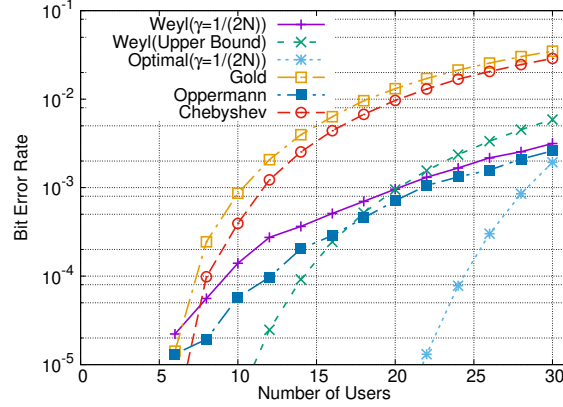


Figure 3.1: Comparison with other sequences in BPSK systems: relation between the number of users and BER:  $E_b/N_0 = 25[\text{dB}]$ ,  $N=31$

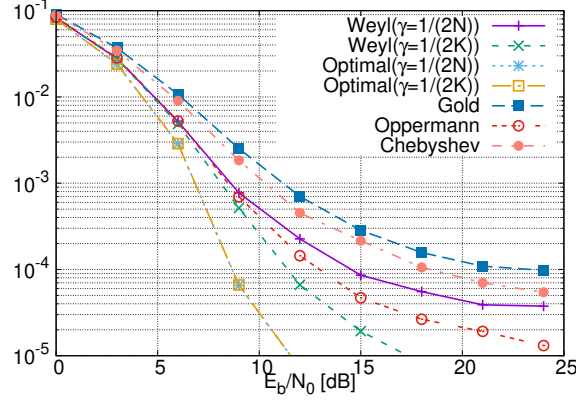


Figure 3.2: Comparison with other sequences in BPSK systems: relation between the  $E_b/N_0[\text{dB}]$  and BER:  $K = 7$ ,  $N=31$

$\gamma = \frac{1}{2N}$ . We compare two types of Weyl spreading sequences, whose  $\sigma_k$  is randomly assigned to user  $k$  and whose  $\frac{\sigma_k}{N}$  is orderly assigned as the  $k$ -th element of the Van der Corput sequence. Note that the first  $K$  elements of the Van der Corput sequence are used as  $\frac{\sigma_k}{N}$  when the number of the users is  $K$ . Figure 3.5 shows the relation between the number of users and BER when  $E_b/N_0 = 25[\text{dB}]$ . The BER of the spreading sequences obtained by Eq. (3.45) is lower than one of the spreading sequences to which we randomly assign  $\sigma_k$ . The BER curve of sequences obtained by Eq. (3.45) is not smooth. This result is caused by systematic assignment of  $\sigma_k$ . From Fig. 3.5, we conclude that the Weyl spreading sequences will have better performance if we successfully assign  $\sigma_k$ .

### VI.III Comparison with SP Sequence

The Song-Park (SP) sequences have been proposed in [56]. We set the length  $N = 30$  and the number of the users  $K = 7$ . Then, the maximum number of the users of the SP sequences is 14. Thus, we compare them with



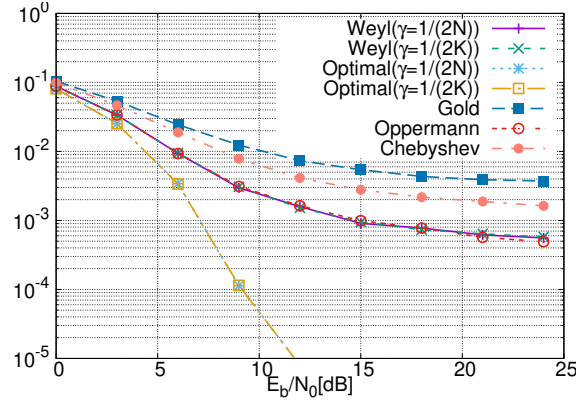


Figure 3.3: Comparison with other sequences in QPSK systems: relation between the  $E_b/N_0$ [dB] and BER:  $K = 7$ ,  $N=31$

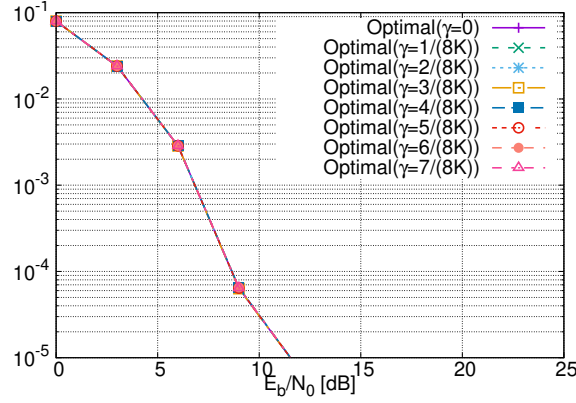


Figure 3.4: BER of optimal sequences with various  $\gamma$  in BPSK systems: relation between the  $E_b/N_0$ [dB] and BER:  $K = 7$ ,  $N=31$

four types of the Weyl spreading sequences. We choose the parameters as  $(K_{\max}, \gamma) = \{(30, 1/(2N)), (30, 1/(2K)), (14, 1/(2N)), (14, 1/(2K))\}$ . The parameter  $K_{\max} = 14$  represents the situation where the maximum number of the users is 14. Thus, the Weyl spreading sequences  $\{\tilde{w}_{k,n}(14, \gamma)\}$  have the same feature to the SP sequences. Figure 3.6 shows the relation between the  $E_b/N_0$ [dB] and BER. The BER of the Weyl spreading sequences whose  $K_{\max} = 30$  is higher than one of the SP sequences. However, the BER of the Weyl spreading sequences whose  $K_{\max} = 14$  and one of the SP sequences are the same. Further, the BER of the Weyl spreading sequences whose  $K_{\max} = 14$  is the same BER in different  $\gamma$ . These results could suggest that the optimal parameter  $\gamma$  will depend on  $N$ ,  $K$  and  $K_{\max}$ . The BER of the global solutions is lowest and each BER of them is the same in  $\gamma = \frac{1}{2N}$  and  $\frac{1}{2K}$ . This result corresponds to our conclusion that the BER of the optimal sequences is independent of  $\gamma$ .

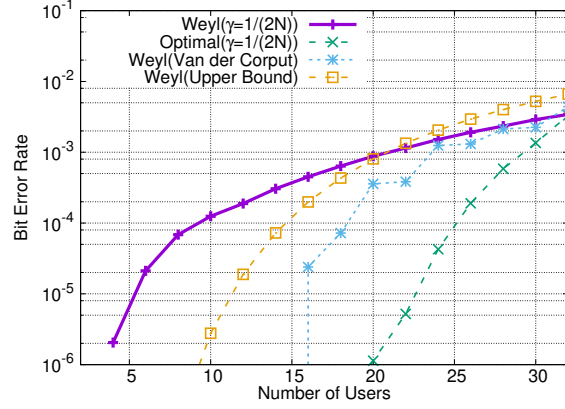


Figure 3.5: Comparison with systematic approaches in BPSK systems: relation between the number of the users and BER:  $N = 32$

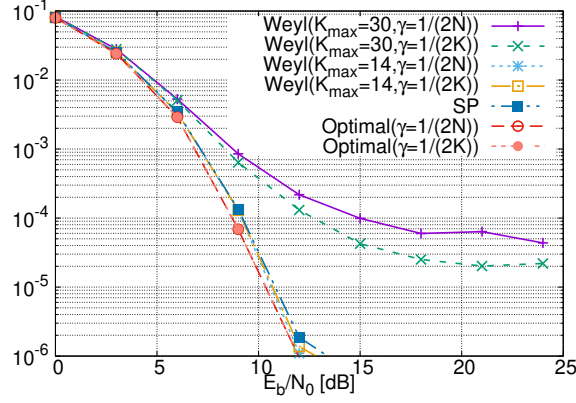


Figure 3.6: Comparison with SP sequence in BPSK systems: relation between the  $E_b/N_0$ [dB] and BER:  $K = 7$ ,  $N = 30$

## VII Summary and Discussions

In this chapter, we have defined the Weyl sequence class and shown the features of the sequences in the class. We have constructed the optimization problem: minimize the upper bound of the absolute value of the whole inter-symbol interference and derived the global solutions. From this solution, we can derive other sequences, the Sarwate's sequences, and the SP sequences. We also have evaluated their SINR in a special case and shown the simulation results for an asynchronous CDMA system. From these results, the global solution is significantly efficient when the number of the users  $K$  is fixed. Moreover, the performance of the global solution is independent of the parameter  $\gamma$ .

In the global solution of the problem (P), the parameter  $\gamma$  is any real number. However, its BER depends on  $\gamma$  when we let the maximum number of users  $K_{\max}$  be  $N$  or any number other than  $K$ . The remained issue is to investigate the optimal  $\gamma$  and how to assign  $\sigma_k$  successfully to the user  $k$ .

### 3.A Appendix A

In this appendix, we prove that the global optimal solutions of  $(P)$ ,  $\rho_i^*$  and  $t_{i,k}^*$  are given by

$$\begin{aligned}\rho_i^* &= \gamma + \frac{i-1}{K} \quad (i = 1, 2, \dots, K), \\ t_{i,k}^* &= \min \left\{ \frac{|k-i|}{K}, 1 - \frac{|k-i|}{K} \right\},\end{aligned}\tag{3.47}$$

where  $\gamma$  is a real number.

Since the problem  $(P)$  is a convex, it is necessary and sufficient for the global solution to satisfy the KKT conditions, Eqs. (3.27)-(3.29). From Eq. (3.29), when  $\rho_i^*$  satisfies Eq. (3.47), it is clearly that

$$\begin{aligned}\nu_i &= 0 \quad (i = 1, 2, \dots, K-1), \\ o_{i,k} &= 0 \quad (i < k)\end{aligned}\tag{3.48}$$

since  $e_i(\mathbf{x}^*) < 0$  and  $h_{i,k}(\mathbf{x}^*) < 0$ . We let  $\xi_1 = \xi_K = 0$ . Thus, it is sufficient to consider only two kinds of the Lagrange multipliers,  $\lambda_{i,k}$  and  $\mu_{i,k}$ . They satisfy the following equation which is obtained from Eq. (3.27):

$$\begin{aligned}-\sum_{i < k} \frac{\pi \cos(\pi t_{i,k}^*)}{\sin^2(\pi t_{i,k}^*)} \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_{i,k} \end{pmatrix} + \sum_{i < k} \lambda_{i,k} \begin{pmatrix} \mathbf{e}_i - \mathbf{e}_k \\ \mathbf{e}_{i,k} \end{pmatrix} \\ + \sum_{i < k} \mu_{i,k} \begin{pmatrix} -\mathbf{e}_i + \mathbf{e}_k \\ \mathbf{e}_{i,k} \end{pmatrix} &= \mathbf{0},\end{aligned}\tag{3.49}$$

where  $\mathbf{e}_i \in \mathbb{R}^K$  have 1 in the  $i$ -th element and 0 in the others and  $\mathbf{e}_{i,k} \in \mathbb{R}^{K(K-1)/2}$  have 1 in the  $\{i(2K-i-1)/2 + k - K\}$ -th element and 0 in the others. From Eq. (3.49), we consider two vector equations. One is the first  $K$ -dimensional vector equation of Eq. (3.49) and the other is the last  $K(K-1)/2$ -dimensional vector equation. They are expressed as

$$\sum_{i < k} (\lambda_{i,k} - \mu_{i,k})(\mathbf{e}_i - \mathbf{e}_k) = \mathbf{0},\tag{3.50}$$

$$\sum_{i < k} \left( \frac{\pi \cos(\pi t_{i,k}^*)}{\sin^2(\pi t_{i,k}^*)} - \lambda_{i,k} - \mu_{i,k} \right) \mathbf{e}_{i,k} = \mathbf{0}.\tag{3.51}$$

Then, we define  $\alpha(t_{i,k}^*)$  as

$$\alpha(t_{i,k}^*) = \frac{\pi \cos(\pi t_{i,k}^*)}{\sin^2(\pi t_{i,k}^*)}.\tag{3.52}$$

Note that  $\alpha(t_{i,k}^*) \geq 0$  since  $0 < t_{i,k}^* \leq \frac{1}{2}$ . From the definition of  $t_{i,k}^*$ ,  $\alpha(t_{i,k}^*)$  only depends on the absolute value of difference,  $|k-i|$ . We therefore rewrite  $\alpha(t_{i,k}^*)$  as

$$\alpha(t_{i,k}^*) = \tilde{\alpha}(|k-i|).\tag{3.53}$$

The variable  $\tilde{\alpha}(|k-i|)$  has the property that

$$\tilde{\alpha}(k) = \tilde{\alpha}(K-k) \quad (1 \leq k \leq K).\tag{3.54}$$

This result is obtained from the definition of  $t_{i,k}^*$ . We consider the two types of  $K$ :  $K$  is an odd number or  $K$  is an even number.

### 3.A.I $K$ is an odd number

For all  $i$  and  $k$  ( $i < k$ ),  $\rho_i^*$ ,  $\rho_k^*$  and  $t_{i,k}^*$  satisfy either only  $c_{i,k}(\mathbf{x}^*) = 0$  or  $d_{i,k}(\mathbf{x}^*) = 0$ . They satisfy

$$\begin{aligned} c_{i,k}(\mathbf{x}^*) &= 0, d_{i,k}(\mathbf{x}^*) < 0, (k-i < K/2), \\ d_{i,k}(\mathbf{x}^*) &= 0, c_{i,k}(\mathbf{x}^*) < 0, (k-i > K/2), \\ \lambda_{i,k} &= \begin{cases} \tilde{\alpha}(k-i) & (k-i < K/2) \\ 0 & (k-i > K/2) \end{cases}, \\ \mu_{i,k} &= \begin{cases} 0 & (k-i < K/2) \\ \tilde{\alpha}(k-i) & (k-i > K/2) \end{cases}. \end{aligned} \quad (3.55)$$

We consider the  $n$ -th element of the left side of Eq. (3.50).

$$\begin{aligned} & \sum_{n < k} (\lambda_{n,k} - \mu_{n,k}) - \sum_{i < n} (\lambda_{i,n} - \mu_{i,n}) \\ &= \sum_{\substack{n < k \\ k-n < K/2}} \lambda_{n,k} + \sum_{\substack{i < n \\ n-i > K/2}} \mu_{i,n} - \sum_{\substack{i < n \\ n-i < K/2}} \lambda_{i,n} - \sum_{\substack{n < k \\ k-n > K/2}} \mu_{n,k} \\ &= \sum_{\substack{n < k \\ k-n < K/2}} \tilde{\alpha}(k-n) + \sum_{\substack{i < n \\ n-i > K/2}} \tilde{\alpha}(n-i) \\ &\quad - \sum_{\substack{i < n \\ n-i < K/2}} \tilde{\alpha}(n-i) - \sum_{\substack{n < k \\ k-n > K/2}} \tilde{\alpha}(k-n) \\ &= \sum_{\substack{n < k \\ k-n < K/2}} \tilde{\alpha}(k-n) + \sum_{\substack{i < n \\ n-i > K/2}} \tilde{\alpha}(K+i-n) \\ &\quad - \sum_{\substack{i < n \\ n-i < K/2}} \tilde{\alpha}(n-i) - \sum_{\substack{n < k \\ k-n > K/2}} \tilde{\alpha}(K+n-k) \\ &= \sum_{\substack{n < k \\ k-n < K/2}} \tilde{\alpha}(k-n) + \sum_{\substack{i < n \\ n-i < K/2}} \tilde{\alpha}(n-i) \\ &\quad - \sum_{\substack{i < n \\ n-i < K/2}} \tilde{\alpha}(n-i) - \sum_{\substack{n < k \\ k-n < K/2}} \tilde{\alpha}(k-n) = 0. \end{aligned} \quad (3.56)$$

From Eq. (3.55), for all the integers  $i$  and  $k$ , the term in summation of the left side of Eq. (3.51) equals 0. From the above proof, all the Lagrange multipliers satisfy Eq. (3.29).

### 3.A.II $K$ is an even number

The Lagrange multipliers  $\rho_i^*$ ,  $\rho_k^*$  and  $t_{i,k}^*$  satisfy

$$\begin{aligned} c_{i,k}(\mathbf{x}^*) &= 0, d_{i,k}(\mathbf{x}^*) < 0, \quad (k-i < K/2), \\ d_{i,k}(\mathbf{x}^*) &= 0, c_{i,k}(\mathbf{x}^*) < 0, \quad (k-i > K/2), \\ d_{i,k}(\mathbf{x}^*) &= 0, c_{i,k}(\mathbf{x}^*) = 0, \quad (k-i = K/2). \end{aligned} \quad (3.57)$$

When  $k - i = K/2$ , they satisfy  $c_{i,k}(\mathbf{x}^*) = 0$  and  $d_{i,k}(\mathbf{x}^*) = 0$ . Thus, we set

$$\begin{aligned}\lambda_{i,k} &= \begin{cases} \tilde{\alpha}(k-i) & (k-i < K/2), \\ \tilde{\alpha}(k-i)/2 & (k-i = K/2), \\ 0 & (k-i > K/2) \end{cases} \\ \mu_{i,k} &= \begin{cases} 0 & (k-i < K/2), \\ \tilde{\alpha}(k-i)/2 & (k-i = K/2), \\ \tilde{\alpha}(k-i) & (k-i > K/2) \end{cases}\end{aligned}\quad (3.58)$$

Similar to the case that  $K$  is an odd number, we consider the  $n$ -th element of left side of Eq. (3.50).

$$\begin{aligned}& \sum_{n < k} (\lambda_{n,k} - \mu_{n,k}) - \sum_{i < n} (\lambda_{i,n} - \mu_{i,n}) \\&= \sum_{\substack{n < k \\ k-n < K/2}} \lambda_{n,k} - \sum_{\substack{n < k \\ k-n > K/2}} \mu_{n,k} - \sum_{\substack{i < n \\ n-i < K/2}} \lambda_{i,n} + \sum_{\substack{i < n \\ n-i > K/2}} \mu_{i,n} \\&+ \sum_{\substack{n < k \\ k-n = K/2}} \lambda_{n,k} - \sum_{\substack{n < k \\ k-n = K/2}} \mu_{n,k} - \sum_{\substack{i < n \\ n-i = K/2}} \lambda_{i,n} + \sum_{\substack{i < n \\ n-i = K/2}} \mu_{i,n}.\end{aligned}\quad (3.59)$$

The terms of the difference equaling  $K/2$  vanish. Therefore, we obtain

$$\begin{aligned}& \sum_{n < k} (\lambda_{n,k} - \mu_{n,k}) - \sum_{i < n} (\lambda_{i,n} - \mu_{i,n}) \\&= \sum_{\substack{n < k \\ k-n < K/2}} \lambda_{n,k} - \sum_{\substack{n < k \\ k-n > K/2}} \mu_{n,k} - \sum_{\substack{i < n \\ n-i < K/2}} \lambda_{i,n} + \sum_{\substack{i < n \\ n-i > K/2}} \mu_{i,n} \\&= \sum_{\substack{n < k \\ k-n < K/2}} \tilde{\alpha}(k-n) - \sum_{\substack{n < k \\ k-n > K/2}} \tilde{\alpha}(k-n) \\&- \sum_{\substack{i < n \\ n-i < K/2}} \tilde{\alpha}(n-i) + \sum_{\substack{i < n \\ n-i > K/2}} \tilde{\alpha}(n-i) \\&= \sum_{\substack{n < k \\ k-n < K/2}} \tilde{\alpha}(k-n) - \sum_{\substack{n < k \\ k-n > K/2}} \tilde{\alpha}(K-k+n) \\&- \sum_{\substack{i < n \\ n-i < K/2}} \tilde{\alpha}(n-i) + \sum_{\substack{i < n \\ n-i > K/2}} \tilde{\alpha}(K-n+i) \\&= \sum_{\substack{n < k \\ k-n < K/2}} \tilde{\alpha}(k-n) - \sum_{\substack{n < k \\ k-n < K/2}} \tilde{\alpha}(k-n) \\&- \sum_{\substack{i < n \\ n-i < K/2}} \tilde{\alpha}(n-i) + \sum_{\substack{i < n \\ n-i < K/2}} \tilde{\alpha}(n-i) = 0.\end{aligned}\quad (3.60)$$

Thus, we have proven that Eq. (3.60) equals to 0. It is clearly that the left side of Eq. (3.51) equals 0 when  $k - i \neq K/2$ . When  $k - i = K/2$ , it follows that

$$\tilde{\alpha}(K/2) - \frac{\tilde{\alpha}(K/2)}{2} - \frac{\tilde{\alpha}(K/2)}{2} = 0. \quad (3.61)$$

Thus, for all the integers  $i$  and  $k$ , Eq. (3.51) is satisfied.

From the proofs A and B, we have proven that the existence of the Lagrange multipliers which satisfy Eq. (3.29). Therefore,  $\rho_i^*$  and  $t_{i,k}^*$  are the global solutions of the problem (P).

### 3.B Appendix B

In this appendix, with the spreading sequences  $\{\tilde{w}_{k,n}(N, \gamma)\}$ , we prove that SINR of the user  $i$  is given by

$$\text{SINR}_i = \left\{ R_i + \frac{N_0}{2E} \right\}^{-1/2}, \quad (3.62)$$

where

$$R_i = \frac{(K-1)}{18N^2} \left\{ 2(N+1) + (N-2) \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right) \right\}. \quad (3.63)$$

We assume that the element  $\sigma_k$  is a random variable uniformly distributed in  $\{0, 1, 2, \dots, N-1\}$ . This assumption is fulfilled when the ratio  $K/N$  is close to 1, that is, the number of users is sufficiently large since SINR is not the reciprocal of the average of the inter-symbol interference over the users. However, with the spreading sequences  $\{\tilde{w}_{k,n}(N, \gamma)\}$ , they are the same when the number of users  $K$  equals  $N$  (see Eq. (3.74)). Thus, it is conceivable that the assumption is established when the ratio  $K/N$  is close to 1.

The correlation function  $C_{i,k}(l)$  of the spreading sequences in Eq. (3.32) is

$$C_{i,k}(l) = \begin{cases} -Z_{\sigma_i, \sigma_k} \Phi_{\gamma, \sigma_i, \sigma_k}(l) & 0 \leq l \leq N-1, \\ Z_{\sigma_i, \sigma_k} \Phi_{\gamma, \sigma_i, \sigma_k}(l) & 1-N \leq l < 0, \\ 0 & |l| \geq N, \end{cases} \quad (3.64)$$

where

$$Z_{\sigma_i, \sigma_k} = \frac{\exp \left( 2\pi j \frac{\sigma_k - \sigma_i}{N} \right)}{1 - \exp \left( 2\pi j \frac{\sigma_k - \sigma_i}{N} \right)} \quad (3.65)$$

and

$$\Phi_{\gamma, \sigma_i, \sigma_k}(l) = \exp \left( -2\pi j l \left( \gamma + \frac{\sigma_k}{N} \right) \right) - \exp \left( -2\pi j l \left( \gamma + \frac{\sigma_i}{N} \right) \right). \quad (3.66)$$

Thus, we obtain the squared absolute value of  $C_{i,k}(l)$ :

$$|C_{i,k}(l)|^2 = \frac{1 - \cos \left( 2\pi l \frac{\sigma_k - \sigma_i}{N} \right)}{1 - \cos \left( 2\pi \frac{\sigma_k - \sigma_i}{N} \right)}. \quad (3.67)$$

On the other hand, the following relations are satisfied:

$$\begin{aligned} \sum_{l=0}^{N-1} |C_{i,k}(l-N)|^2 &= \sum_{l=0}^{N-1} |C_{i,k}(l-N+1)|^2 = \sum_{l=0}^{N-1} |C_{i,k}(l)|^2 \\ &= \sum_{l=0}^{N-1} |C_{i,k}(l+1)|^2 = \frac{N}{1 - \cos \left( 2\pi \frac{\sigma_k - \sigma_i}{N} \right)}, \end{aligned} \quad (3.68)$$

$$\begin{aligned}
& \sum_{l=0}^{N-1} \operatorname{Re}[C_{i,k}(l-N) \overline{C_{i,k}(l-N+1)}] \\
&= \frac{N \left\{ \cos \left( 2\pi \left( \gamma + \frac{\sigma_k}{N} \right) \right) + \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right) \right\}}{2 \left( 1 - \cos \left( 2\pi \frac{\sigma_k - \sigma_i}{N} \right) \right)},
\end{aligned} \tag{3.69}$$

and

$$\begin{aligned}
& \sum_{l=0}^{N-1} \operatorname{Re} \left[ C_{i,k}(l-N) \overline{C_{i,k}(l-N+1)} \right] \\
&= \sum_{l=0}^{N-1} \operatorname{Re} \left[ C_{i,k}(l) \overline{C_{i,k}(l+1)} \right].
\end{aligned} \tag{3.70}$$

In the above equations, we used the assumption  $\sigma_i \neq \sigma_k$ . From Eqs. (3.68)-(3.70),  $r_{i,k}$  in Eq. (3.38) is given by

$$\begin{aligned}
r_{i,k} &= \frac{N}{1 - \cos \left( 2\pi \frac{\sigma_k - \sigma_i}{N} \right)} \\
&\quad \cdot \left\{ 4 + \cos \left( 2\pi \left( \gamma + \frac{\sigma_k}{N} \right) \right) + \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right) \right\}.
\end{aligned} \tag{3.71}$$

When we calculate the sum of Eq. (3.71), the first term of it is given by

$$\sum_{\substack{k=1 \\ k \neq i}} \frac{4N}{1 - \cos \left( 2\pi \frac{\sigma_k - \sigma_i}{N} \right)} = \sum_{\substack{k=1 \\ k \neq i}} \frac{2N}{\sin^2 \left( \pi \frac{\sigma_k - \sigma_i}{N} \right)}. \tag{3.72}$$

The integer  $\sigma_k \in \{0, 1, 2, \dots, N-1\}$  is a random variable and satisfies  $\sigma_k \neq \sigma_i$  when  $k \neq i$ . Thus,  $\sigma_k$  is expressed as

$$\sigma_k = (\sigma_i + q) \bmod N, \quad q \in \{1, 2, \dots, N-1\}. \tag{3.73}$$

and we can treat  $q$  as a random variable instead of  $\sigma_k$ . The integer  $q$  is uniformly distributed in  $\{1, 2, \dots, N-1\}$ . Thus, the average of Eq. (3.72) is

$$\begin{aligned}
& \operatorname{E} \left\{ \sum_{\substack{k=1 \\ k \neq i}} \frac{2N}{\sin^2 \left( \pi \frac{\sigma_k - \sigma_i}{N} \right)} \right\} \\
&= \sum_{\substack{k=1 \\ k \neq i}} \operatorname{E} \left\{ \frac{2N}{\sin^2 \left( \pi \frac{\sigma_k - \sigma_i}{N} \right)} \right\} \\
&= \sum_{\substack{k=1 \\ k \neq i}} \frac{1}{N-1} \sum_{q=1}^{N-1} \frac{2N}{\sin^2 \left( \pi \frac{q}{N} \right)} \\
&= \frac{N-1}{N-1} \sum_{q=1}^{N-1} \frac{2N}{\sin^2 \left( \pi \frac{q}{N} \right)},
\end{aligned} \tag{3.74}$$

where E is the average over  $\sigma_k$ .

In [58], it is shown that

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2(\pi \frac{k}{n})} = \frac{n^2 - 1}{3} = \frac{(n-1)(n+1)}{3}. \quad (3.75)$$

Thus, Eq. (3.74) is equivalent to

$$\frac{K-1}{N-1} \sum_{q=1}^{N-1} \frac{2N}{\sin^2(\pi \frac{q}{N})} = \frac{2N(N+1)(K-1)}{3}. \quad (3.76)$$

From the above result, we obtain the following relation

$$\mathbb{E} \left\{ \sum_{\substack{k=1 \\ k \neq i}} \frac{4N}{1 - \cos(2\pi \frac{\sigma_k - \sigma_i}{N})} \right\} = \frac{2N(N+1)(K-1)}{3}. \quad (3.77)$$

The average of the second term of the sum of Eq. (3.71) is given by

$$\begin{aligned} & \sum_{\substack{k=1 \\ k \neq i}} \mathbb{E} \left\{ \frac{N \cos(2\pi(\gamma + \frac{\sigma_k}{N}))}{1 - \cos(2\pi \frac{\sigma_k - \sigma_i}{N})} \right\} \\ &= \frac{K-1}{N-1} \sum_{q=1}^{N-1} \frac{N}{2 \sin^2(\pi \frac{q}{N})} \cos \left( 2\pi \left( \gamma + \frac{\sigma_i + q}{N} \right) \right) \\ &= \frac{N(K-1)}{N-1} \sum_{q=1}^{N-1} \left\{ \frac{\cos(2\pi(\gamma + \frac{\sigma_i}{N})) \cos(2\pi \frac{q}{N})}{2 \sin^2(\pi \frac{q}{N})} \right. \\ & \quad \left. - \frac{\sin(2\pi(\gamma + \frac{\sigma_i}{N})) \sin(2\pi \frac{q}{N})}{2 \sin^2(\pi \frac{q}{N})} \right\} \\ &= \frac{N(K-1)}{N-1} \sum_{q=1}^{N-1} \left\{ \frac{\cos(2\pi(\gamma + \frac{\sigma_i}{N})) (1 - 2 \sin^2(\pi \frac{q}{N}))}{2 \sin^2(\pi \frac{q}{N})} \right. \\ & \quad \left. - \frac{\sin(2\pi(\gamma + \frac{\sigma_i}{N})) \cos(\pi \frac{q}{N})}{\sin(\pi \frac{q}{N})} \right\}. \end{aligned} \quad (3.78)$$

Note that it is clear that

$$\sum_{q=1}^{N-1} \frac{\cos(\pi \frac{q}{N})}{\sin(\pi \frac{q}{N})} = 0. \quad (3.79)$$

Therefore, Eq. (3.78) is rewritten as

$$\begin{aligned} & \frac{N(K-1)}{N-1} \cos(2\pi(\gamma + \frac{\sigma_i}{N})) \sum_{q=1}^{N-1} \left\{ \frac{1}{2 \sin^2(\pi \frac{q}{N})} - 1 \right\} \\ &= N(K-1) \left( \frac{N+1}{6} - 1 \right) \cos(2\pi(\gamma + \frac{\sigma_i}{N})) \\ &= \frac{N(N-5)(K-1)}{6} \cos(2\pi(\gamma + \frac{\sigma_i}{N})). \end{aligned} \quad (3.80)$$



Thus, the sum of the average of the second term in Eq. (3.71) is written as

$$\begin{aligned} & \sum_{\substack{k=1 \\ k \neq i}}^K \mathbb{E} \left\{ \frac{N \cos \left( 2\pi \left( \gamma + \frac{\sigma_k}{N} \right) \right)}{1 - \cos \left( 2\pi \frac{\sigma_k - \sigma_i}{N} \right)} \right\} \\ &= \frac{N(N-5)(K-1)}{6} \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right) \end{aligned} \quad (3.81)$$

Similarly, we obtain the average of the sum of the third term of Eq. (3.71):

$$\begin{aligned} & \mathbb{E} \left\{ \sum_{\substack{k=1 \\ k \neq i}} \frac{N \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right)}{1 - \cos \left( 2\pi \frac{\sigma_k - \sigma_i}{N} \right)} \right\} \\ &= \frac{K-1}{N-1} \sum_{q=1}^{N-1} \frac{N \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right)}{2 \sin^2 \left( \pi \frac{q}{N} \right)} \\ &= \frac{N(N+1)(K-1)}{6} \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right). \end{aligned} \quad (3.82)$$

Finally, we obtain the following relation

$$\begin{aligned} & \mathbb{E} \left\{ \sum_{\substack{k=1 \\ k \neq i}} \frac{N \left( \cos \left( 2\pi \left( \gamma + \frac{\sigma_k}{N} \right) \right) + \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right) \right)}{1 - \cos \left( 2\pi \frac{\sigma_k - \sigma_i}{N} \right)} \right\} \\ &= \frac{N(N-2)(K-1)}{3} \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right). \end{aligned} \quad (3.83)$$

From Eq. (3.77) and Eq. (3.83), we arrive at SINR of the user  $i$  with the spreading sequence  $(\tilde{w}_{k,n})$

$$\text{SINR}_i = \left\{ R_i + \frac{N_0}{2E} \right\}^{-1/2}, \quad (3.84)$$

where

$$R_i = \frac{(K-1)}{18N^2} \left\{ 2(N+1) + (N-2) \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right) \right\}. \quad (3.85)$$

## Chapter 4

# SINR Formula for Asynchronous CDMA System with Quadratic Forms

### I Introduction

As seen in the previous chapter, spreading sequences are utilized in code division multiple access (CDMA) systems. We can increase SINR with improving spreading sequences since it has been known that SINR is written in terms of spreading sequences [13].

In a sense of designing spreading sequences, our ultimate goal is to obtain sequences which achieve the maximum SINR. In synchronous CDMA systems without time delays, it is known that the Welch bound equality (WBE) sequences realize the maximal capacity [59]. When the time delays are given and fixed, the way to find the optimal spreading sequences has been suggested [60]. However, in asynchronous CDMA systems, the optimal spreading sequences have not been found. Moreover, the bound of SINR is still unknown. Therefore, asynchronous CDMA systems have been investigated [61] [62] [63].

To achieve high SINR for asynchronous CDMA systems, there are many papers on designing spreading sequences, the Gold codes [38]. These sequences are obtained from M-sequences. Therefore, the Gold codes are obtained from shift registers. In [41] [64] [44] [45] [65], it is proposed to use chaotic dynamical systems to design spreading sequences. There are some works to obtain dynamical systems as generators of sequences whose correlation is low [42] [66] [67] [68]. For these chaos-based DS-CDMA systems, the performance in fading channels is investigated in [42] and [69] [70]. As alternative systems of spread spectrum telecommunication systems, Differential Chaos Shift Keying (DCSK) systems are proposed [71]. In DCSK systems, there are many recent works. We refer the reader to [72] [73] [74] [75]. In some of these works, the form of spreading sequences has been often assumed. In the previous chapter, we have assumed that all the sequences belong to the Weyl class, which has been defined

in Chapter 3.

While the bound of SINR is unknown, the properties of sequences have been investigated. Note that these properties are derived without any assumptions about the form of sequences. In CDMA systems, cross-correlation is treated as Multiple-Access Interference (MAI) and auto-correlation is related to synchronization at the receiver side and fading noise. Sarwate [47] has shown that there is an avoidable limitation trade-off as a relation between the peak of cross-correlation and the second peak of auto-correlation. The limitation is achieved by the Frank-Zadoff-Chu (FZC) sequences [48] [49]. Besides, Welch has shown [54] [76] that there are lower bounds on maximum magnitudes of inner products of a set of unit-norm complex-vectors. This limitation is called the Welch bound and the sequences which satisfy the equality of the Welch bound are called the Welch Bound Equality (WBE) sequences. The WBE sequences have been investigated in [59] [77].

In this chapter, we discuss how to obtain the sequences which achieve large SINR without any assumptions about the form of sequences. We consider a Rician fading channel, and obtain the worst case of SINR and derive spreading sequences as solutions of the optimization problem: maximize SINR.

Although the expressions of SINR have been obtained in [45] and [13] as forms including the real-part operators, their expressions are complicated. Therefore, it is not straightforward to solve the optimization problem with their expressions, and then an expression of SINR which does not have real-part operators is demanded. We derive such an expression of SINR. Moreover, the main terms of our expression consist of periodic correlation terms and odd periodic correlation terms. From this expression, we construct the optimization problem: maximize SINR. It turns out that this problem is not convex and a way to obtain the global maximizer is not in evidence. Then, we obtain the relaxed problem with a semidefinite relaxation technique [78]. In certain parameter regions, this problem is convex. In the other parameter regions, this problem is a generalized convex multiplicative programming problem [79]. There is an algorithm to solve a generalized convex multiplicative programming problem. However, it is expected to take a lot of computational time to solve since our problem has the large number of terms. As an alternative way, we obtain the solutions with an optimization solver. Note that the global maximizer can be obtained when only a certain user is focused on and there is no fading. In such a situation, the maximum SINR and the maximum capacity can be analyzed. Finally, we compare them in BER with other sequences.

## II Asynchronous CDMA Model

In this section, we fix our model used thorough this chapter and mathematical symbols that will be used in the following sections. We consider the following asynchronous binary phase shift keying (BPSK) CDMA model [13]. For more details of this model, we refer the reader to [80]. Let  $N$  be the length of spreading sequences. The user  $k$ 's data signal  $b_k(t)$  is expressed as

$$b_k(t) = \sum_{n=-\infty}^{\infty} b_{k,n} p_T(t - nT), \quad (4.1)$$

where  $b_{k,n} \in \{-1, 1\}$  is the  $n$ -th component of symbols which the user  $k$  send,  $T$  is the duration of one symbol and  $p_T(t)$  is a rectangular pulse written as

$$p_T(t) = \begin{cases} 1 & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}.$$

The user  $k$ 's code waveform  $s_k(t)$  is expressed as

$$s_k(t) = \sum_{n=-\infty}^{\infty} s_{k,n} p_{T_c}(t - nT_c), \quad (4.2)$$

where  $s_{k,n}$  is the  $n$ -th component of the user  $k$ 's spreading sequence and  $T_c$  is the width of each chip, such that  $NT_c = T$ . We assume that the sequence  $(s_{k,n})$  has the period  $N$ , that is,  $s_{k,n} = s_{k,n+N}$ . Moreover, we assume the power normalization condition:

$$\sum_{n=1}^N |s_{k,n}|^2 = N. \quad (4.3)$$

This condition is often used [47] [54].

The user  $k$ 's transmitted signal  $\zeta_k(t)$  is written as

$$\zeta_k(t) = \sqrt{2P} \operatorname{Re}[s_k(t)b_k(t)\exp(j\omega_c t + j\theta_k)], \quad (4.4)$$

where  $P$  is the common signal power to all the users,  $\omega_c$  is the common carrier frequency to all the users and  $\theta_k$  is the phase of the user  $k$ .

If there is no strong stable component in the received signal, then the direct sequence form of spread spectrum communication would not generally be suitable [80]. Therefore, we consider a Rician fading channel. The received signal  $r(t)$  is written as

$$r(t) = \sum_{k=1}^K \operatorname{Re}[u_k(t - \tau_k)\exp(j\omega_c t + j\psi_k)] + n(t), \quad (4.5)$$

where  $\psi_k = \theta_k - \omega_c \tau_k$ ,  $n(t)$  is additive white Gaussian noise (AWGN) and  $u_k(t)$  is written as

$$u_k(t) = \gamma_k \int_{-\infty}^{\infty} h_k(\tau, t) x_k(t - \tau) d\tau + x_k(t), \quad (4.6)$$

$$x_k(t) = \sqrt{2P} s_k(t) b_k(t). \quad (4.7)$$

The first term of Eq. (4.6) is the component of faded signals and the second term is the component of a direct wave. The function  $h_k(\tau, t)$  is a zero-mean complex Gaussian random process and  $\gamma_k$  is a non-negative real parameter, which represents the transmission coefficient for the user  $k$ 's signal. In general,  $h(\tau, t)$  is often approximated by [12] [81] [82]

$$h_k(\tau, t) = \frac{1}{\sqrt{N}} \sum_{n=1}^N a_n^{(k)} \exp(2\pi j f_n(t)) \delta(\tau - \tau_n^{(k)}), \quad (4.8)$$

where

$$f_n(t) = f_{D_n}^{(k)} t + f_{\text{hop}}^{(k)}(t) \tau_n^{(k)},$$

$a_n^{(k)}$  is an attenuation coefficient,  $f_{D_n}^{(k)}$  is the Doppler frequency,  $f_{\text{hop}}^{(k)}(t)$  is the carrier frequency shift,  $\tau_n^{(k)}$  is the delay time of the  $n$ -th delayed signal and  $\delta(x)$  is the delta function.

If the received signal  $r(t)$  is the input to a correlation receiver matched to  $\zeta_i(t)$ , then the corresponding output  $Z_i$  is written as

$$Z_i = \int_0^T r(t) \text{Re}[s_i(t - \tau_i) \exp(j\omega_c t + j\psi_i)] dt. \quad (4.9)$$

Without loss of generality, we assume  $\tau_i = 0$  and  $\theta_i = 0$  and hence  $\psi_i = 0$ . With a low-pass filter, we can ignore double frequency terms, and rewrite Eq. (4.9) as

$$\begin{aligned} Z_i &= \frac{1}{2} \sum_{k=1}^K \int_0^T \text{Re}[u_k(t - \tau_k) \overline{s_i(t)} \exp(j\psi_k)] dt \\ &\quad + \int_0^T n(t) \text{Re}[s_i(t) \exp(j\omega_c t)] dt, \end{aligned} \quad (4.10)$$

where  $\bar{z}$  is the complex conjugate of  $z$  and

$$\overline{s_i(t)} = \sum_{n=-\infty}^{\infty} \overline{s_{i,n}} p_{T_c}(t - nT_c). \quad (4.11)$$

In obtaining Eq. (4.10), we have used the identity

$$2 \text{Re}[z_1] \text{Re}[z_2] = \text{Re}[z_1 z_2] + \text{Re}[z_1 \bar{z}_2], \quad (4.12)$$

where  $z_1, z_2 \in \mathbb{C}$ .

Similar to [13], we assume that the phases  $\psi_k$ , time delays  $\tau_k$  and symbols  $b_{k,n}$  are independent random variables and they are uniformly distributed on  $[0, 2\pi)$ ,  $[0, T)$  and  $\{-1, 1\}$ , respectively. Without loss of generality, we assume that  $b_{i,0} = +1$ .

To evaluate SINR, we define

$$\mu_{i,k}(\tau; t) = b_k(t - \tau) s_k(t - \tau) \overline{s_i(t)} \quad (4.13)$$

and

$$\xi_{i,k_1,k_2}(\tau_1, \tau_2; t_1, t_2) = \mu_{i,k_1}(\tau_1; t_1) \overline{\mu_{i,k_2}(\tau_2; t_2)}. \quad (4.14)$$

For notational convenience, we write  $\mu_{i,i}$  as  $\mu_i$ , and  $\xi_{i,i,i}$  as  $\xi_i$ . We divide  $Z_i$  into the four signals, the user  $i$ 's desired signal  $D_i$ , the user  $i$ 's faded signal  $F_i$ , the interference signal  $I_i$  and the AWGN signal  $N_i$ . They are expressed as

$$\begin{aligned} D_i &= \sqrt{\frac{P}{2}} \int_0^T b_i(t) dt, \quad F_i = \sqrt{\frac{P}{2}} \text{Re}[\tilde{F}_i] \\ I_i &= \sqrt{\frac{P}{2}} \sum_{\substack{k=1 \\ k \neq i}}^K (\text{Re}[\gamma_k \tilde{I}_{i,k}] + \text{Re}[\tilde{I}'_{i,k}]) \\ N_i &= \int_0^T n(t) \text{Re}[s_i(t) \exp(j\omega_c t)] \end{aligned} \quad (4.15)$$

where

$$\begin{aligned}\tilde{F}_i &= \int_0^T \int_{-\infty}^{\infty} \gamma_i h_i(\tau, t) \mu_i(\tau; t) d\tau dt \\ \tilde{I}_{i,k} &= \int_0^T \int_{-\infty}^{\infty} h_k(\tau, t - \tau_k) \mu_{i,k}(\tau_k + \tau; t) \exp(j\psi_k) d\tau dt \\ \tilde{I}'_{i,k} &= \int_0^T \mu_{i,k}(\tau_k; t) \exp(j\psi_k) dt.\end{aligned}$$

From these expressions,  $Z_i$  is expressed as

$$Z_i = D_i + F_i + I_i + N_i. \quad (4.16)$$

### III Evaluation of SINR

Since  $E\{F_i\} = E\{I_i\} = E\{N_i\} = 0$  and  $E\{D_i\} = T\sqrt{P/2}$ , we have  $E\{Z_i\} = T\sqrt{P/2}$ , where  $E\{X\}$  is the average of  $X$ . We assume that each Gaussian process  $h_k(\tau, t)$  is independent and  $h_k(\tau, t)$ ,  $\psi_k$ ,  $\tau_k$  and  $b_{k,n}$  are independent. Then, SINR of the user  $i$  is defined as

$$\text{SINR}_i = \sqrt{\frac{PT^2/2}{\text{Var}\{F_i\} + \text{Var}\{I_i\} + \text{Var}\{N_i\}}}, \quad (4.17)$$

where  $\text{Var}\{X\}$  is the variance of  $X$ . In this section, we focus on the estimation of the lower bound of Eq. (4.17) under some assumptions. It is known [13] [80] that the variance of  $N_i$  is

$$\text{Var}\{N_i\} = \frac{1}{4}N_0T \quad (4.18)$$

if  $n(t)$  has a two-sided spectral density denoted as  $\frac{1}{2}N_0$ .

We make assumptions about the channel, the variables and the Gaussian process  $h_k(\tau, t)$  that

1. the Fourier transform of  $h_k(\tau, t)$  and its inverse Fourier transform exist.
2. the channel is a wide-sense-stationary uncorrelated-scattering (WSSUS) channel [83].
3. the channel is a frequency selective fading channel.
4. the Gaussian process  $h_k(\tau, t)$  satisfies  $h_k(\tau, t) = 0$  when  $\tau < 0$  [82].
5. there is an integer  $M_k$  which satisfies  $h_k(\tau, t) = 0$  when  $\tau > M_kT$ .
6. the variable  $\tau$  satisfies that  $nT + lT_c \leq \tau_k < nT + (l+1)T_c$ , where  $n$  and  $l$  are the integers which satisfy  $0 \leq n$  and  $0 \leq l < N$ .
7. the phases  $\psi_k$ , time delays  $\tau_k$  and symbols  $b_{k,n}$  are independent random variables and they are uniformly distributed on  $[0, 2\pi)$ ,  $[0, T)$  and  $\{-1, 1\}$ , respectively.

The first assumption is required to define a WSSUS channel. The second and third assumptions are often used in analysis of wireless communications. The fourth assumption is equivalent to that the channel is causal. The fifth assumption is equivalent to the one that the delayed signal becomes to zero in finite-time. From Eq. (4.8), the faded signal is composed of the sum of delayed signals which are affected by the Doppler shift. Delayed signals are attenuated as time passes. To incorporate such attenuation, it is often assumed that the probability of time delay  $\tau$  obeys an exponential distribution in some models [81]. Then, we ignore the delayed signals whose delay time is sufficiently large since they are attenuated and their power gets sufficiently small. The sixth assumption is often used [13] [80]. The last assumption is written in Section II.

In WSSUS channels, the covariance function of  $h_k(\tau, t)$  is expressed as [80]

$$\begin{aligned}\Sigma_k(\tau_1, \tau_2; t_1, t_2) &= \text{E}[h_k(\tau_1, t_1)\overline{h_k(\tau_2, t_2)}] \\ &= \rho_k(\tau_1, t_1 - t_2)\delta(\tau_1 - \tau_2).\end{aligned}\quad (4.19)$$

Adding to this condition, in a selective fading channel, covariance function  $\Sigma_k$  is expressed as [80]

$$\begin{aligned}\Sigma_k(\tau_1, \tau_2; t_1, t_2) &= \rho_k(\tau_1, 0)\delta(\tau_1 - \tau_2) \\ &= g_k(\tau_1)\delta(\tau_1 - \tau_2).\end{aligned}\quad (4.20)$$

In the above equation, we have defined  $g_k(\tau_1) = \rho_k(\tau_1, 0)$ . From Eq. (4.20), the covariance function  $\Sigma_k$  is independent of  $t_1$  and  $t_2$ .

First, we calculate the variance of  $F_i$ . With Eq. (4.12),  $\text{Var}\{F_i\}$  is written as

$$\text{Var}\{F_i\} = \frac{P}{4} \text{E}\{\text{Re}[\tilde{F}_i]^2\} + \frac{P}{4} \text{E}\{|\tilde{F}_i|^2\}.\quad (4.21)$$

Here,  $\text{E}\{\text{Re}[\tilde{F}_i]^2\}$  and  $\text{E}\{|\tilde{F}_i|^2\}$  are expressed as

$$\begin{aligned}\text{E}\{\text{Re}[\tilde{F}_i]^2\} &= \gamma_i^2 \cdot \text{E}_{\mathbf{b}_i} \left\{ \text{Re} \left[ \int_0^T \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\Sigma}_i(\tau_1, \tau_2; t_1, t_2) \right. \right. \\ &\quad \left. \left. \cdot \mu_i(\tau_1; t_1) \mu_i(\tau_2; t_2) d\tau_1 d\tau_2 dt_1 dt_2 \right] \right\}, \\ \text{E}\{|\tilde{F}_i|^2\} &= \gamma_i^2 \cdot \text{E}_{\mathbf{b}_i} \left\{ \int_0^T \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Sigma_i(\tau_1, \tau_2; t_1, t_2) \right. \\ &\quad \left. \cdot \xi_i(\tau_1, \tau_2; t_1, t_2) d\tau_1 d\tau_2 dt_1 dt_2 \right\},\end{aligned}\quad (4.22)$$

where

$$\tilde{\Sigma}_i(\tau_1, \tau_2; t_1, t_2) = \text{E}\{h_i(\tau_1, t_1)h_i(\tau_2, t_2)\}$$

and  $\text{E}_{\mathbf{b}_i}\{X\}$  is the average over all the bits of the user  $i$ . To simplify the expressions above, we introduce the average of  $X$  with respect to a distribution function  $\mathcal{F}$ , where  $X$  and  $\mathcal{F}$  depend on a variable  $\eta$ . We denote this average as  $E_\eta\{X\}$ .

In [80] and [12], it is shown that we can use

$$\mathbb{E}\{h_k(\tau_1, t_1)h_k(\tau_2, t_2)\} = 0. \quad (4.23)$$

This result is obtained from the demodulation of RF signals. From Eqs. (4.20)-(4.23), we have

$$\text{Var}\{F_i\} = \frac{P}{4}\gamma_i^2 \cdot \mathbb{E}_{\mathbf{b}_i} \left\{ \int_{-\infty}^{\infty} g_i(\tau) \int_0^T \int_0^T \xi_i(\tau, \tau; t_1, t_2) dt_1 dt_2 d\tau \right\}. \quad (4.24)$$

The double integral term above is written as

$$\int_0^T \int_0^T \xi_i(\tau, \tau; t_1, t_2) dt_1 dt_2 = \left| \int_0^T \mu_i(\tau; t) dt \right|^2 = \Gamma_i(\tau) \geq 0, \quad (4.25)$$

where  $\Gamma_i(\tau)$  has been defined. Note that  $\Gamma_i(\tau)$  is the squared absolute value of the correlation in an asynchronous CDMA system. From the assumptions 4 and 5, we obtain

$$g_i(\tau) = 0 \quad \text{for } \tau < 0, \quad g_i(\tau) = 0 \quad \text{for } \tau > M_i T. \quad (4.26)$$

It is clear that  $g_i(\tau)$  is non-negative since

$$g_i(\tau) = \rho_i(\tau, 0) = \mathbb{E}\{h_i(\tau, 0)\overline{h_i(\tau, 0)}\} \geq 0.$$

Further, we can assume that  $g_i(\tau)$  has the upper bound  $C_i$  in  $[0, M_i T]$ . This is proven in Appendix A. In designing sequences, we have no knowledge about the form of  $g_i(\tau)$ . For this reason, we evaluate the upper bound of  $\text{Var}\{F_i\}$  with the product of two terms, one is related to  $g_i(\tau)$  and the other is related to spreading sequences. When the upper bound decreases, it is expected that  $\text{Var}\{F_i\}$  gets small. From Hölder's inequality, we evaluate Eq. (4.24) as

$$\begin{aligned} \text{Var}\{F_i\} &= \frac{P}{4}\gamma_i^2 \cdot \mathbb{E}_{\mathbf{b}_i} \left\{ \int_{-\infty}^{\infty} g_i(\tau) \Gamma_i(\tau) d\tau \right\} \\ &\leq \frac{P}{4}\gamma_i^2 C_i \cdot \mathbb{E}_{\mathbf{b}_i} \left\{ \int_0^{M_i T} \Gamma_i(\tau) d\tau \right\}. \end{aligned} \quad (4.27)$$

The equality is attained if  $g_i(\tau)$  is the rectangular function. This is the worst case where  $\text{Var}\{F_i\}$  is maximized. From the assumption 6, the time delay  $\tau$  satisfies  $n_i T + l_i T_c \leq \tau < n_i T + (l_i + 1)T_c$ , where  $n_i$  and  $l_i$  are the integers which satisfy  $0 \leq n_i < M_i$  and  $0 \leq l_i < N$ . Note that  $n_i T + NT_c = (n_i + 1)T$ . Since the correlation in an asynchronous CDMA system is the superposition of the correlations in a chip-synchronous CDMA system, the function  $\Gamma_i(\tau)$  can be written as

$$\Gamma_i(\tau) = \left| \int_0^T \mu_i(\tau; t) dt \right|^2 = \left| R_i(\tau, n_i, l_i) + \hat{R}_i(\tau, n_i, l_i) \right|^2 \quad (4.28)$$



where

$$\begin{aligned}
R_i(\tau, n, l) &= (\tau - nT - lT_c) \\
&\cdot \left\{ b_{i, -n-1} \sum_{m=1}^l \overline{s_{i,m}} s_{i, N-l+m} + b_{i, -n} \sum_{m=1}^{N-l} \overline{s_{i, l+m}} s_{i, m} \right\} \\
\hat{R}_i(\tau, n, l) &= (nT + (l+1)T_c - \tau), \\
&\cdot \left\{ b_{i, -n-1} \sum_{m=1}^{l+1} \overline{s_{i,m}} s_{i, N-l+m-1} + b_{i, -n} \sum_{m=1}^{N-l-1} \overline{s_{i, l+m+1}} s_{i, m} \right\}.
\end{aligned} \tag{4.29}$$

Note that  $R_i(\tau, n, l)$  and  $\hat{R}_i(\tau, n, l)$  are expressed as the auto-correlation functions in chip-synchronous CDMA systems. From Eq. (4.29), it is sufficient to consider only two adjacent bits,  $b_{i, -n_i-1}$  and  $b_{i, -n_i}$ . From the independence of each bit  $b_{i, -n_i}$ , Eq. (4.27) can be written as

$$\begin{aligned}
\text{Var}\{F_i\} &\leq \frac{P}{4} \gamma_i^2 C_i \cdot \mathbf{E}_{\mathbf{b}_i} \left\{ \int_0^{M_i T} \Gamma_i(\tau) d\tau \right\} \\
&= \frac{P}{4} \gamma_i^2 C_i M_i \cdot \mathbf{E}_{\mathbf{b}_i} \left\{ \sum_{l_i=0}^{N-1} \int_{l_i T_c}^{(l_i+1)T_c} \Gamma_i(\tau, 0, l_i) d\tau \right\},
\end{aligned} \tag{4.30}$$

where

$$\Gamma_i(\tau, n, l) = \left| R_i(\tau, n, l) + \hat{R}_i(\tau, n, l) \right|^2.$$

Since  $P\gamma_i^2 C_i M_i$  is a constant, it is sufficient for reducing the upper bound of  $\text{Var}\{F_i\}$  to focus on the sum term in the right hand side of Eq. (4.30).

Similar to the fading terms, we evaluate the interference term  $I_i$ . The variance of  $I_i$  is written as

$$\text{Var}\{I_i\} = \frac{P}{4} \sum_{\substack{k=1 \\ k \neq i}}^K \left[ \gamma_k^2 \text{Var}\{|\tilde{I}_{i,k}|\} + \text{Var}\{|\tilde{I}'_{i,k}|\} \right]. \tag{4.31}$$

In the above equation, we have used Eq. (4.12) and Eq. (4.23). It is clear that

$$E_{\psi_k} \left\{ \text{Re} \left[ \left( \tilde{I}'_{i,k} \right)^2 \right] \right\} = 0. \tag{4.32}$$

In Eq. (4.31),  $\tilde{I}_{i,k}$  is the faded interference term and  $\tilde{I}'_{i,k}$  is the term of a direct wave. With Eq. (4.20), the variances of them are expressed as

$$\begin{aligned}
\text{Var}\{|\tilde{I}_{i,k}|\} &= \mathbf{E}_{\mathbf{b}_k, \tau_k} \left\{ \int_{-\infty}^{\infty} \int_0^T \int_0^T g_k(\tau) \right. \\
&\quad \cdot \xi_{i,k,k}(\tau + \tau_k, \tau + \tau_k; t_1, t_2) dt_1 dt_2 d\tau \Big\}, \\
\text{Var}\{|\tilde{I}'_{i,k}|\} &= \mathbf{E}_{\mathbf{b}_k, \tau_k} \left\{ \int_0^T \int_0^T \xi_{i,k,k}(\tau_k, \tau_k; t_1, t_2) dt_1 dt_2 \right\}.
\end{aligned} \tag{4.33}$$

From the assumptions 6 and 7,  $\tau_k$  satisfies that  $l_k T_c \leq \tau_k < (l_k + 1)T_c$ , where  $l_k$  ( $0 \leq l_k < N$ ) is an integer. The double integral term is written as

$$\begin{aligned} & \int_0^T \int_0^T \xi_{i,k,k}(\tau_k, \tau_k; t_1, t_2) dt_1 dt_2 \\ &= \left| R_{i,k}(\tau_k, 0, l_k) + \hat{R}_{i,k}(\tau_k, 0, l_k) \right|^2, \end{aligned} \quad (4.34)$$

where

$$\begin{aligned} R_{i,k}(\tau, n, l) &= (\tau - nT - lT_c) \\ &\cdot \left\{ b_{k,-n-1} \sum_{m=1}^l \overline{s_{i,m}} s_{k,N-l+m} + b_{k,-n} \sum_{m=1}^{N-l} \overline{s_{i,l+m}} s_{k,m} \right\}, \\ \hat{R}_{i,k}(\tau, n, l) &= (nT + (l+1)T_c - \tau) \\ &\cdot \left\{ b_{k,-n-1} \sum_{m=1}^{l+1} \overline{s_{i,m}} s_{k,N-l+m-1} + b_{k,-n} \sum_{m=1}^{N-l-1} \overline{s_{i,l+m+1}} s_{k,m} \right\}. \end{aligned} \quad (4.35)$$

Note that  $R_{i,k}(\tau, n, l)$  and  $\hat{R}_{i,k}(\tau, n, l)$  are cross-correlation functions in chip-synchronous CDMA systems. We define

$$\Gamma_{i,k}(\tau, n, l) = \left| R_{i,k}(\tau, n, l) + \hat{R}_{i,k}(\tau, n, l) \right|^2, \quad (4.36)$$

so that  $\text{Var}\{|\tilde{I}'_{i,k}|\}$  is concisely written as

$$\text{Var}\{|\tilde{I}'_{i,k}|\} = \frac{1}{T} \cdot \mathbb{E}_{\mathbf{b}_k} \left\{ \sum_{l_k=0}^{N-1} \int_{l_k T_c}^{(l_k+1)T_c} \Gamma_{i,k}(\tau_k, 0, l_k) d\tau_k \right\}. \quad (4.37)$$

We consider the variance of  $|\tilde{I}_{i,k}|$ . Similar to the faded signal term, from the assumption 6,  $\tau$  satisfies that  $n'_k T + l'_k T_c \leq \tau < n'_k T + (l'_k + 1)T_c$ , where  $l'_k$  and  $n'_k$  are the integers which satisfy  $0 \leq l'_k < N$  and  $n'_k \geq 0$ . Since each bit  $b_{k,-n}$  is independent, it is sufficient to consider only the two adjacent bits. In other words, it is sufficient to consider only the bits  $b_{k,-n'_k}$  and  $b_{k,-n'_k-1}$ . Thus, taking average over  $\tau_k$ , we obtain

$$\begin{aligned} & \frac{1}{T} \cdot \mathbb{E}_{\mathbf{b}_k} \left\{ \int_0^T \int_0^T \int_0^T \xi_{i,k,k}(\tau + \tau_k, \tau + \tau_k; t_1, t_2) dt_1 dt_2 d\tau_k \right\} \\ &= \frac{1}{T} \cdot \mathbb{E}_{\mathbf{b}_k} \left\{ \sum_{l_k=0}^{N-1} \int_{l_k T_c}^{(l_k+1)T_c} \Gamma_{i,k}(\tau_k, n'_k, l_k) d\tau_k \right\} = \text{Var}\{|\tilde{I}'_{i,k}|\}. \end{aligned} \quad (4.38)$$

In the above equations, we have set  $n'_k = 0$  to obtain the last equality. Then, we can express  $\text{Var}\{|\tilde{I}_{i,k}|\}$  as the product of  $\text{Var}\{|\tilde{I}'_{i,k}|\}$  and the integral covariance term. With the above results, we have

$$\text{Var}\{|\tilde{I}_{i,k}|\} = \frac{1}{T} L_k \cdot \mathbb{E}_{\mathbf{b}_k} \left\{ \sum_{l_k=0}^{N-1} \int_{l_k T_c}^{(l_k+1)T_c} \Gamma_{i,k}(\tau_k, 0, l_k) d\tau_k \right\}, \quad (4.39)$$

where

$$L_k = \int_{-\infty}^{\infty} g_k(\tau) d\tau = \int_0^{M_k T} g_k(\tau) d\tau. \quad (4.40)$$

In the worst case for  $\text{Var}\{F_i\}$ , where  $g_i(\tau)$  is the rectangular function,  $L_k$  is written as

$$L_k = M_k C_k T. \quad (4.41)$$

From these calculations, the variance of  $I_i$  is

$$\begin{aligned} \text{Var}\{I_i\} = & \frac{P}{4T} \sum_{\substack{k=1 \\ k \neq i}}^K (1 + \gamma_k^2 L_k) \\ & \cdot \mathbb{E}_{\mathbf{b}_k} \left\{ \sum_{l_k=0}^{N-1} \int_{l_k T_c}^{(l_k+1)T_c} \Gamma_{i,k}(\tau_k, 0, l_k) d\tau_k \right\}. \end{aligned} \quad (4.42)$$

To increase the lower bound of SINR, it is necessary to reduce the sum and integral term since  $(1 + \gamma_k^2 L_k)$  is constant.

## IV Proposed Expression of SINR

In this section, our goal is to calculate Eq. (4.30) and Eq. (4.42) and to derive an expression of the upper bound of SINR, which does not have real-part operators. In [29] [30], it has been shown that the correlation of a chip-synchronous CDMA system can be written in a quadratic form. With this expression, Eq. (4.29) is rewritten as

$$\begin{aligned} R_i(\tau, n, l) &= (\tau - nT - lT_c) \mathbf{s}_i^* B_{b_{i,-n-1}, b_{i,-n}}^{(l)} \mathbf{s}_i, \\ \hat{R}_i(\tau, n, l) &= (nT + (l+1)T_c - \tau) \mathbf{s}_i^* B_{b_{i,-n-1}, b_{i,-n}}^{(l+1)} \mathbf{s}_i, \end{aligned} \quad (4.43)$$

where  $\mathbf{z}^*$  is a complex conjugate transpose of  $\mathbf{z}$ ,

$$\begin{aligned} \mathbf{s}_k &= (s_{k,1}, s_{k,2}, \dots, s_{k,N})^T, \\ B_{b_{k,n-1}, b_{k,n}}^{(l)} &= \begin{pmatrix} O & b_{k,n-1} E_l \\ b_{k,n} E_{N-l} & O \end{pmatrix}. \end{aligned} \quad (4.44)$$

In the above equations,  $\mathbf{s}^T$  is the transpose of  $\mathbf{s}$ ,  $E_l$  is the identity matrix of size  $l$  and  $O$  is a zero matrix. Similar to Eq. (4.43), Eq. (4.35) is rewritten as

$$\begin{aligned} R_{i,k}(\tau, n, l) &= (\tau - nT - lT_c) \mathbf{s}_i^* B_{b_{k,-n-1}, b_{k,-n}}^{(l)} \mathbf{s}_k, \\ \hat{R}_{i,k}(\tau, n, l) &= (nT + (l+1)T_c - \tau) \mathbf{s}_i^* B_{b_{k,-n-1}, b_{k,-n}}^{(l+1)} \mathbf{s}_k. \end{aligned} \quad (4.45)$$

Thus, we can rewrite  $\Gamma_{i,k}(\tau_k, 0, l_k)$  in Eq. (4.42) as

$$\begin{aligned} & \Gamma_{i,k}(\tau_k, 0, l_k) \\ &= \left| (\tau_k - l_k T_c) \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k)} \mathbf{s}_k + ((l_k + 1)T_c - \tau_k) \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k+1)} \mathbf{s}_k \right|^2 \\ &= (\tau_k - l_k T_c)^2 \left| \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k)} \mathbf{s}_k \right|^2 + ((l_k + 1)T_c - \tau_k)^2 \left| \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k+1)} \mathbf{s}_k \right|^2 \\ &+ 2(\tau_k - l_k T_c)((l_k + 1)T_c - \tau_k) \\ &\cdot \text{Re} \left[ \left( \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k)} \mathbf{s}_k \right) \overline{\left( \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k+1)} \mathbf{s}_k \right)} \right]. \end{aligned} \quad (4.46)$$

Replacing  $k$  with  $i$ , we obtain the expression of  $\Gamma_i(\tau, 0, l_i)$  in Eq. (4.30).

Let us define  $\boldsymbol{\alpha}^{(k)}$  and  $\boldsymbol{\beta}^{(k)}$  as [30]

$$\boldsymbol{\alpha}^{(k)} = V \mathbf{s}_k, \boldsymbol{\beta}^{(k)} = \hat{V} \mathbf{s}_k, \quad (4.47)$$

where each  $(m, n)$ -th element of  $V$  and  $\hat{V}$  is written as

$$\begin{aligned} V_{m,n} &= \frac{1}{\sqrt{N}} \exp \left( -2\pi j \frac{mn}{N} \right), \\ \hat{V}_{m,n} &= \frac{1}{\sqrt{N}} \exp \left( -2\pi j n \left( \frac{m}{N} + \frac{1}{2N} \right) \right), \end{aligned} \quad (4.48)$$

respectively. We denote the  $m$ -th elements of  $\boldsymbol{\alpha}^{(k)}$  and  $\boldsymbol{\beta}^{(k)}$  as  $\alpha_m^{(k)}$  and  $\beta_m^{(k)}$ , respectively. Note that the matrices  $B_{1,1}^{(l)}$  and  $B_{-1,1}^{(l)}$  are decomposed as

$$B_{1,1}^{(l)} = V^* \Lambda^{(l)} V, \quad B_{-1,1}^{(l)} = \hat{V}^* \hat{\Lambda}^{(l)} \hat{V}. \quad (4.49)$$

In the above equations,  $\Lambda^{(l)}$  and  $\hat{\Lambda}^{(l)}$  are diagonal matrices with the eigenvalues of  $B_{1,1}^{(l)}$  and  $B_{-1,1}^{(l)}$ , respectively. The  $(m, m)$ -th elements of  $\Lambda^{(l)}$  and  $\hat{\Lambda}^{(l)}$  are written as

$$\lambda_m^{(l)} = \exp \left( -2\pi j l \frac{m}{N} \right), \quad \hat{\lambda}_m^{(l)} = \exp \left( -2\pi j l \left( \frac{m}{N} + \frac{1}{2N} \right) \right), \quad (4.50)$$

respectively. Thus, the four types of the correlation  $\mathbf{s}_i^* B_{b_k, -1, b_k, 0}^{(l_k)} \mathbf{s}_k$  are expressed as

$$\begin{aligned} \mathbf{s}_i^* B_{1,1}^{(l)} \mathbf{s}_k &= -\mathbf{s}_i^* B_{-1,-1}^{(l)} \mathbf{s}_k = \sum_{m=1}^N \lambda_m^{(l)} \overline{\alpha_m^{(i)}} \alpha_m^{(k)}, \\ \mathbf{s}_i^* B_{-1,1}^{(l)} \mathbf{s}_k &= -\mathbf{s}_i^* B_{1,-1}^{(l)} \mathbf{s}_k = \sum_{m=1}^N \hat{\lambda}_m^{(l)} \overline{\beta_m^{(i)}} \beta_m^{(k)}. \end{aligned} \quad (4.51)$$

With these expressions, we calculate Eq. (4.30) and Eq. (4.42). First, calculating the integral of  $\Gamma_{i,k}(\tau_k, 0, l_k)$ , we have

$$\begin{aligned} & \int_{l_k T_c}^{(l_k+1)T_c} \Gamma_{i,k}(\tau_k, 0, l_k) d\tau_k \\ &= \frac{1}{3} T_c^3 \left| \mathbf{s}_i^* B_{b_k, -1, b_k, 0}^{(l_k)} \mathbf{s}_k \right|^2 + \frac{1}{3} T_c^3 \left| \mathbf{s}_i^* B_{b_k, -1, b_k, 0}^{(l_k+1)} \mathbf{s}_k \right|^2 \\ &+ \frac{1}{3} T_c^3 \operatorname{Re} \left[ \left( \mathbf{s}_i^* B_{b_k, -1, b_k, 0}^{(l_k)} \mathbf{s}_k \right) \overline{\left( \mathbf{s}_i^* B_{b_k, -1, b_k, 0}^{(l_k+1)} \mathbf{s}_k \right)} \right]. \end{aligned} \quad (4.52)$$

When we take the average of each term in Eq. (4.52) over the bits  $b_{k,-1}$  and

$b_{k,0}$ , the each resultant averaged quantity is expressed as

$$\begin{aligned}
& \mathbb{E}_{\mathbf{b}_k} \left\{ \left| \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k)} \mathbf{s}_k \right|^2 \right\} \\
&= \frac{1}{2} \left\{ \left| \sum_{m=1}^N \lambda_m^{(l_k)} \overline{\alpha_m^{(i)}} \alpha_m^{(k)} \right|^2 + \left| \sum_{m=1}^N \hat{\lambda}_m^{(l_k)} \overline{\beta_m^{(i)}} \beta_m^{(k)} \right|^2 \right\}, \\
& \mathbb{E}_{\mathbf{b}_k} \left\{ \left| \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k+1)} \mathbf{s}_k \right|^2 \right\} \\
&= \frac{1}{2} \left\{ \left| \sum_{m=1}^N \lambda_m^{(l_k+1)} \overline{\alpha_m^{(i)}} \alpha_m^{(k)} \right|^2 + \left| \sum_{m=1}^N \hat{\lambda}_m^{(l_k+1)} \overline{\beta_m^{(i)}} \beta_m^{(k)} \right|^2 \right\}, \quad (4.53) \\
& \mathbb{E}_{\mathbf{b}_k} \left\{ \operatorname{Re} \left[ \left( \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k)} \mathbf{s}_k \right) \overline{\left( \mathbf{s}_i^* B_{b_{k,-1}, b_{k,0}}^{(l_k+1)} \mathbf{s}_k \right)} \right] \right\} \\
&= \frac{1}{2} \operatorname{Re} \left[ \left( \sum_{m=1}^N \lambda_m^{(l_k)} \overline{\alpha_m^{(i)}} \alpha_m^{(k)} \right) \overline{\left( \sum_{m'=1}^N \lambda_{m'}^{(l_k+1)} \overline{\alpha_{m'}^{(i)}} \alpha_{m'}^{(k)} \right)} \right] \\
&+ \frac{1}{2} \operatorname{Re} \left[ \left( \sum_{m=1}^N \hat{\lambda}_m^{(l_k)} \overline{\beta_m^{(i)}} \beta_m^{(k)} \right) \overline{\left( \sum_{m'=1}^N \hat{\lambda}_{m'}^{(l_k+1)} \overline{\beta_{m'}^{(i)}} \beta_{m'}^{(k)} \right)} \right].
\end{aligned}$$

With these expressions, we can calculate the sum and integral term of Eq. (4.42) as follows. Taking the sum of Eqs. (4.53) over  $l_k$ , we rewrite Eq. (4.42) as

$$\operatorname{Var}\{I_i\} = \frac{PT^2}{12N^2} \sum_{\substack{k=1 \\ k \neq i}}^K (1 + \gamma_k^2 L_k) \sum_{m=1}^N S_m^{i,k}, \quad (4.54)$$

where

$$\begin{aligned}
S_m^{i,k} &= \left| \alpha_m^{(i)} \right|^2 \left| \alpha_m^{(k)} \right|^2 \left( 1 + \frac{1}{2} \cos \left( 2\pi \frac{m}{N} \right) \right) \\
&+ \left| \beta_m^{(i)} \right|^2 \left| \beta_m^{(k)} \right|^2 \left( 1 + \frac{1}{2} \cos \left( 2\pi \left( \frac{m}{N} + \frac{1}{2N} \right) \right) \right). \quad (4.55)
\end{aligned}$$

Replacing  $k$  with  $i$ , we obtain the expression of  $\Gamma_i(\tau, 0, l_i)$ , which is defined in Eq. (4.30). Then, we find that Eq. (4.30) is rewritten with  $S_m^{i,k}$  as

$$\operatorname{Var}\{F_i\} \leq \frac{PT^3}{12N^2} \gamma_i^2 C_i M_i \sum_{m=1}^N S_m^{i,i}. \quad (4.56)$$

From the above expressions, we arrive at the formula for the lower bound of SINR of the user  $i$

$$\operatorname{SINR}_i \geq \left\{ \frac{1}{6N^2} \sum_{k=1}^K Z_{i,k} \sum_{m=1}^M S_m^{i,k} + \frac{N_0}{2PT} \right\}^{-1/2}, \quad (4.57)$$

where

$$Z_{i,k} = \begin{cases} \gamma_i^2 C_i M_i T & i = k \\ 1 + \gamma_k^2 L_k & i \neq k \end{cases}.$$

When  $Z_{i,k} = 1$  ( $i \neq k$ ) and  $Z_{i,i} = 0$ , Eq. (4.57) is equivalent to the expression in [45] and [13].

The first term of Eq. (4.55) is related to the periodic correlation since the coefficients  $\alpha_m^{(k)}$  appear in the periodic correlation. The last term of Eq. (4.55) is related to the odd periodic correlation since the coefficients  $\beta_m^{(k)}$  appear in the odd periodic correlation. In Section V, we discuss the relation between SINR and correlation in detail.

## V Relation between SINR and Mean-Square Correlation

We show the relation between the expression of SINR and the mean-square correlations. The mean-square correlations are proposed as indices for performance of spreading sequences. In particular, the mean-square cross-correlation is used for advantage of spreading sequences [84]. The mean-square cross-correlation function  $R_{CC}$  and the mean-square auto-correlation function  $R_{AC}$  are defined as [50]

$$R_{CC} = \frac{1}{K} \sum_{i=1}^K R_{CC}^{(i)}, \quad R_{AC} = \frac{1}{K} \sum_{i=1}^K R_{AC}^{(i)}, \quad (4.58)$$

where

$$R_{CC}^{(i)} = \frac{1}{(K-1)} \frac{1}{N^2} \sum_{\substack{k=1 \\ k \neq i}}^K \sum_{l=1-N}^{N-1} |C_{i,k}(l)|^2, \quad (4.59)$$

$$R_{AC}^{(i)} = \frac{1}{N^2} \sum_{l=1-N, l \neq 0}^{N-1} |C_i(l)|^2,$$

$$C_{i,k}(l) = \begin{cases} \sum_{n=1}^{N-l} \overline{s_{i,n+l}} s_{k,n} & 0 \leq l \leq N-1 \\ \sum_{n=1}^{N+l} \overline{s_{i,n}} s_{k,n-l} & 1-N \leq l < 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.60)$$

and  $C_i(l)$  denotes  $C_{i,i}(l)$ . From [84], Eq. (4.58) is rewritten as

$$R_{CC}^{(i)} = \frac{1}{2(K-1)} \frac{1}{N^2} \sum_{\substack{k=1 \\ k \neq i}}^K \sum_{l=0}^{N-1} \left\{ |\theta_{i,k}(l)|^2 + |\hat{\theta}_{i,k}(l)|^2 \right\}, \quad (4.61)$$

$$R_{AC}^{(i)} = \frac{1}{2N^2} \sum_{l=1}^{N-1} \left\{ |\theta_i(l)|^2 + |\hat{\theta}_i(l)|^2 \right\},$$

where  $\theta_{i,k}(l)$  and  $\hat{\theta}_{i,k}(l)$  are the periodic correlation function and the odd periodic correlation function, respectively. These correlation functions are defined as

$$\theta_{i,k}(l) = C_{i,k}(l) + C_{i,k}(l-N), \quad \hat{\theta}_{i,k}(l) = C_{i,k}(l) - C_{i,k}(l-N), \quad (4.62)$$

and  $\theta_i(l)$  and  $\hat{\theta}_i(l)$  denote  $\theta_{i,i}(l)$  and  $\hat{\theta}_{i,i}(l)$ , respectively. From [30],  $\theta_{i,k}(l)$  and  $\hat{\theta}_{i,k}(l)$  are expressed as

$$\theta_{i,k}(l) = \sum_{m=1}^N \lambda_m^{(l)} \overline{\alpha_m^{(i)}} \alpha_m^{(k)}, \quad \hat{\theta}_{i,k}(l) = \sum_{m=1}^N \hat{\lambda}_m^{(l)} \overline{\beta_m^{(i)}} \beta_m^{(k)}. \quad (4.63)$$

With these expressions, Eq. (4.59) is rewritten as

$$R_{CC}^{(i)} = \frac{1}{2(K-1)} \frac{1}{N} \cdot \sum_{\substack{k=1 \\ k \neq i}}^K \sum_{m=1}^N \left\{ |\alpha_m^{(i)}|^2 |\alpha_m^{(k)}|^2 + |\beta_m^{(i)}|^2 |\beta_m^{(k)}|^2 \right\}, \quad (4.64)$$

$$R_{AC}^{(i)} = \frac{1}{2N} \sum_{m=1}^N \left\{ |\alpha_m^{(i)}|^4 + |\beta_m^{(i)}|^4 \right\} - 1. \quad (4.65)$$

In obtaining the above equations, we have used the relation that  $|\theta_{i,k}(0)|^2 = |\hat{\theta}_{i,k}(0)|^2 = N^2$ . Thus, we derive the bounds of Eq. (4.57) as

$$\begin{aligned} & \left[ \frac{1}{2N} \left\{ Z_{i,i}(R_{AC}^{(i)} + 1) + Z_{i,U}(K-1)R_{CC}^{(i)} \right\} + \frac{N_0}{2PT} \right]^{-1/2} \\ & \leq \left\{ \frac{1}{6N^2} \sum_{k=1}^K Z_{i,k} \sum_{m=1}^M S_m^{i,k} + \frac{N_0}{2PT} \right\}^{-1/2} \\ & \leq \left[ \frac{1}{6N} \left\{ Z_{i,i}(R_{AC}^{(i)} + 1) + Z_{i,L}(K-1)R_{CC}^{(i)} \right\} + \frac{N_0}{2PT} \right]^{-1/2}, \end{aligned} \quad (4.66)$$

where  $Z_{i,U} = \max_{k \neq i} Z_{i,k}$  and  $Z_{i,L} = \min_{k \neq i} Z_{i,k}$ . In the above inequality, we have used  $-1 \leq \cos(x) \leq 1$ . Note that the term of  $R_{AC}^{(i)}$  corresponds to the faded signal term obtained in Eq. (4.56). Moreover, the term of  $R_{CC}^{(i)}$  corresponds to the interference term obtained in Eq. (4.54). Therefore, Eq. (4.66) shows that the effects of faded signals and multiple-access interference are reduced when the mean-square correlations  $R_{AC}^{(i)}$  and  $R_{CC}^{(i)}$  get small, respectively. Thus, it is necessary for designing spreading sequences to reduce mean-square correlations,  $R_{AC}^{(i)}$  and  $R_{CC}^{(i)}$ .

## VI Optimization Problem for SINR

In this section, our goal is to derive the optimization problem which maximizes SINR for the worst case. Here, the parameters  $N$ ,  $K$ ,  $Z_{i,k}$  are fixed. We treat  $Z_{i,k}$  as the weights among all the users. Therefore, the parameters  $Z_{i,k}$  are supposed to satisfy a following condition:

$$Z_{i,k} = \begin{cases} Z_{AC} & i = k \\ Z_{CC} & i \neq k \end{cases}, \quad (4.67)$$

where  $Z_{AC}$  and  $Z_{CC}$  are non-negative real parameters corresponding to auto-correlation and cross-correlation, respectively. We ignore the Gaussian noise term since it has no relation to spreading sequences. To maximize the lower

bound of SINR of the user  $i$ , we should minimize the first term of the denominator of Eq. (4.57). We consider the optimization problem ( $\tilde{P}$ ):

$$\begin{aligned}
(\tilde{P}) \quad & \min \sum_{k=1}^K Z_{i,k} \sum_{m=1}^N S_m^{i,k} \\
\text{subject to} \quad & \boldsymbol{\alpha}^{(k)} = V \mathbf{s}_k, \quad \boldsymbol{\beta}^{(k)} = \hat{V} \mathbf{s}_k \quad (k = 1, 2, \dots, K), \\
& \left\| \boldsymbol{\alpha}^{(k)} \right\|^2 = \left\| \boldsymbol{\beta}^{(k)} \right\|^2 = N \quad (k = 1, 2, \dots, K),
\end{aligned} \tag{4.68}$$

where  $\|\mathbf{x}\|$  is the Euclidean norm of the vector  $\mathbf{x}$ .

In the problem ( $\tilde{P}$ ), we take into account only the user  $i$ . For the purpose of practical applications, we should consider all the users for designing spreading sequences. In this case, we consider the problem that we maximize the sum of  $\text{SINR}_i$ . However, it is not straightforward to obtain its solution since its objective function is complicated. To overcome such a difficulty, we consider the following problem ( $\tilde{P}'$ ).

$$\begin{aligned}
(\tilde{P}') \quad & \min \sum_{i=1}^K \sum_{k=1}^K Z_{i,k} \sum_{m=1}^N S_m^{i,k} \\
\text{subject to} \quad & \boldsymbol{\alpha}^{(k)} = V \mathbf{s}_k, \quad \boldsymbol{\beta}^{(k)} = \hat{V} \mathbf{s}_k \quad (k = 1, 2, \dots, K), \\
& \left\| \boldsymbol{\alpha}^{(k)} \right\|^2 = \left\| \boldsymbol{\beta}^{(k)} \right\|^2 = N \quad (k = 1, 2, \dots, K).
\end{aligned} \tag{4.69}$$

Note that there is a relation between the problem ( $\tilde{P}$ ) and ( $\tilde{P}'$ ):

$$\frac{1}{K} \sum_{i=1}^K \text{SINR}_i \geq \left[ \frac{1}{\sum_{i=1}^K \{\text{Denom}(\text{SINR}_i)\}^2} \right]^{1/2}, \tag{4.70}$$

where  $\text{Denom}\{\text{SINR}_i\}$  is the denominator of  $\text{SINR}_i$ . This result is obtained from the relation between the arithmetic mean and the harmonic mean. In the problem ( $\tilde{P}'$ ), we evaluate a lower bound of the sum of  $\text{SINR}_i$ .

With Eq. (4.47), the element of the objective function in the problem ( $\tilde{P}'$ ) is rewritten as

$$S_m^{i,k} = (\mathbf{s}_i^* Q_m \mathbf{s}_i) (\mathbf{s}_k^* Q_m \mathbf{s}_k) + \left( \mathbf{s}_i^* \hat{Q}_m \mathbf{s}_i \right) \left( \mathbf{s}_k^* \hat{Q}_m \mathbf{s}_k \right), \tag{4.71}$$

where  $Q_m$  and  $\hat{Q}_m$  are the symmetric positive semidefinite matrices written as

$$Q_m = V^* C_m V, \quad \hat{Q}_m = \hat{V}^* \hat{C}_m \hat{V}. \tag{4.72}$$

In the above equations,  $C_m$  and  $\hat{C}_m$  are the matrices whose  $(m, m)$ -th elements are given by

$$\begin{aligned}
(C_m)_{m,m} &= \sqrt{1 + \frac{1}{2} \cos \left( 2\pi \frac{m}{N} \right)}, \\
(\hat{C}_m)_{m,m} &= \sqrt{1 + \frac{1}{2} \cos \left( 2\pi \left( \frac{m}{N} + \frac{1}{2N} \right) \right)},
\end{aligned} \tag{4.73}$$

and the other elements are zero.



With the above expressions, the problem  $(\tilde{P})'$  is rewritten as the problem  $(P)$ :

$$(P) \quad \min \sum_{i=1}^K \sum_{k=1}^K Z_{i,k} \sum_{m=1}^N S_m^{i,k} \quad (4.74)$$

subject to  $\|\mathbf{s}_k\|^2 = N \quad (k = 1, 2, \dots, K).$

Note that the problem  $(P)$  has the variables  $\mathbf{s}_k$ , not  $\boldsymbol{\alpha}_k$  and  $\boldsymbol{\beta}_k$ .

## VII Semidefinite Relaxation

The problem  $(P)$  is not convex since its objective function is not convex and equality constraints are not linear functions. In this section, with a semidefinite relaxation (SDR) technique [78], we transform the problem  $(P)$  to a problem whose constraints are linear functions. Here, it is known that the global solution of the relaxed problem gives the lower bound of the original problem. Moreover, we show that the relaxed problem is convex in certain parameter regions. In the other parameter regions, the relaxed problem is a generalized convex multiplicative programming problem [79]. There is an algorithm to solve it. However, it is expected to take a lot of computational time to solve the problem because of the large number of the terms.

The elements of the objective function in the problem  $(P)$  is rewritten as

$$S_m^{i,k} = \text{Tr}(Q_m \mathbf{s}_i \mathbf{s}_i^*) \text{Tr}(Q_m \mathbf{s}_k \mathbf{s}_k^*) + \text{Tr}(\hat{Q}_m \mathbf{s}_i \mathbf{s}_i^*) \text{Tr}(\hat{Q}_m \mathbf{s}_k \mathbf{s}_k^*), \quad (4.75)$$

where  $\text{Tr}(X)$  is the trace of  $X$ . Let us define  $X_k = \mathbf{s}_k \mathbf{s}_k^*$ . Then, the matrix  $X_k$  is a Hermitian positive semidefinite matrix since  $X_k$  is a Gram matrix. With this expression, Eq. (4.75) is rewritten as

$$S_m^{i,k} = \text{Tr}(Q_m X_i) \text{Tr}(Q_m X_k) + \text{Tr}(\hat{Q}_m X_i) \text{Tr}(\hat{Q}_m X_k). \quad (4.76)$$

Note that a Hermitian positive semidefinite matrix  $X_k$  is a rank-one matrix from its definition. Therefore, the problem  $(P)$  is rewritten as

$$(P) \quad \min \sum_{i=1}^K \sum_{k=1}^K Z_{i,k} \sum_{m=1}^N S_m^{i,k}$$

subject to  $\text{Tr}(X_k) = N \quad (k = 1, 2, \dots, K),$  (4.77)

$X_k \succcurlyeq 0 \quad (k = 1, 2, \dots, K),$

$\text{rank}(X_k) = 1 \quad (k = 1, 2, \dots, K).$

Here, we use  $X \succcurlyeq 0$  to indicate that  $X$  is a Hermitian positive semidefinite matrix. It is known [85] that the set of Hermitian positive semidefinite matrices is a convex cone. Therefore, the “difficult constraints” [78] in the problem  $(P)$  are the rank constraints,  $\text{rank}(X_k) = 1$ . These constraints are not convex. Since rank constraints can greatly increase the complexity of the problem in general

[86], we drop the rank constraints to obtain the following relaxed problem

$$\begin{aligned}
(P') \quad & \min \sum_{i=1}^K \sum_{k=1}^K Z_{i,k} \sum_{m=1}^N S_m^{i,k} \\
\text{subject to} \quad & \text{Tr}(X_k) = N \quad (k = 1, 2, \dots, K), \\
& X_k \succeq 0 \quad (k = 1, 2, \dots, K).
\end{aligned} \tag{4.78}$$

In the problem  $(P')$ , all the constraints are convex. Therefore, if the objective function of the problem  $(P')$  is convex, then the problem  $(P')$  is convex. In Appendix B, we prove that the objective function of the problem  $(P')$  is convex when the condition  $Z_{AC} \geq Z_{CC}$  is satisfied. Because of the convexity, the problem  $(P')$  can be solved with interior point methods. In the other parameter regions (i.e.  $Z_{AC} < Z_{CC}$ ), the problem  $(P')$  is a generalized convex multiplicative programming problem [79]. There is an algorithm to solve a generalized convex multiplicative programming problem. However, it is expected to take a lot of computational time to solve since the objective function of the problem  $(P')$  has the large number of the terms. Therefore, we do not use this algorithm to solve the problem  $(P')$ .

If we obtain the solution of the problem  $(P')$ , then the rank of the solution is not always one. Therefore, it is required to approximate the solution to the rank-one solution when the non-rank-one solution is obtained. Let us define  $X_k^*$  as the solution of the problem  $(P')$  and  $r_k = \text{rank}(X_k^*)$ . Then,  $X_k^*$  is decomposed as

$$X_k^* = \sum_{n=1}^{r_k} \lambda_n^{(k)} \mathbf{q}_n^{(k)} \left( \mathbf{q}_n^{(k)} \right)^*, \tag{4.79}$$

where  $\lambda_1^{(k)} \geq \dots \geq \lambda_{r_k}^{(k)} > 0$  are the eigenvalues and  $\mathbf{q}_1^{(k)}, \dots, \mathbf{q}_{r_k}^{(k)}$  are the corresponding eigenvectors of  $X_k^*$ . In the least two-norm sense, the best rank-one approximation  $X_k^*$  to  $\tilde{X}_{k,1}^*$  is given by

$$\tilde{X}_{k,1}^* = \lambda_1^{(k)} \mathbf{q}_1^{(k)} \left( \mathbf{q}_1^{(k)} \right)^*. \tag{4.80}$$

However, we do not use Eq. (4.80), since the trace conditions in the problem  $(P')$  are necessary. Therefore, as a rank-one approximated solution, we obtain the approximated solution  $X_{k,1}^*$ ,

$$X_{k,1}^* = N \mathbf{q}_1^{(k)} \left( \mathbf{q}_1^{(k)} \right)^*. \tag{4.81}$$

Then, we obtain the approximated solution  $\sqrt{N} \mathbf{q}_1^{(k)}$  as a spreading sequence  $\mathbf{s}_k$ .

## VIII Optimal Sequence and SINR for Desired User in No Fading

In previous sections, we have considered all the users in channels. In this section, we consider the only one desired user  $i$  and derive an optimal spreading sequence in no fading situation. Since the optimal spreading sequence is derived, the maximum SINR and the maximum capacity for the user  $i$  are obtained.

In this section, we make the following assumptions

1. there is no fading effect.
2. the spreading sequences for the other users are given and fixed.
3. Gaussian channel noise is given.
4. interference noise follows Gaussian.
5. interference noise is independent of Gaussian channel noise.

From the assumption 1 and Eqs. (4.57) (4.71), when the spreading sequence  $\mathbf{s}_i$  is given, SINR for the user  $i$  is written as

$$\text{SINR}(\mathbf{s}_i)_i = \left\{ \frac{1}{6N^2} \sum_{\substack{k=1 \\ k \neq i}}^K \sum_{m=1}^N S_m^{i,k} + \frac{N_0}{2PT} \right\}^{-1/2}, \quad (4.82)$$

where

$$S_m^{i,k} = (\mathbf{s}_i^* Q_m \mathbf{s}_i) (\mathbf{s}_k^* Q_m \mathbf{s}_k) + (\mathbf{s}_i^* \hat{Q}_m \mathbf{s}_i) (\mathbf{s}_k^* \hat{Q}_m \mathbf{s}_k). \quad (4.83)$$

From assumptions 2 and 3, the above quantity depends on only the  $\mathbf{s}_i$ . Therefore, to maximize SINR, we consider the following optimization problem

$$\begin{aligned} (P_i) \quad & \min \sum_{\substack{k=1 \\ k \neq i}}^K \sum_{m=1}^N (\mathbf{s}_i^* Q_m \mathbf{s}_i) (\mathbf{s}_k^* Q_m \mathbf{s}_k) + (\mathbf{s}_i^* \hat{Q}_m \mathbf{s}_i) (\mathbf{s}_k^* \hat{Q}_m \mathbf{s}_k) \\ & \text{subject to } \|\mathbf{s}_i\|^2 = N. \end{aligned} \quad (4.84)$$

It is clear that maximum SINR is obtained from the above optimization problem.

To analyze the optimization problem, we define the following matrix  $\Sigma$

$$\Sigma = \sum_{\substack{k=1 \\ k \neq i}}^K \sum_{m=1}^N (\mathbf{s}_k^* Q_m \mathbf{s}_k) Q_m + (\mathbf{s}_k^* \hat{Q}_m \mathbf{s}_k) \hat{Q}_m. \quad (4.85)$$

From assumption 2, the above matrix  $\Sigma$  is constant since  $\mathbf{s}_k$  is given and fixed for  $k \neq i$ . Further, the matrix  $\Sigma$  is positive semidefinite since the quantities  $(\mathbf{s}_k^* Q_m \mathbf{s}_k)$  and  $(\mathbf{s}_k^* \hat{Q}_m \mathbf{s}_k)$  are non-negative, and the matrices  $Q_m$  and  $\hat{Q}_m$  are positive semidefinite.

With the matrix  $\Sigma$ , the optimization problem  $(P_i)$  is rewritten as

$$\begin{aligned} (P_i) \quad & \min \mathbf{s}_i^* \Sigma \mathbf{s}_i \\ & \text{subject to } \|\mathbf{s}_i\|^2 = N. \end{aligned} \quad (4.86)$$

Further, the above problem is equivalent to the following one

$$\begin{aligned} (P_i) \quad & \min \frac{\mathbf{s}_i^* \Sigma \mathbf{s}_i}{\|\mathbf{s}_i\|^2 / N} \\ & \text{subject to } \|\mathbf{s}_i\|^2 = N. \end{aligned} \quad (4.87)$$

Let the vector  $\mathbf{u}_i$  be  $\mathbf{u}_i = \frac{1}{\sqrt{N}}\mathbf{s}_i$ . Then, the problem  $(P_i)$  is rewritten as

$$(P_i) \quad \min \quad \frac{N \cdot \mathbf{u}_i^* \Sigma \mathbf{u}_i}{\|\mathbf{u}_i\|^2} \quad (4.88)$$

subject to  $\|\mathbf{u}_i\|^2 = 1$ .

It is obvious that the value of the objective function is invariant under the action  $\mathbf{u}_i \mapsto c\mathbf{u}_i$ , where  $c \in \mathbb{C}$  is a non-zero scalar. Thus, we can eliminate the constraint and obtain the following problem

$$(P'_i) \quad \min \quad \frac{N \cdot \mathbf{u}_i^* \Sigma \mathbf{u}_i}{\|\mathbf{u}_i\|^2}. \quad (4.89)$$

This is the Rayleigh quotient of  $N\Sigma$  [87]. It is known that optimal value coincides with the product of  $N$  and the minimum eigenvalue of  $\Sigma$ ,  $\lambda_{\min} \geq 0$  and that the global minimizer of the problem  $(P_i)$  is the eigenvector corresponding to  $\lambda_{\min}$ . Let  $\mathbf{u}$  be such a minimizer. When the minimizer  $\mathbf{u}$  is normalized as  $\|\mathbf{u}\| = 1$ , the optimal spreading sequence for the user  $i$ ,  $\mathbf{s}_i^*$  is written as

$$\mathbf{s}_i^* = \sqrt{N}\mathbf{u}. \quad (4.90)$$

Then, maximum SINR is written as

$$\text{SINR}_i^* = \text{SINR}(\mathbf{s}_i^*)_i = \left\{ \frac{\lambda_{\min}}{6N} + \frac{N_0}{2PT} \right\}^{-1/2}. \quad (4.91)$$

Further, it is known that the channel capacity is written in terms of Signal-to-Noise Ratio (SNR) if an input is continuous and channel noise is Gaussian [5] [15]. From assumptions 4 and 5, the sum of interference noise and channel noise follows Gaussian since the sum of the two independent Gaussian variables follow Gaussian [88]. However, it is known that the channel capacity with a practical scheme is complicated [16]. To overcome this obstacle, we approximate the maximum channel capacity of the user  $i$  by one with a continuous channel since the channel capacity with BPSK scheme is close to one with a continuous channel in low SNR [16]. With this approximation, from Eq. (4.91), the maximum channel capacity for the user  $i$ ,  $C_i^*$  is written as

$$C_i^* = \frac{1}{2} \log \left[ 1 + \left\{ \frac{\lambda_{\min}}{6N} + \frac{N_0}{2PT} \right\}^{-1} \right]. \quad (4.92)$$

As seen in the above discussions, maximum SINR and the channel capacity depend on the minimum eigenvalue of the matrix  $\Sigma$  and these maximums are achieved with the eigenvector corresponding to the minimum eigenvalue.

We have obtained optimal SINR in asynchronous CDMA systems for the desired user. However, it is not straightforward to obtain its explicit form. We derive the bound of the optimal SINR. As seen in the definition of the matrix  $\Sigma$ , the matrix  $\Sigma$  consists of two kinds of the matrices,  $Q_m$  and  $\hat{Q}_m$ . From Eq. (4.72), the matrices  $Q_m$  and  $\hat{Q}_m$  are rank-1 matrices. Therefore, the matrix  $\Sigma$  is written as

$$\Sigma = V^* \Lambda V + \hat{V}^* \hat{\Lambda} \hat{V}, \quad (4.93)$$

where  $\Lambda$  and  $\hat{\Lambda}$  are diagonal matrices whose  $m$ -th diagonal components,  $\lambda_m$  and  $\hat{\lambda}_m$  are written as

$$\begin{aligned}\lambda_m &= \sqrt{1 + \frac{1}{2} \cos\left(2\pi \frac{m}{N}\right)} \sum_{\substack{k=1 \\ k \neq i}}^K (\mathbf{s}_k^* Q_m \mathbf{s}_k) \\ \hat{\lambda}_m &= \sqrt{1 + \frac{1}{2} \cos\left(2\pi \left(\frac{m}{N} + \frac{1}{2N}\right)\right)} \sum_{\substack{k=1 \\ k \neq i}}^K (\mathbf{s}_k^* \hat{Q}_m \mathbf{s}_k).\end{aligned}\tag{4.94}$$

Thus, the quantities  $\lambda_m$  and  $\hat{\lambda}_m$  are the eigenvalues of the matrices  $V^* \Lambda V$  and  $\hat{V}^* \hat{\Lambda} \hat{V}$ , respectively. Note that the quantities  $\lambda_m$  and  $\hat{\lambda}_m$  depend on the spreading sequences  $\mathbf{s}_k$  for  $k \neq i$ .

With the above eigenvalues, the bounds of the optimal SINR for the user  $i$  are derived. First, we derive the upper bound. As seen in Eq. (4.91), the optimal SINR is written in terms of the minimum eigenvalue of  $\Sigma$ ,  $\lambda_{\min}$ . Since  $\lambda_{\min}$  is the minimizer of the Rayleigh quotient of  $\Sigma$ , the following relations are obtained

$$\begin{aligned}\lambda_{\min} &= \min_{\mathbf{u} \neq \mathbf{0}} \frac{\mathbf{u}^* \Sigma \mathbf{u}}{\|\mathbf{u}\|^2} \\ &= \min_{\mathbf{u} \neq \mathbf{0}} \frac{\mathbf{u}^* \left( V^* \Lambda V + \hat{V}^* \hat{\Lambda} \hat{V} \right) \mathbf{u}}{\|\mathbf{u}\|^2} \\ &\geq \min_{\mathbf{u}_1 \neq \mathbf{0}} \frac{\mathbf{u}_1^* V^* \Lambda V \mathbf{u}_1}{\|\mathbf{u}_1\|^2} + \min_{\mathbf{u}_2 \neq \mathbf{0}} \frac{\mathbf{u}_2^* \hat{V}^* \hat{\Lambda} \hat{V} \mathbf{u}_2}{\|\mathbf{u}_2\|^2} \\ &= \min_m \lambda_m + \min_m \hat{\lambda}_m.\end{aligned}\tag{4.95}$$

In the above discussions, we have used Eq. (4.93). Then, the upper bound of the optimal SINR is written as

$$\left\{ \frac{6}{N} (\min_m \lambda_m + \min_m \hat{\lambda}_m) + \frac{N_0}{2PT} \right\}^{-1/2} \geq \text{SINR}_i^*.\tag{4.96}$$

On the other hand, to derive the lower bound of the optimal SINR, we use the following theorem [89].

**Theorem (Weyl).** Let  $A$  and  $B$  be the  $n \times n$  Hermitian matrices whose eigenvalues are written as  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$  and  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$ , respectively. Further, we define the Hermitian matrix  $C = A + B$  whose eigenvalues are written as  $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n$ . Then, the following relation holds for  $k + l - 1 \leq n$

$$\gamma_{k+l-1} \leq \alpha_k + \beta_l.\tag{4.97}$$

The eigenvalues of sums of Hermitian matrices have been investigated in [90] [91] [92]. From the above theorem, there is a following inequality

$$\lambda_{\min} \leq \min \left\{ \min_m \lambda_m + \max_m \hat{\lambda}_m, \max_m \lambda_m + \min_m \hat{\lambda}_m \right\}.\tag{4.98}$$

The above result is obtained when we set  $k = n$  and  $l = 1$  in the theorem. Then, the optimal SINR has a lower bound

$$\left\{ \frac{6}{N} \min \left\{ \min_m \lambda_m + \max_m \hat{\lambda}_m, \max_m \lambda_m + \min_m \hat{\lambda}_m \right\} + \frac{N_0}{2PT} \right\}^{-1/2} \leq \text{SINR}_i^*. \quad (4.99)$$

Therefore, from Eqs. (4.95) and (4.98), the optimal SINR relates to the quantities  $\lambda_m$  and  $\hat{\lambda}_m$ . These results imply that SINR is improved if the quantities  $\lambda_m$  and  $\hat{\lambda}_m$  is reduced.

## IX Numerical Results

We numerically solve the problem ( $P'$ ) with the MATLAB package, PENLAB [93]. In the problem ( $P'$ ), the matrix variables are complex Hermitian matrices. Since PENLAB cannot handle Hermitian matrices, we transform the Hermitian matrices to symmetric matrices [94] [95]. The properties of the transformation are shown in [94] and [95]. Then, the problem ( $P'$ ) is divided into the two cases, one is convex and the other is not convex.

In the parameter regions where  $Z_{AC} \geq Z_{CC}$ , the problem ( $P'$ ) is convex. Then, we find that the optimal solutions of the problem ( $P'$ ) is the same, that is,

$$X_1^* = X_2^* = \cdots = X_K^*. \quad (4.100)$$

In this situation, all the sequences as approximated solutions are the same as Eq. (4.81). Therefore, we obtain each approximated solution as the eigenvectors corresponding to the first  $K$  largest eigenvalues. In the other parameter regions, the problem ( $P'$ ) is not convex. Here, we should notice the following two. First, it is unclear whether our solutions are the global solutions or not. Second, since each of these solutions has one large eigenvalue, we obtain approximated solutions from their eigenvectors.

In each simulation, the system is a binary phase shift keying (BPSK) system. This system is described in Section II. The number of users  $K = 5$ , and the length of sequences  $N = 31$  are fixed. The receiver is a correlation receiver. As compared sequences, the Gold code [38], the second-degree Chebyshev optimal sequences [42], the Oppermann sequences [50], and the chaotic bit sequences (CBS) [67] [66] are used. These sequences are randomly chosen from their sets. In particular, as the triple of the Oppermann sequences,  $\{p, q, r\} = \{1.0, 1.0, 1.275\}$  is chosen. This triple is shown in [50] as the optimal parameters when  $N = 31$  with  $N$  being the length. As the chaotic bit sequences, we choose the second-degree Chebyshev map, and 14-th bit. To obtain sequences from the problem ( $P'$ ), we choose the six types of parameters,  $(Z_{AC}, Z_{CC}) = (0, 1), (1, 0.2), (1, 0.5), (1, 1), (1, 2)$  and  $(1, 5)$ . We measure the average BER, which is defined as

$$\text{BER} = \frac{1}{K} \frac{1}{U} \sum_{k=1}^K \sum_{u=1}^U \text{BER}_{k,u}, \quad (4.101)$$

where  $U$  is the trial numbers,  $u$  is the  $u$ -th trial number and  $\text{BER}_{k,u}$  is the BER of the user  $k$  at the  $u$ -th trial. In our simulations, we set  $U = 1000$ .

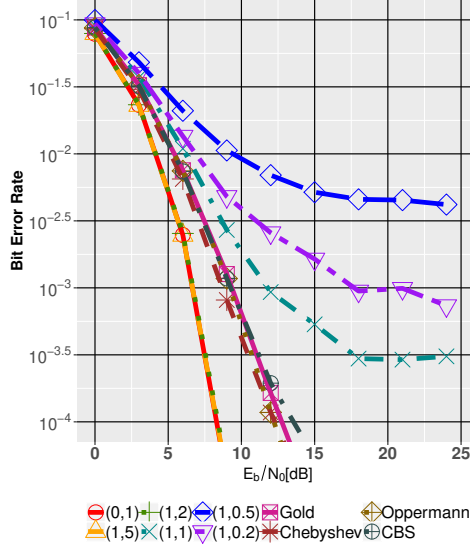


Figure 4.1: Comparison of Bit Error Rate among  $(Z_{AC}, Z_{CC})$ , and known sequences: no fading,  $N = 31$ ,  $K = 5$ , ©2018 IEEE

Figure A.2 shows the BER in no fading environments. In Fig A.2,  $E_b$  denotes the average energy per bit at the receiver which is need for a reliable recovery of the information. The sequences obtained in the parameter regions  $Z_{AC} < Z_{CC}$  achieve the lowest BER in all the sequences. Since each BER of sequences satisfying  $Z_{AC} < Z_{CC}$  is almost same, we conclude that they have the same resistance against interference noise. However, if the parameters satisfy  $Z_{AC} \geq Z_{CC}$ , then BER gets high. This result indicates that the approximated solutions are not suitable when  $Z_{AC} \geq Z_{CC}$ .

Figure 4.2 shows the BER result in fading environments. For each user, there is one direct signal. The multipath delays and the number of multipath components are uniformly distributed in  $[0, 3T]$  and  $[0, 9]$ , respectively. Each power of delayed signals is the same and satisfies  $P_d/P = -15$  [dB], where  $P_d$  is the power of delayed signals. This environment is used in [96]. In this simulation, lower BER is achieved by the sequences whose parameters  $Z_{AC}$  and  $Z_{CC}$  satisfy  $Z_{AC} < Z_{CC}$  and ratio  $Z_{AC}/Z_{CC}$  is higher. In particular, the sequences whose parameters equal (1,2) have the lowest BER in the parameter regions  $Z_{AC} < Z_{CC}$ . Since the BER performances of the sequences derived in the parameter regions  $Z_{AC} < Z_{CC}$  are the same in no fading environments, this result is caused by fading effects. Therefore, we conclude that the sequences have more resistance against fading noise as the ratio  $Z_{AC}/Z_{CC}$  gets higher.

In Section V, we have discussed a relation between SINR and mean-square correlations. From Section V, it is expected that sequences have resistance against interference noise and fading noise when the mean square cross-correlation and mean-square auto-correlation get smaller, respectively. Table 4.1 shows the correlation properties of our sequences and known sequences. The indices  $\theta_a$ ,

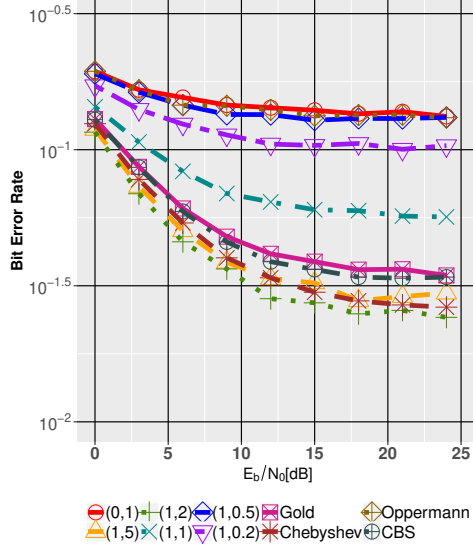


Figure 4.2: Comparison of Bit Error Rate among  $(Z_{AC}, Z_{CC})$ , and known sequences: -15dB fading signals,  $N = 31$ ,  $K = 5$ , ©2018 IEEE

$\theta_c$ ,  $\hat{\theta}_a$ , and  $\hat{\theta}_c$  are defined as [47] [54]

$$\begin{aligned} \theta_a &= \max_{1 \leq i \leq K, l \neq 0} \theta_{i,i}(l), & \theta_c &= \max_{i \neq k, 0 \leq l \leq N-1} \theta_{i,k}(l), \\ \hat{\theta}_a &= \max_{1 \leq i \leq K, l \neq 0} \hat{\theta}_{i,i}(l), & \hat{\theta}_c &= \max_{i \neq k, 0 \leq l \leq N-1} \hat{\theta}_{i,k}(l). \end{aligned} \quad (4.102)$$

To compare the indices in Table I, we choose a set of  $K$  sequences in each known sequence. Note that these indices of the chaotic bit sequences, the Oppermann sequences and the Gold codes have been investigated in [66] and [50]. In Ta-

Table 4.1: table of correlation properties :  $N = 31$ ,  $K = 5$ .

| Sequence                            | $\theta_a$ | $\theta_c$ | $\hat{\theta}_a$ | $\hat{\theta}_c$ | $R_{AC}$ | $R_{CC}$ |
|-------------------------------------|------------|------------|------------------|------------------|----------|----------|
| Gold [38]                           | 9.00       | 9.00       | 13.0             | 15.0             | 0.905    | 0.991    |
| Chebyshev Optimal [42]              | 16.0       | 15.6       | 16.5             | 16.8             | 1.67     | 1.14     |
| Oppermann [50]                      | 29.6       | 6.19       | 30.4             | 4.12             | 18.4     | 0.0932   |
| Chaotic Bit Sequence (CBS) [66][67] | 15.0       | 15.0       | 15.0             | 19.0             | 0.954    | 1.11     |
| $(Z_{AC}, Z_{CC}) = (0, 1)$         | 30.7       | 1.70       | 30.7             | 2.09             | 16.7     | 0.0180   |
| $(Z_{AC}, Z_{CC}) = (1, 5)$         | 29.8       | 3.48       | 29.8             | 1.64             | 6.61     | 0.0254   |
| $(Z_{AC}, Z_{CC}) = (1, 2)$         | 27.4       | 2.84       | 27.4             | 5.13             | 5.63     | 0.0627   |
| $(Z_{AC}, Z_{CC}) = (1, 1)$         | 27.5       | 22.3       | 27.5             | 24.0             | 3.20     | 2.69     |
| $(Z_{AC}, Z_{CC}) = (1, 0.5)$       | 30.8       | 31.0       | 30.9             | 31.0             | 11.8     | 5.11     |
| $(Z_{AC}, Z_{CC}) = (1, 0.2)$       | 30.8       | 26.5       | 30.9             | 23.9             | 11.7     | 3.44     |

ble 4.1, the sequences derived from the parameter regions  $Z_{AC} \geq Z_{CC}$  have



high mean-square auto-correlations and cross-correlations. As seen in Figure A.2, these sequences do not have advantage in BER. On the other hand, the sequences derived in the parameter regions  $Z_{AC} < Z_{CC}$  have the lowest mean-square cross-correlations in all the sequences. Therefore, the sequences derived in the parameter regions  $Z_{AC} < Z_{CC}$  have the lowest BER in no fading environments. Furthermore, if the ratio  $Z_{AC}/Z_{CC}$  gets higher, then the mean-square auto-correlation gets lower. In Figure 4.2, the sequences whose parameters equal (1, 2) have the lowest BER in all the sequences. This result shows that the sequences have resistance against fading noise when the ratio  $Z_{AC}/Z_{CC}$  gets high. As seen in Figure 4.2, in some fading effects, BER gets lower as the ratio  $Z_{AC}/Z_{CC}$  gets higher. From these results, the parameters  $Z_{AC}$  and  $Z_{CC}$  correspond to the resistance against the fading effects and multiple-access interference, respectively. Furthermore, we can obtain the desired sequences by choosing the ratio  $Z_{AC}/Z_{CC}$ .

## X Summary and Discussions

In this chapter, we have derived the optimization problem and numerically solved it to obtain spreading sequences. First, we have shown an expression of SINR with periodic correlation terms and odd periodic correlation terms. Then, the relation between SINR and mean-square correlations has been elucidated. With a semidefinite relaxation technique, we have obtained the relaxed problem. Also, we have shown that this problem is convex in certain parameter regions. In the other parameter regions, this problem is a generalized convex multiplicative programming problem. Finally, we have obtained its solutions with PENLAB and have evaluated BER for the solutions. From the results of BER, we have concluded that the desired sequences can be generated by choosing the ratio  $Z_{AC}/Z_{CC}$ .

One of the remained issues is to obtain an efficient algorithm to find optimal solutions and their explicit forms. We have numerically obtained the solutions as sequences with the parameters  $N = 31$  and  $K = 5$ . It is expected to generate sequences flexibly with varied parameters if the explicit form of optimal solutions is obtained. Another issue is to find the optimal way to choosing the ratio  $Z_{AC}/Z_{CC}$  since it is expected that the optimal ratio  $\bar{Z}_{AC}/\bar{Z}_{CC}$  depends on the fading function  $h_k(\tau, t)$  and its covariance function  $\Sigma_k(\tau_1, \tau_2; t_1, t_2)$ .

Note that our relaxed optimization problem is convex when we focus on only one variable  $X_i$  and fix the other variables  $X_k$  ( $k \neq i$ ). Thus, our optimization problem can be solved with Alternating Direction Method of Multipliers technique [97].

Possible future works are to investigate the BER performance with our sequences in mono-scenario  $K = 1$ , to evaluate our sequence in the selection criteria of chaotic sequences [98], and to compare the performance of CDMA systems using our sequences with one of DCSK systems.

## 4.A Appendix A

In this appendix, we prove that the covariance function  $g_k(\tau)$  has an upper bound discussed in Section III. The function  $h_k(f, t)$  is written as

$$h_k(t, \tau) = \int_{-\infty}^{\infty} H_k(f, t) \exp(2\pi j f \tau) df, \quad (4.103)$$

where  $H_k(f, t)$  is the Fourier transformation of  $h_k(\tau, t)$ . From the assumption 1 in Section III, the function  $h_k(\tau, t)$  is transformed to  $H_k(f, t)$  by the Fourier transformation and  $H_k(f, t)$  is transformed to  $h_k(\tau, t)$  by the inverse Fourier transformation. For the existence of them, it is necessary to satisfy

$$\int_{-\infty}^{\infty} |h_k(\tau, t)| d\tau < \infty, \quad \int_{-\infty}^{\infty} |H_k(f, t)| df < \infty, \quad (4.104)$$

for all  $t$ . Then, the absolute value of the function  $h_k(\tau, t)$  has an upper bound

$$|h_k(\tau, t)| = \left| \int_{-\infty}^{\infty} H_k(f, t) \exp(2\pi j f \tau) df \right| \leq \int_{-\infty}^{\infty} |H_k(f, t)| df < \infty \quad (4.105)$$

for all  $\tau$  and  $t$ . Thus, the covariance function  $g_k(\tau)$  is evaluated as

$$g_k(\tau) = E\{h_k(\tau, 0) \overline{h_k(\tau, 0)}\} \leq E\{C\} = C < \infty, \quad (4.106)$$

where

$$C = \sup_{\tau} |h_k(\tau, 0)|^2 < \infty. \quad (4.107)$$

In the above equations, we have used the assumption that  $g_k(\tau)$  is independent of the variable  $t$ , that is, the channel is a selective fading channel. We thus proved that  $g_k(\tau)$  has an upper bound.

## 4.B Appendix B

In this appendix, we prove that the following function

$$\sum_{i=1}^K \sum_{k=1}^K Z_{i,k} \sum_{m=1}^N \text{Tr}(Q_m X_i) \text{Tr}(Q_m X_k) + \text{Tr}(\hat{Q}_m X_i) \text{Tr}(\hat{Q}_m X_k) \quad (4.108)$$

is convex if  $Z_{AC} \geq Z_{CC}$  and  $X_k$  is a Hermitian positive semidefinite matrix (see Section VII). We decompose  $Z_{AC}$  as  $Z_{AC} = Z_{CC} + Z$ , where  $Z$  is a non-negative real value. Then, Eq. (4.108) is rewritten as

$$\begin{aligned} & Z_{CC} \left\{ \sum_{m=1}^N \text{Tr} \left( Q_m \sum_{i=1}^K X_i \right)^2 + \text{Tr} \left( \hat{Q}_m \sum_{i=1}^K X_i \right)^2 \right\} \\ & + Z \left\{ \sum_{i=1}^K \sum_{m=1}^N \text{Tr}(Q_m X_i)^2 + \text{Tr}(\hat{Q}_m X_i)^2 \right\}. \end{aligned} \quad (4.109)$$

The matrices  $Q_m$ , and  $\hat{Q}_m$  are symmetric positive semidefinite matrices. Therefore, when  $X$  is a Hermitian positive semidefinite matrix,  $\text{Tr}(Q_m X)$ , and  $\text{Tr}(\hat{Q}_m X)$  are convex and non-negative. From Theorem 5.1 in [99], each term of Eq. (4.109) is convex. Since the sum of convex functions is convex in general, Eq. (4.109) is convex.

## Chapter 5

# Randomization Method in Partial Transmit Sequence Technique

### I Introduction

In Chapter 1, we have discussed the relation between SNDR and PAPR with OFDM systems. Since OFDM signals are generated with Inverse Fourier Transformation [12], OFDM systems have an advantage that OFDM systems can deal with multi-path delay. It has been known that effects of multi-path delay in frequency selective channels can be removed with guard interval techniques and zero padding techniques [8]. Since OFDM systems can remove the effects of multi-path delay, Multiple-Input Multiple-Output systems with OFDM systems have been investigated [100].

While there are some advantages in OFDM systems, there are two main problems. One is that signals of OFDM systems have relatively large side-lobes [101]. This problem is caused since OFDM signals consists of sine waves. The other is that signals of OFDM systems have large PAPR, which is the ratio of the maximum value of RF signal powers to the average value of them. Approximately, the output power grows linearly for low values of the input powers. However, for input signals with large power, the growth of the output power is non-linear. Then, in-band distortion and out-of-band distortion are caused for a large input power [9]. With symbols chosen independently, PAPR for OFDM signals has been investigated in [102] [103]. With dependent symbols, for example, Bose-Chaudhuri-Hocquenghem (BCH) codes, their PAPR has been investigated [104]. Further, the performance of OFDM systems with non-linear amplifiers has been investigated in [6] [7].

To reduce PAPR, many methods have been proposed and explored. For example, a selected mapping method [105], a balancing method [106], an active constellation extension method [107], a tone injection method [108], a complement block coding method [109], a constellation reshaping method [110], an iterative filtering method [111], and a compounding method [112]. These methods are summarized in [10] [11] [113].

One of the methods to reduce PAPR is a Partial Transmit Sequence (PTS) technique [114] [115] [116] [117]. PTS techniques are to multiply symbols by phases to reduce PAPR. It is necessary to transmit the vector as side information to the receiver. This yields that OFDM signals are not distorted with an ideal amplifier since we only modulate symbols. Then, their side lobes stay unchanged.

With PTS techniques, there is a significant task for reducing PAPR, how to reduce the calculation amount. In [114], a suitable vector is chosen as one which achieves the lowest PAPR from all of the candidates. Then, the calculation amount exponentially gets larger as the length of a vector increases. To reduce such calculations, some methods have been known. In [118], the neighborhood search algorithm has been proposed. With this method, we can obtain a local optimal solution. Another method is a phase random method [119]. This method consists of generating random vectors whose phase is uniformly distributed in the set of the candidates.

In this chapter, we propose a method to search the vector which achieves low PAPR. The main point of our method is to obtain the vector from a set of random vectors generated from the Gaussian distribution. Therefore, our method is similar to a phase random method [119]. Then, we derive the optimization problem to reduce PAPR and obtain a solution from the relaxed problem. We regard the solution as a covariance matrix and we can determine the Gaussian distribution.

Our method is a probabilistic scheme since the vectors are generated from the Gaussian distribution. There are some probabilistic schemes in PAPR reduction techniques. In [105], for a selective mapping method, it has been proposed to generate phases randomly to obtain independent vectors. In [120], for a PTS technique, it has been proposed to choose sub-blocks randomly. Therefore, our method is one of such schemes.

The rest of this chapter is organized as follows: Section II shows the definitions of PAPR and Peak-to-Mean Envelope Power Ratio (PMEPR). In the literature, these two notions sometimes are assumed to coincide, however, these two are different since PAPR and PMEPR are defined with RF signals and base-band signals, respectively. In this Section, we clarify the property of signals considered in this chapter. Section III shows a partial transmit sequence technique and an optimization problem. It is not straightforward to solve this optimization problem since the feasible region of this optimization problem is discrete. Therefore, in Section IV, we show a semidefinite relaxation technique to solve it. With this technique, we can obtain approximate solutions. In Section V, we consider random vectors generated from the Gaussian distribution whose covariance matrix is the solution of the relaxed problem. Then, in Section VI, we show the relation between our randomization method and a phase random method, which is a conventional method. In Section VII, since our problem stated in Sections IV and V has the large number of constraints, we propose another optimization problem to reduce the upper bound of PAPR. In this problem, the number of constraints is less than one of that in Section IV and V. Finally, we compare the PAPR of our method with existing methods.

## II OFDM System Model and PAPR

In this section, we fix the model and the quantities used throughout this chapter. A complex baseband OFDM signal is written as [12]

$$s(t) = \sum_{k=1}^K A_k \exp\left(2\pi j \frac{k-1}{T} t\right), \quad 0 \leq t < T, \quad (5.1)$$

where  $A_k$  is a  $k$ -th transmitted symbol,  $K$  is the number of symbols,  $j$  is the unit imaginary number, and  $T$  is a duration of symbols. It is known that OFDM signals are generated by Inverse Fast Fourier Transformation (IFFT) [12]. As seen in Eq. (6.1), the OFDM signals have no cyclic prefixes. With the cyclic prefix technique, the PAPR is preserved since the cyclic prefix does not induce any new peaks [18]. Therefore, we consider such OFDM signals written in Eq. (6.1).

A Radio Frequency (RF) OFDM signal  $\zeta(t)$  is written with Eq. (6.1) as

$$\begin{aligned} \zeta(t) &= \text{Re}\{s(t) \exp(2\pi j f_c t)\} \\ &= \text{Re}\left\{\sum_{k=1}^K A_k \exp\left(2\pi j \left(\frac{k-1}{T} + f_c\right) t\right)\right\}, \end{aligned} \quad (5.2)$$

where  $\text{Re}\{z\}$  is the real part of  $z$ , and  $f_c$  is a carrier frequency. With RF signals, PAPR is defined as [121] [122]

$$\text{PAPR} = \max_{0 \leq t < T} \frac{\left| \text{Re}\left\{\sum_{k=1}^K A_k \exp\left(2\pi j \left(\frac{k-1}{T} + f_c\right) t\right)\right\} \right|^2}{P_{\text{av}}}, \quad (5.3)$$

where  $P_{\text{av}}$  corresponds to the average power of signals,  $P_{\text{av}} = \sum_{k=1}^K \text{E}\{|A_k|^2\}$ , and  $\text{E}\{X\}$  is the average of  $X$ . Similarly, with baseband signals, PMEPR is defined as [121] [122]

$$\text{PMEPR} = \max_{0 \leq t < T} \frac{\left| \sum_{k=1}^K A_k \exp\left(2\pi j \frac{k-1}{T} t\right) \right|^2}{P_{\text{av}}}. \quad (5.4)$$

In the literature, PAPR and PMEPR have often been evaluated as probabilities, since PAPR and PMEPR depend on symbols  $A_k$  that can be regarded as random variables [102] [104].

Obviously, PAPR does not always correspond to PMEPR. Further, from Eqs. (6.3) and (6.4), PAPR does not exceed PMEPR. In [18], under some conditions described below, it has been proven that the following relations are established

$$\left(1 - \frac{\pi^2 K^2}{2r^2}\right) \cdot \text{PMEPR} \leq \text{PAPR} \leq \text{PMEPR}, \quad (5.5)$$

where  $r$  is an integer such that  $f_c = r/T$ . The conditions that Eq. (5.5) holds are  $K \ll r$  and  $\exp(2\pi j K/r) \approx 1$ . In addition to these, another relation has been shown in [11]. Equation (5.5) implies that PMEPR approximately equals

PAPR for sufficiently large  $f_c$ . It is often the case that PMEPR is evaluated instead of PAPR [102]. In what follows, we assume that the carrier frequency  $f_c$  is sufficiently large, that is, we consider baseband OFDM signals instead of RF signals.

### III Partial Transmit Sequence Technique

With OFDM systems, the Partial Transmit Sequence (PTS) technique has been proposed to reduce PAPR. In this section, we show the model and the details of PTS techniques. PTS techniques need a vector to reduce PAPR. A disadvantage of PTS techniques is that the large amount of calculation is necessary in some situations. The details of PTS techniques are described in [11] [114] [123].

The symbols in this chapter and how to derive index sets for symbols are given as follows. We assume that the symbols  $A_k$  are given and the number of the symbols is  $K$ . Further, the index set  $\Lambda = \{1, 2, \dots, K\}$  corresponds to the set of unordered symbols  $\{A_1, A_2, \dots, A_K\}$ . To apply a PTS technique, we divide the index set  $\Lambda$  into  $P$  disjoint subsets,  $\Lambda_1, \dots, \Lambda_P$ , that is,

$$\Lambda = \Lambda_1 \cup \dots \cup \Lambda_P, \quad \Lambda_k \cap \Lambda_m = \emptyset \text{ if } k \neq m \quad (5.6)$$

for  $k, m = 1, 2, \dots, P$ . There are some discussions about how to divide the index set. We refer the reader to [124] [125] [120].

To express the instantaneous power, we define some quantities as below. For each subsets of symbols, we introduce a rotation vector  $\mathbf{b} = (b_1, b_2, \dots, b_P)^\top$ , where  $\mathbf{x}^\top$  is the transpose of  $\mathbf{x}$ . The vector  $\mathbf{b}$  is chosen as one satisfying  $b_p = \exp(j\theta_p)$  for  $p = 1, 2, \dots, P$ , where  $\theta_p \in [0, 2\pi)$ . This  $\mathbf{b}$  plays various roles throughout this chapter. For convenience, let us define the quantities

$$A_k^{(p)} = \begin{cases} A_k & k \in \Lambda_p \\ 0 & \text{otherwise} \end{cases} \quad (5.7)$$

for  $k = 1, 2, \dots, K$  and  $p = 1, 2, \dots, P$ . With  $A_k^{(p)}$ , a modified baseband OFDM signal  $\hat{s}(t)$  is written as

$$\hat{s}(t) = \sum_{p=1}^P \sum_{k=1}^K A_k^{(p)} b_p \exp\left(2\pi j \frac{k-1}{T} t\right). \quad (5.8)$$

Note that the average power of modified signals is equivalent to one of the original OFDM signals since  $|b_p| = 1$ . With a matrix and vectors, the above equation is rewritten as

$$\hat{s}(t) = \mathbf{v}_t^\top \mathbf{A} \mathbf{b}, \quad (5.9)$$

where

$$\mathbf{v}_t = \begin{pmatrix} v_{1,t} & v_{2,t} & \dots & v_{K,t} \end{pmatrix}^\top, \quad \mathbf{A} = \begin{pmatrix} A_1^{(1)} & A_1^{(2)} & \dots & A_1^{(P)} \\ A_2^{(1)} & A_2^{(2)} & \dots & A_2^{(P)} \\ \vdots & \vdots & \ddots & \vdots \\ A_K^{(1)} & A_K^{(2)} & \dots & A_K^{(P)} \end{pmatrix} \quad (5.10)$$

and

$$v_{k,t} = \exp\left(2\pi j \frac{k-1}{T} t\right). \quad (5.11)$$

Note that the matrix  $A$  is a  $K \times P$  matrix. With these quantities, the instantaneous power  $|\hat{s}(t)|^2$  is written as

$$|\hat{s}(t)|^2 = \mathbf{b}^* A^* (\mathbf{v}_t^*)^\top \mathbf{v}_t^\top A \mathbf{b}, \quad (5.12)$$

where  $\mathbf{z}^*$  and  $X^*$  are the complex conjugate transposes of  $\mathbf{z}$  and  $X$ , respectively. We denote by  $C_t$  the matrix  $A^* (\mathbf{v}_t^*)^\top \mathbf{v}_t^\top A$ . Note that  $C_t$  is a positive semidefinite matrix since  $C_t$  is a Gram matrix and each value of  $b_p$  is chosen as one achieving the lowest PAPR.

At the receiver side, to recover symbols, it is necessary for the receiver to know the explicit values of  $\mathbf{b}$ . Note that Signal to Noise Ratio (SNR) is preserved if the receiver knows  $\mathbf{b}$ . To let the receiver know, the vector  $\mathbf{b}$  has to be transmitted as side-information. To reduce information content, the value of  $b_p$  is usually restricted to

$$b_p \in \left\{1, \exp\left(2\pi j \frac{1}{L}\right), \dots, \exp\left(2\pi j \frac{L-1}{L}\right)\right\}, \quad (5.13)$$

where  $L$  is a positive integer. We define the set  $\Omega_L$  as

$$\Omega_L = \left\{1, \exp\left(2\pi j \frac{1}{L}\right), \dots, \exp\left(2\pi j \frac{L-1}{L}\right)\right\}. \quad (5.14)$$

From the above definition, the vector  $\mathbf{b}$  belongs  $\Omega_L^P$ , where  $\Omega_L^P$  is a  $P$ -tuple of  $\Omega_L$ . Then, the information content of  $\mathbf{b}$  is  $(P-1)\log_2 L$  [bits] since we can set  $b_1 = 1$  without loss of generality. It is obvious that the number of elements in the set  $\Omega_L^{P-1}$  is  $L^{P-1}$ . Let  $\mathbf{b}^*$  be the optimal vector which achieves the lowest PAPR in  $\Omega_L^P$ . Then, it turns out that  $\mathbf{b}^*$  is the global solution of the optimization problem

$$\begin{aligned} (Q_L) \quad & \min \max_{0 \leq t < T} |\hat{s}(t)|^2 \\ \text{subject to} \quad & b_p \in \Omega_L \quad (p = 1, 2, \dots, P). \end{aligned} \quad (5.15)$$

Our aim is to find the vector  $\mathbf{b}^*$ . To this end, there are two main obstacles to solve the problem  $(Q_L)$ .

One obstacle is that the time  $t$  is continuous. In [18], with baseband OFDM signals  $s(t)$  defined in Eq. (6.1), it has been shown that there is the following relation between continuous signals and sampled signals

$$\max_{0 \leq t < T} |s(t)| < \sqrt{\frac{J^2}{J^2 - \pi^2/2}} \max_{0 \leq n < JK} \left| s\left(\frac{nT}{JK}\right) \right|, \quad (5.16)$$

where  $J$  is an integer satisfying  $J > \pi/\sqrt{2}$ . Equation (5.16) implies that PMEPR can be estimated precisely from signals sampled with a sufficiently large oversampling factor. For maxima of continuous signals and sampled signals, other relations have been shown in [126] [127]. The integer  $J$  is often called oversampling factor [6], and is often chosen as  $J \geq 4$ . How to choose the oversampling factor  $J$  has been discussed in [18].

With sampled signals, the problem  $(Q_L)$  is rewritten as

$$\begin{aligned}
(\hat{Q}_L) \quad & \min \lambda \\
\text{subject to} \quad & \mathbf{b}^* C_{nT/(JK)} \mathbf{b} \leq \lambda \quad (n = 0, 1, \dots, JK - 1) \\
& \lambda \in \mathbb{R}, \quad b_p \in \Omega_L \quad (p = 1, 2, \dots, P).
\end{aligned} \tag{5.17}$$

Note that the variables in the problem  $(\hat{Q}_L)$  are  $\mathbf{b}$  and  $\lambda$ .

The other obstacle is that the feasible region  $\Omega_L$  is discrete. In [114], a brute-force search has been used to find the global solution  $\mathbf{b}^*$ . With this method, we have to find the vector  $\mathbf{b}^*$  from  $L^{P-1}$  candidates, and the calculation amount exponentially gets larger as  $P$  increases. In [118], the neighborhood search algorithm has been proposed. With this method, we can obtain a local optimal solution. However, it is only known that its calculation amount is proportional to  ${}_{P-1}C_r \cdot L^r$ , where  $r$  is an integer parameter expressing the distance of a neighborhood and  ${}_aC_b$  is a binomial coefficient. Another existing method is a phase random method [119]. This method consists of generating random vectors whose phase is uniformly distributed in the region  $\Omega_L^P$ , from which we obtain a solution.

## IV Semidefinite Relaxation

Since it is not straightforward to obtain the global solution, we propose an efficient method to obtain a solution which achieves low PAPR. Optimization problems, such as the problem  $(\hat{Q}_L)$ , appear in MIMO detection [128]. Thus, we can use these methods that have already been developed to our problem. One of such existing methods uses a semidefinite relaxation technique [78]. In this section, we obtain a solution with such semidefinite relaxation techniques.

We apply semidefinite relaxation techniques to the problem  $(\hat{Q}_L)$ . Our main aim is to change the variable  $\mathbf{b}$  to a positive semidefinite matrix  $X$ . The ways to solve the problem  $(\hat{Q}_L)$  depend on  $\Omega_L$ . Therefore, we consider each problem for various cases of  $L$ .

### IV.I Optimization Problem for $L = 2$

First, we consider the problem  $(\hat{Q}_L)$  for  $L = 2$ ,  $(\hat{Q}_2)$ . Then,  $\Omega_2 = \{-1, 1\}$ . Note that  $b^2 = 1$  for  $b \in \Omega_2$ . If we define the matrix  $X = \mathbf{b}\mathbf{b}^\top$ , then  $X$  is a positive semidefinite matrix whose rank is 1 and the problem  $(\hat{Q}_2)$  is rewritten as

$$\begin{aligned}
(\hat{Q}_2) \quad & \min \lambda \\
\text{subject to} \quad & \text{Tr}(C_{nT/(JK)} X) \leq \lambda \quad (n = 0, 1, \dots, JK - 1) \\
& \text{rank}(X) = 1, \quad X_{p,p} = 1 \quad (p = 1, 2, \dots, P) \\
& X \succcurlyeq 0, \quad X \in \mathbb{S}_P, \quad \lambda \in \mathbb{R},
\end{aligned} \tag{5.18}$$

where  $\text{Tr}(X)$  is the trace of  $X$ ,  $\text{rank}(X)$  is the rank of  $X$ ,  $X \succcurlyeq 0$  indicates that  $X$  is a positive semidefinite matrix and  $\mathbb{S}_P$  is the set of symmetric matrices of dimension  $P$ . Due to the constraint  $\text{rank}(X) = 1$ , the problem  $(\hat{Q}_2)$  is not convex. Note that the set of positive semidefinite matrices is convex [85]. By



dropping the rank constraint, we obtain the relaxed optimization problem

$$\begin{aligned}
& (\hat{Q}'_2) \quad \min \lambda \\
& \text{subject to} \quad \text{Tr}(C_{nT/(JK)}X) \leq \lambda \quad (n = 0, 1, \dots, JK - 1) \\
& \quad X_{p,p} = 1 \quad (p = 1, 2, \dots, P) \\
& \quad X \succcurlyeq 0, \quad X \in \mathbb{S}_P, \quad \lambda \in \mathbb{R}.
\end{aligned} \tag{5.19}$$

The above problem  $(\hat{Q}'_2)$  can be solved since the problem  $(\hat{Q}'_2)$  is convex. Let  $X_2^*$  be the global solution of the problem  $(\hat{Q}'_2)$ . If the rank of  $X_2^*$  is 1, then we obtain the global solution of the problem  $(\hat{Q}_2)$ , denoted by  $\mathbf{b}_2^*$ . However, the rank of  $X_2^*$  is not always 1. To deal with this, we obtain an approximate solution from  $X_2^*$  as follows. First, the solution  $X_2^*$  is decomposed as

$$X_2^* = \sum_{i=1}^{r_2} \lambda_i \mathbf{q}_i \mathbf{q}_i^*, \tag{5.20}$$

where  $r_2 = \text{rank}(X_2^*)$ ,  $\lambda_i$  is the eigenvalue of  $X_2^*$ ,  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r_2}$  and  $\mathbf{q}_i$  is the respective eigenvector. Then, in a least-two-norm sense, the approximate solution whose rank 1 is obtained as  $\hat{X}_2^* = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^*$ . From this approximate solution, we systematically obtain the solution of the original problem  $(\hat{Q}_2)$  as  $\sqrt{\lambda_1} \mathbf{q}_1$ . However, this solution is not always in the feasible region of the problem  $(\hat{Q}_2)$ . To have an approximate solution in the feasible region, for the problem  $(\hat{Q}_2)$ , we need to project the solution onto the feasible region. We finally arrive at the  $p$ -th element of the approximate solution of the problem  $(\hat{Q}_2)$  as

$$\hat{b}_{2,p} = \text{sgn}(q_{1,p}), \tag{5.21}$$

where  $q_{1,p}$  is the  $p$ -th element of  $\mathbf{q}_1$  and

$$\text{sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}. \tag{5.22}$$

#### IV.II Optimization Problem for $L = 4$

Similar to the case  $L = 2$ , we obtain the approximate solution of the problem  $(\hat{Q}_4)$ . For  $L = 4$ , the set  $\Omega_4$  is written as  $\Omega_4 = \{1, \exp(j\pi/2), -1, \exp(j3\pi/4)\}$ . To obtain the relaxed problem, we rewrite the problem  $(\hat{Q}_4)$  as follows. First, let us define the set

$$\hat{\Omega}_4 = \{+1 + j, +1 - j, -1 + j, -1 - j\}, \tag{5.23}$$

which can be expressed as

$$\hat{\Omega}_4 = \{\sqrt{2} \exp(j\pi/4) \cdot a \mid a \in \Omega_4\}. \tag{5.24}$$

Note that  $\text{Re}\{b\}^2 = \text{Im}\{b\}^2 = 1$  for  $b \in \hat{\Omega}_4$  and that the elements in  $\hat{\Omega}_4$  are complex. Second, we rewrite the set  $\hat{\Omega}_4$  in terms of real parts and imaginary parts. To this end, we introduce the following transformations for  $\mathbf{z} \in \mathbb{C}^n$  and  $Z \in \mathbb{H}_n$ , where  $\mathbb{H}_n$  is the set of Hermitian matrices of dimension  $n$  [95],

$$\mathcal{T}(\mathbf{z}) = \begin{pmatrix} \text{Re}\{\mathbf{z}\} \\ \text{Im}\{\mathbf{z}\} \end{pmatrix}, \quad \text{and} \quad \mathcal{T}(Z) = \begin{pmatrix} \text{Re}\{Z\} & -\text{Im}\{Z\} \\ \text{Im}\{Z\} & \text{Re}\{Z\} \end{pmatrix}.$$

Note that  $\mathcal{T}(X) \in \mathbb{S}_{2n}$  if  $X \in \mathbb{H}_n$  [129]. Finally, we arrive at the relaxed problem for  $L = 4$ .

$$\begin{aligned}
& (\hat{Q}'_4) \quad \min \lambda \\
& \text{subject to} \quad \text{Tr}(\hat{C}_{nT/(JK)}X) \leq \lambda \quad (n = 0, 1, \dots, JK - 1) \\
& \quad X_{p,p} = 1 \quad (p = 1, 2, \dots, 2P) \\
& \quad X \succcurlyeq 0, \quad X \in \mathbb{S}_{2P}, \quad \lambda \in \mathbb{R},
\end{aligned} \tag{5.25}$$

where  $\hat{C}_t = \mathcal{T}(C_t)$ . Note that the problem  $(\hat{Q}'_4)$  is equivalent to  $(\hat{Q}_4)$  if we impose the rank constraints to the problem  $(\hat{Q}'_4)$  and the problem  $(\hat{Q}'_4)$  is convex. Let  $X_4^*$  be the global solution of the problem  $(\hat{Q}'_4)$ . From  $X_4^*$ , we can obtain an approximate solution by the following procedure. First, we decompose  $X_4^*$  as

$$X_4^* = \sum_{i=1}^{r_4} \lambda_i \mathbf{q}_i \mathbf{q}_i^*, \tag{5.26}$$

where  $r_4 = \text{rank}(X_4^*)$ ,  $\lambda_i$  is the eigenvalue of  $X_4^*$ ,  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r_4}$  and  $\mathbf{q}_i$  is the respective eigenvector. From the vector  $\mathbf{q}_1$ , we can obtain the approximate solution  $\hat{\mathbf{b}}_4 \in \mathbb{C}^P$  written as

$$\hat{b}_{4,p} = \frac{1}{\sqrt{2}} \exp(-j\pi/4) (\text{sgn}(q_p) + j \cdot \text{sgn}(q_{p+P})), \tag{5.27}$$

where  $\hat{b}_{4,p}$  and  $q_p$  are the  $p$ -th elements of  $\hat{\mathbf{b}}_4$  and  $\mathbf{q}_1$ , respectively.

#### IV.III Optimization Problem for General $L$

For general  $L$ , we consider the relaxed problem

$$\begin{aligned}
& (\hat{Q}'_L) \quad \min \lambda \\
& \text{subject to} \quad \text{Tr}(C_{nT/(JK)}X) \leq \lambda \quad (n = 0, 1, \dots, JK - 1) \\
& \quad X_{p,p} = 1 \quad (p = 1, 2, \dots, P) \\
& \quad X \succcurlyeq 0, \quad X \in \mathbb{H}_P, \quad \lambda \in \mathbb{R}.
\end{aligned} \tag{5.28}$$

Note that the set of Hermitian semidefinite positive matrices is convex [78] and the above problem  $(\hat{Q}'_L)$  is convex. The problem  $(\hat{Q}'_L)$  is not equivalent to the problem  $(\hat{Q}_L)$  for  $L \neq 2, 4$  if the rank constraint is imposed. Similar to the problem for  $L = 2$  and  $L = 4$ , let  $X_L^*$  be the global solution of the problem  $(\hat{Q}'_L)$ . Then,  $X_L^*$  is decomposed as

$$X_L^* = \sum_{i=1}^{r_L} \lambda_i \mathbf{q}_i \mathbf{q}_i^*, \tag{5.29}$$

where  $r_L = \text{rank}(X_L^*)$ ,  $\lambda_i$  is the eigenvalue of  $X_L^*$ ,  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r_L}$  and  $\mathbf{q}_i$  is the respective eigenvector. From the vector  $\mathbf{q}_1$ , we can obtain the approximate solution  $\hat{\mathbf{b}}_L \in \mathbb{C}^P$  as

$$\hat{\mathbf{b}}_L = \arg \min_{\mathbf{b} \in \Omega_L^P} \|\mathbf{q} - \mathbf{b}\|, \tag{5.30}$$

where  $\mathbf{q} = \sqrt{\lambda_1} \mathbf{q}_1$  and  $\|\mathbf{z}\|$  is the Euclidean norm of  $\mathbf{z}$ .

## V Randomization Method

In Section IV, we have discussed the relaxed problems and how to obtain the approximate solutions. However, clearly, approximate solutions are not suitable if the global solutions of the relaxed problems have some large eigenvalues, that is, the ranks of solutions are not regarded as unity.

In this section, we introduce a randomization method. This method has been used to analyze how far the optimal value of relaxed problems is from one of original problems [130]. With this method, we obtain solutions as random values which are generated from Gaussian distribution. Similar to discussions in Section IV, we consider each problem for various cases of  $L$ . Further, in the end of this section, we discuss the computational complexity of our randomization method.

### V.I Randomization for $L = 2$

First, we consider the problem for  $L = 2$ . Let  $\boldsymbol{\xi}$  be a random vector generated from the Gaussian distribution  $\mathcal{N}(\mathbf{0}, X)$  with zero mean and a covariance matrix  $X$ . The definition and properties of a Gaussian distribution have been shown in [131].

To find an approximate solution, we rewrite the problem  $(\hat{Q}'_2)$  as follows. With

$$\mathbb{E}\{\boldsymbol{\xi}^\top C_t \boldsymbol{\xi}\} = \text{Tr}(C_t X), \quad (5.31)$$

the problem  $(\hat{Q}'_2)$  can be written as

$$\begin{aligned} (\hat{Q}'_2) \quad & \min \lambda \\ \text{subject to} \quad & \mathbb{E}\{\boldsymbol{\xi}^\top C_{nT/(JK)} \boldsymbol{\xi}\} \leq \lambda \quad (n = 0, 1, \dots, JK - 1) \\ & X_{p,p} = 1 \quad (p = 1, 2, \dots, P) \\ & X \succcurlyeq 0, \quad X \in \mathbb{S}_P, \quad \lambda \in \mathbb{R}, \\ & \boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, X). \end{aligned} \quad (5.32)$$

Note that the variables of the above problem are  $X$  and  $\lambda$ . Then, it is clear that the optimal matrix  $X_2^*$  defined in Section IV is the optimal matrix of the above problem in a sense of a covariance matrix. This result suggests that a suitable solution can be obtained from a set of random vectors generated from the Gaussian distribution  $\mathcal{N}(\mathbf{0}, X_2^*)$  [132]. We can then obtain the approximate solution as follows.

1. Solve the problem  $(\hat{Q}'_2)$  and obtain the covariance matrix  $X_2^*$ .
2. Generate random vectors  $\{\boldsymbol{\xi}\}$  from the Gaussian distribution  $\mathcal{N}(\mathbf{0}, X_2^*)$  and project them onto the feasible region of the original problem  $(\hat{Q}_L)$ , that is, for  $L = 2$ , obtain the projected solutions

$$\hat{b}_p = \text{sgn}(\xi_p) \quad (p = 1, 2, \dots, P), \quad (5.33)$$

where  $\xi_p$  is the  $p$ -th element of  $\boldsymbol{\xi}$ .

3. Choose the solution which achieves the minimum PAPR among all the random vectors and regard it as an approximate solution.

## V.II Randomization for $L = 4$

Similar to the case for  $L = 2$ , we can obtain the covariance matrix  $X_4^*$  for  $L = 4$  and obtain random vectors  $\{\boldsymbol{\xi}\}$  generated from  $\mathcal{N}(\mathbf{0}, X_4^*)$ . Since the dimension of the vectors  $\{\boldsymbol{\xi}\}$  is  $2P$ , the projected one is written as

$$\hat{b}_p = \frac{1}{\sqrt{2}} \exp(-j\pi/4) (\text{sgn}(\xi_p) + j \cdot \text{sgn}(\xi_{p+P})) \quad (5.34)$$

for  $p = 1, 2, \dots, P$ . From these random vectors, we choose an approximate solution which achieves the minimum PAPR among them.

## V.III Randomization with General $L$

For general  $L$ , we can obtain the complex covariance matrix  $X_L^*$  as a solution of the problem  $(\hat{Q}_L')$ . Similar to the methods for  $L = 2$  and  $L = 4$ , our goal is to choose the solution from random vectors. The main part of our method is to obtain an approximate solution from random vectors generated from the complex Gaussian distribution  $\mathcal{CN}(\mathbf{0}, X_L^*)$ . The definition and the detail of a complex Gaussian distribution have been shown in [17]. There have been some methods to obtain an approximate solution from random vectors [133] [134] [135], and our method is a special case of an algorithm in [135]. From the complex Gaussian distribution  $\mathcal{CN}(\mathbf{0}, X_L^*)$ , we can obtain the random vectors  $\{\boldsymbol{\xi}\}$ . Then, we have to transform the random vectors  $\{\boldsymbol{\xi}\}$  into feasible ones as solutions of the problem  $(\hat{Q}_L)$ . Our transformation method is written as follows. Let  $f_L$  be

$$f_L(z) = \begin{cases} 1 & \text{Arg } z \in [0, \frac{1}{L}2\pi) \\ \omega_L & \text{Arg } z \in [\frac{1}{L}2\pi, \frac{2}{L}2\pi) \\ \vdots & \\ \omega_L^l & \text{Arg } z \in [\frac{l}{L}2\pi, \frac{l+1}{L}2\pi) \\ \vdots & \\ \omega_L^{L-1} & \text{Arg } z \in [\frac{L-1}{L}2\pi, 2\pi) \end{cases}, \quad (5.35)$$

where  $z \in \mathbb{C}$ ,  $\omega_L = \exp(2\pi j/L)$  and  $\text{Arg } z$  is the angle of  $z$ . With the function  $f_L$ , the random vector  $\boldsymbol{\xi}$  generated from the complex Gaussian distribution  $\mathcal{CN}(\mathbf{0}, X_L^*)$  is transformed to

$$\hat{b}_p = f_L(\xi_p) \quad (p = 1, 2, \dots, P). \quad (5.36)$$

It is clear that  $\hat{b}_p \in \Omega_L$ . Therefore,  $\hat{\mathbf{b}}$  is a feasible solution of the problem  $(\hat{Q}_L)$ . With the above method, we can obtain the feasible solutions from random vectors generated from the complex Gaussian distribution  $\mathcal{CN}(\mathbf{0}, X_L^*)$ . Then, we choose the approximate solution from them which achieves the minimum PAPR among the set of the random vectors. Our method is summarized in Algorithm 1.

## V.IV Computational Complexity

There are many discussions about computational complexities of PAPR reduction techniques. In this section, we discuss the computational complexity of our

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**Algorithm 1:** Randomization Method with Semidefinite Relaxation

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- 1 Obtain the relaxed problem with semidefinite relaxation techniques.
  - 2 Obtain the positive semidefinite matrix  $X^*$  as the optimal solution of the relaxed problem.
  - 3 Determine the Gaussian distribution  $\mathcal{N}(\mathbf{0}, X^*)$  (with  $L \neq 2, 4$ ,  $\mathcal{CN}(\mathbf{0}, X^*)$  is determined). Then, generate  $N$  samples from the Gaussian distribution as the candidates of the solution.
  - 4 Project the samples onto the feasible region, and obtain the projected samples.
  - 5 Choose the solution  $\mathbf{b}^*$  from the projected samples which achieves the minimum PAPR. Then, output  $\mathbf{b}^*$  as the solution.
- 

randomization method under the condition that the disjoint index subsets  $\Lambda_p$  are given. The remaining complexities consists of three parts: (i) to generate random vectors, (ii) to choose the optimal vector from the random vectors, and (iii) to obtain the solution of the optimization problem.

In generating random vectors, computational complexity is in proportion to the number of random vectors. The computational complexity turns out to be  $N \cdot c_{\text{gen}}$ , where  $N$  and  $c_{\text{gen}}$  are the number of random vectors and the computational cost for generating each random vector, respectively. Similarly, in choosing the optimal vector from given random vectors, its computational complexity is  $N \cdot JK \cdot c_{\text{cal}}$ , where  $c_{\text{cal}}$  is the computational cost for calculating the amplitude of each samples,  $J$  and  $K$  have been defined in Section III and II, respectively.

When we solve the optimization problem to obtain the solution, the computational complexity for solving the optimization problem is not clear since our optimization problem is not an ordinary semidefinite relaxation problem written in [78]. In this paper, we assume that the computational complexity of our optimization problem is nearly equivalent to one of ordinary semidefinite relaxation problems. Then, the worst case complexity is  $\mathcal{O}(\max\{m, n\}^4 n^{1/2} \log(1/\epsilon))$ , where  $\mathcal{O}$  represents the order,  $m$  is the number of constraints,  $n$  is the dimension of the matrix, and  $\epsilon > 0$  is a solution accuracy [78]. For the general  $L$ , since the optimization problem  $(\hat{Q}'_L)$ , the worst case complexity is  $\mathcal{O}((JK + P)^4 P^{1/2} \log(1/\epsilon))$ .

## VI Relation between Our Method and Phase Random Method

In Section V, we have shown our randomization method. Similar to our method, a phase random method has been proposed [119]. This method uses random vectors whose phase is uniformly distributed in  $\Omega_L$ . In this Section, we discuss the relation between our method and a phase random method.

First, we explain a phase random method. We define the probability mass function as

$$\Pr\{z = \omega_L^l\} = \frac{1}{L} \quad (l = 0, 1, \dots, L-1), \quad (5.37)$$

where  $\omega_L = \exp(2\pi j/L)$ , which has been defined in Section V. Thus,  $\omega_L \in \Omega_L$

and phases are uniformly distributed in  $\Omega_L$ . With this probability, a phase random method is summarized in Algorithm 2.

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**Algorithm 2:** Phase Random Method

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- 1 Let the number of the samples be  $N$ . Generate the variable  $z_{n,p} \in \Omega_L$  with the probability  $\Pr\{z = \omega_L^l\}$  for  $n = 1, 2, \dots, N$  and  $p = 1, 2, \dots, P - 1$ .
  - 2 Obtain the set of vectors  $\{\mathbf{z}\}$  as  $\mathbf{z}_n = (1, z_{n,1}, z_{n,2}, \dots, z_{n,P-1})^\top$ .
  - 3 Choose the vector  $\mathbf{z}^*$  which achieves the lowest PAPR from  $\{\mathbf{z}\}$ .
- 

Further, let us discuss the complex Gaussian distribution. From [17], if  $\mathbf{z} \in \mathbb{C}^n$  follows  $\mathcal{CN}(\boldsymbol{\mu}, \Sigma)$ , then the probability density function of  $\mathcal{T}(\mathbf{z}) \in \mathbb{R}^{2n}$  is the Gaussian distribution  $\mathcal{N}(\mathcal{T}(\boldsymbol{\mu}), \frac{1}{2}\mathcal{T}(\Sigma))$ . Therefore, we can consider a real-value Gaussian distribution instead of a complex Gaussian distribution.

Let us consider the complex Gaussian distribution  $\mathcal{CN}(\mathbf{0}, I_P)$ , where  $I_P$  is the identity matrix whose size is  $P$ . It is clear that the matrix  $\mathcal{T}(I_P)$  is the identity matrix whose size is  $2P$ . From the above discussion, and the covariance matrix is identity matrix, each variable of  $\mathbf{z}$  generated from  $\mathcal{CN}(\mathbf{0}, I_P)$  is uncorrelated. It is known in [88] that uncorrelatedness is equivalent to independence for normal variables. Therefore, it is sufficient to consider a vector  $\mathbf{z}$  whose element is generated from the complex Gaussian distribution  $\mathcal{CN}(0, 1)$ . The variable  $z$  which is the element of  $\mathbf{z}$  can be decomposed as

$$z = x + jy, \quad (5.38)$$

where  $x$  and  $y$  are real numbers following the independent Gaussian distribution  $\mathcal{N}(0, 1/2)$ , respectively.

Let us define  $r \geq 0$  and  $\theta \in [0, 2\pi)$  so that

$$x + jy = r \exp(j\theta) \quad (5.39)$$

Then, since  $x$  and  $y$  are normal variables following  $\mathcal{N}(0, 1/2)$ , the probability density of  $\theta \in [0, 2\pi]$  is [88]

$$p(\theta) = \frac{1}{2\pi}, \quad (5.40)$$

from which, the phase of a variable  $z$  generated from  $\mathcal{CN}(0, 1)$  is uniformly distributed.

From the above discussions and the definition of the function  $f_L(z)$ , the probability mass function of  $f_L(z)$  is written as

$$\Pr\{f_L(z) = \omega_L^l\} = \frac{1}{L}. \quad (5.41)$$

This result implies that a phase random method is equivalent to our method whose covariance matrix is the identity matrix with the function  $f_L(z)$ .

## VII Reducing Upper Bound of PAPR

We have discussed how to obtain a covariance matrix to determine a Gaussian distribution. In Section V, we have obtained the optimization problem  $(\hat{Q}_L)$ .

This problem contains the oversampling parameter  $J$ . As seen in Section III, measured PAPR calculated from sampled signals converges to the true value of PAPR as  $J \rightarrow \infty$ . Therefore, a sufficiently large  $J$  is necessary to evaluate PAPR tightly. Then, however, as seen in Section V, the number of constraints in the optimization problem  $(\hat{Q}'_L)$  gets larger as  $J$  increases. In such a situation, the optimization problem  $(\hat{Q}'_L)$  gets complicated.

To overcome this obstacle, instead of PAPR, we consider an optimization problem to reduce the upper bound of PAPR which does not depend on time  $t$ . From this problem, we obtain a covariance matrix as the solution.

In this Section, we consider a general  $L$ . Then, specifying  $L = 2, 4$ , we can verify the same results for ones obtained in this Section with the techniques discussed in Section IV as follows. With  $L = 2$ , the set of matrices is the symmetric matrices  $\mathbb{S}_P$ , and with  $L = 4$ , we replace a positive semidefinite matrix  $X$  with  $\mathcal{T}(X)$ .

The upper bound of the signal envelope has been shown with Eq. (6.1) as [28]

$$|s(t)|^2 \leq \sum_{k=1}^K |A_k|^2 + 2 \sum_{i=1}^{K-1} |\rho(i)|, \quad (5.42)$$

where

$$\rho(i) = \sum_{k=1}^{K-i} A_k \bar{A}_{k+i} \quad (5.43)$$

and  $\bar{z}$  is the complex conjugate of  $z$ . We let  $\rho(K) = 0$  for latter convenience. The right hand side of Eq. (5.42) is independent of the time  $t$ . Let us define  $\boldsymbol{\rho}' = (\rho(1), \rho(2), \dots, \rho(K-1))^\top$ . Note that the first term in right side of Eq. (5.42),  $\sum_{k=1}^K |A_k|^2$ , corresponds to  $\rho(0)$  and this term is not varied with PTS techniques since each element of a vector  $b_n$  satisfies  $|b_n| = 1$ .

From the above discussion, without taking into account convexity, it is expected to decrease PAPR when we reduce  $\|\boldsymbol{\rho}'\|_{l_1}$ , where  $\|\mathbf{z}\|$  is the  $l_1$ -norm of  $\mathbf{z}$ . However, it is not the case since each  $|\rho(i)|$  is not convex if we regard  $A_k$  as variables. Therefore, we use  $l_2$ -norm of  $\boldsymbol{\rho}'$ ,  $\|\boldsymbol{\rho}'\|_{l_2}$ . From the Cauchy-Schwarz inequality, it follows that

$$\|\boldsymbol{\rho}'\|_{l_1} \leq \sqrt{K-1} \|\boldsymbol{\rho}'\|_{l_2}. \quad (5.44)$$

Therefore,  $\|\boldsymbol{\rho}'\|_{l_1}$  is expected to decrease when  $\|\boldsymbol{\rho}'\|_{l_2}$  decreases.

Let us consider the vector  $\hat{\boldsymbol{\rho}} = (\rho(0), \sqrt{2}\rho(1), \dots, \sqrt{2}\rho(K-1))^\top$ . It is clear that minimizing  $\|\hat{\boldsymbol{\rho}}\|_{l_2}$  is equivalent to minimizing  $\|\sqrt{2}\boldsymbol{\rho}'\|_{l_2}$  since  $\rho(0)$  is constant. Then,  $\|\hat{\boldsymbol{\rho}}\|_{l_2}^2$  is written as

$$\begin{aligned} \|\hat{\boldsymbol{\rho}}\|_{l_2}^2 &= 2 \sum_{k=1}^{K-1} |\rho(k)|^2 + |\rho(0)|^2 \\ &= \sum_{k=0}^{K-1} |\rho(k)|^2 + \sum_{k=0}^{K-1} |\rho(K-k)|^2 \\ &= \frac{1}{2} \left\{ \sum_{k=0}^{K-1} |\rho(k) + \overline{\rho(K-k)}|^2 + \sum_{k=0}^{K-1} |\rho(k) - \overline{\rho(K-k)}|^2 \right\}. \end{aligned} \quad (5.45)$$

From the above equations,  $\|\hat{\rho}\|_{l_2}^2$  is divided into a periodic correlation term and an odd periodic correlation term. With Eq. (5.8), these terms are written as

$$\begin{aligned}\rho(k) + \overline{\rho(K-k)} &= \mathbf{b}^* A^* B_{1,1}^{(k)} A \mathbf{b}, \\ \rho(k) - \overline{\rho(K-k)} &= \mathbf{b}^* A^* B_{-1,1}^{(k)} A \mathbf{b},\end{aligned}\quad (5.46)$$

where the matrices  $B_{1,1}^{(k)}$  and  $B_{-1,1}^{(k)}$  are written as

$$B_{1,1}^{(k)} = \begin{pmatrix} O & I_k \\ I_{K-k} & O \end{pmatrix}, \quad B_{-1,1}^{(k)} = \begin{pmatrix} O & -I_k \\ I_{K-k} & O \end{pmatrix}. \quad (5.47)$$

Since these matrices are regular, they can be transformed to diagonal matrices. With this general discussion, these matrices are decomposed as [136]

$$B_{1,1}^{(k)} = V^* D^{(k)} V, \quad B_{-1,1}^{(k)} = \hat{V}^* \hat{D}^{(k)} \hat{V}, \quad (5.48)$$

where  $V$  and  $\hat{V}$  are unitary matrices whose  $(m, n)$ -th elements are

$$\begin{aligned}V_{m,n} &= \frac{1}{\sqrt{K}} \exp\left(-2\pi j \frac{mn}{K}\right), \\ \hat{V}_{m,n} &= \frac{1}{\sqrt{K}} \exp\left(-2\pi j n \left(\frac{m}{K} + \frac{1}{2K}\right)\right),\end{aligned}\quad (5.49)$$

and  $D^{(k)}$  and  $\hat{D}^{(k)}$  are diagonal matrices whose  $n$ -th diagonal elements are

$$\begin{aligned}D_n^{(k)} &= \exp\left(-2\pi j k \frac{n}{K}\right), \\ \hat{D}_n^{(k)} &= \exp\left(-2\pi j k \left(\frac{n}{K} + \frac{1}{2K}\right)\right).\end{aligned}\quad (5.50)$$

With these expressions, Eq. (5.45) is written as

$$\|\hat{\rho}\|_{l_2}^2 = \frac{K}{2} \left\{ \sum_{k=1}^K |\alpha_k|^4 + \sum_{k=1}^K |\beta_k|^4 \right\}, \quad (5.51)$$

where  $\alpha_k$  and  $\beta_k$  are the  $k$ -th element of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  written as  $\boldsymbol{\alpha} = V A \mathbf{b}$  and  $\boldsymbol{\beta} = \hat{V} A \mathbf{b}$ , respectively. With the variable  $\mathbf{b}$ , the above equation is written as

$$\|\hat{\rho}\|_{l_2}^2 = \frac{K}{2} \sum_{k=1}^K \left\{ (\mathbf{b}^* A^* V^* G_k V A \mathbf{b})^2 + (\mathbf{b}^* A^* \hat{V}^* G_k \hat{V} A \mathbf{b})^2 \right\}, \quad (5.52)$$

where  $G_k$  is a matrix whose  $(k, k)$ -th element is unity and the other elements are zero. Note that  $G_k = G_k^* G_k$ . Then, the matrices  $A^* V^* G_k V A$  and  $A^* \hat{V}^* G_k \hat{V} A$  are positive semidefinite matrices since they are the Gram matrices. Further, Eq. (5.52) is convex with respect to the variable  $\mathbf{b}$ . This is proven in Appendix A. From the above discussions, it follows that the squared  $l_2$ -norm of  $\hat{\rho}$  is a convex function with respect to the variable  $\mathbf{b}$ . Combining these discussions above, we obtain the optimization problem,

$$\begin{aligned} & (Q_{l_2}) \quad \min F(\mathbf{b}) \\ \text{subject to} \quad & b_p \in \Omega_L \quad (p = 1, 2, \dots, P),\end{aligned}\quad (5.53)$$



where

$$F(\mathbf{b}) = \sum_{k=1}^K \left\{ (\mathbf{b}^* A^* V^* G_k V A \mathbf{b})^2 + \left( \mathbf{b}^* A^* \hat{V}^* G_k \hat{V} A \mathbf{b} \right)^2 \right\}. \quad (5.54)$$

To overcome the obstacle caused by the discreteness of  $\Omega_L$ , we obtain a relaxed problem with semidefinite relaxation techniques. This convex problem is written as

$$\begin{aligned} (Q'_{l_2}) \quad & \min \hat{F}(X) \\ \text{subject to} \quad & X_{p,p} = 1 \quad (p = 1, 2, \dots, P) \\ & X \succcurlyeq 0, \quad X \in \mathbb{H}_P, \end{aligned} \quad (5.55)$$

where

$$\hat{F}(\mathbf{X}) = \sum_{k=1}^K \left\{ (A^* V^* G_k V A X)^2 + \left( A^* \hat{V}^* G_k \hat{V} A X \right)^2 \right\}. \quad (5.56)$$

From the above discussions, how to obtain the optimal solution as a positive semidefinite matrix has been shown. Then, we discuss the relations between our randomization method and the relaxed problem  $(Q'_{l_2})$ . Let  $X^*$  and  $\{\boldsymbol{\xi}\}$  be the global solution of the problem  $(Q'_{l_2})$  and the random vectors generated from the Gaussian distribution  $\mathcal{CN}(\mathbf{0}, X^*)$ , respectively. They satisfy  $\mathbb{E}\{\boldsymbol{\xi}\boldsymbol{\xi}^*\} = X^*$ . Then, it follows that

$$\begin{aligned} & \sum_{k=1}^K \left\{ \text{Tr} (A^* V^* G_k V A X^*)^2 + \text{Tr} \left( A^* \hat{V}^* G_k \hat{V} A X^* \right)^2 \right\} \\ & \leq \sum_{k=1}^K \mathbb{E} \left\{ (\boldsymbol{\xi}^* A^* V^* G_k V A \boldsymbol{\xi})^2 + \left( \boldsymbol{\xi}^* A^* \hat{V}^* G_k \hat{V} A \boldsymbol{\xi} \right)^2 \right\} \\ & \leq 3 \sum_{k=1}^K \left\{ \text{Tr} (A^* V^* G_k V A X^*)^2 + \text{Tr} \left( A^* \hat{V}^* G_k \hat{V} A X^* \right)^2 \right\}. \end{aligned} \quad (5.57)$$

The above relations are proven in Appendix B. Our main aim is to find  $X_{l_2}^*$  minimizing  $\mathbb{E}\{F(\boldsymbol{\xi})\}$  under the constraints, where  $\boldsymbol{\xi} \sim \mathcal{CN}(\mathbf{0}, X_{l_2}^*)$ . Two inequalities are involved in Eq. (5.57). The first inequality in Eq (5.57) implies that the global solution of the relaxed problem  $X^*$  does not always correspond to  $X_{l_2}^*$ . However, the last inequality in Eq (5.57) implies that  $X^*$  will be an appropriate solution for our randomization method since  $X^*$  will make  $\mathbb{E}\{F(\boldsymbol{\xi})\}$  small where  $\boldsymbol{\xi} \sim \mathcal{CN}(\mathbf{0}, X^*)$ . From the above discussions, the global solution of the relaxed problem  $X^*$  is not the optimal covariance matrix with our randomization method minimizing upper bound of PAPR. However,  $X^*$  will achieve low PAPR with our randomization method.

## VIII Numerical Results

In this section, from the perspective of PAPR and Bit Error Rate (BER), we compare the performances of our randomization methods with ones of the phase random method and the neighborhood method [118]. We numerically solve the problems  $(\hat{Q}'_L)$  and  $(Q'_{l_2})$  with CVX [137] and obtain approximate solutions

with the two kinds of methods, a  $l_2$  approximation method discussed in Section IV and a random method discussed in Sections V and VII, respectively. As the parameters, the number of carriers  $K = 256$  and the oversampling parameter  $J = 16$  are chosen. We obtain PAPR curves with three kinds of parameters,  $(P, L) = (16, 2), (8, 4)$  and  $(8, 8)$ . The oversampling parameter  $J$  is also used in calculating PAPR (see Eq. (5.16)). As the modulation scheme, each symbol is independently chosen from 16QAM symbols. The index sets  $\Lambda_n$  are randomly chosen from the set

$\{1, 2, \dots, K\}$  and they satisfy  $|\Lambda_n| = \frac{K}{P}$  for  $n = 1, 2, \dots, P$ , where  $|\Lambda_n|$  is the number of components in the set  $\Lambda_n$ . It has been known that PAPR with random index sets is lower than one with adjacent index sets [120]. With our randomization methods discussed in Sections V and VII, and the phase random method, we generate 10 and 70 samples as solution candidates and choose the optimal solution from such candidates (see Algorithm 1). For the brute force method and the other methods, we draw the PAPR curves from 200 results and 2000 results, respectively. Note that the PAPR curve with the brute force method is optimal. For the neighborhood method, we set the Hamming weight parameter  $r = 1$  and the maximum iteration parameter  $I = 5, 4$  and  $3$  in  $(P, L) = (16, 2), (8, 4)$  and  $(8, 8)$ , respectively. Note that the number of candidates in the neighborhood method is roughly calculated as  $I(P-1)(L-1)$  when  $r = 1$ . With these parameters, the number of candidates in the neighborhood method is roughly calculated as 70. Thus, the search complexity of the neighborhood method is nearly equivalent to one of our methods.

Figures 5.1, 5.2 and 5.3 show each PAPR curve with original OFDM systems, the brute force method [114], the  $l_2$  approximation method discussed in Section IV, the randomization method discussed in Section V, the reducing upper-bound method discussed in Section VII, the phase random method [119] and the neighborhood method [118]. In the legends, “ $l_2$  approximation”, “Ours (PAPR)”, “Ours (Upper Bound)” and “Neighborhood” mean the  $l_2$  approximation method, the randomization method discussed in Section V, the reducing upper-bound method discussed in Section VII and the neighborhood method, respectively. In Fig. 5.3, the PAPR curve with the brute force method is not drawn since it is not straightforward to obtain the optimal vector due to its significantly large calculation amount. From these figures, the PAPR curve with the  $l_2$  approximation method is far from one with the brute force method. This result shows that the optimal solution of the relaxed problem is far from a rank-1 matrix and it tends to have some large eigenvalues. Therefore, we conclude that the  $l_2$  approximation method is not suitable for PTS techniques.

With randomization methods, there are two PAPR curves obtained from 10 random vectors and 70 random vectors. In both numbers of random vectors, the PAPRs of our two randomization methods are lower than one of phase random techniques. As seen in Section VI, the phase random method is equivalent to our method with the identical matrix as a covariance matrix. Therefore, the performance of randomization methods can be improved when a suitable covariance matrix is chosen. Further, with 70 samples, PAPRs of our two randomization methods are lower than one of the neighborhood method. Since the search region of the neighborhood method is roughly equivalent to ones of our methods, these results show that our methods can find a suitable vector more efficiently than the neighborhood method.

In Section VII, we have discussed the method to reduce the upper bound

of PAPR. From the numerical results, PAPR with the reducing upper bound method is larger than one of the randomization method discussed in Section V. However, in a sense of solving optimization problems, the complexity with the reducing upper bound method is lower than one with the randomization method discussed in Section V. The reason is as follows. The main point of this method is that the problem reducing upper bound of PAPR is independent of the oversampling parameter  $J$ . With this and the number of constraints is independent of  $J$ , the complexity of the solver does not increase as  $J$  increase. On the other hand, in the randomization method discussed in Section V, the number of constraints gets larger as  $J$  increases. Further, as seen in Eq. (5.16), a sufficiently large  $J$  is necessary for calculating PAPR tightly. Then, in the randomization method discussed in Section V, the number of constraints is larger than one of the reducing upper bound method. From the discussion in Section V, computational complexity gets larger as  $J$  increases. Therefore, the reducing upper bound method can achieve low PAPR with low complexity.

To explore the performances with our method and the phase random method, we evaluate each BER. To this end, the amplifier model is chosen as the Rapp model [20], which is described below. Let the input signal be presented in polar coordinates,

$$x(t) = \rho(t) \exp(j\theta(t)). \quad (5.58)$$

Then, the output signal is written as

$$\zeta(x(t)) = \gamma(\rho(t)) \cdot \exp(j \cdot (\theta(t) + \Phi(\rho(t)))), \quad (5.59)$$

where  $\gamma$  and  $\Phi$  are functions of the amplitude  $\rho(t)$ . In the Rapp model, these two functions are chosen as

$$\gamma(\rho) = \frac{\rho}{(1 + (\frac{\rho}{r})^p)^{\frac{1}{2p}}}, \quad \Phi(\rho) = 0, \quad (5.60)$$

where  $\rho$  is an amplitude,  $r$  is the clipping level and  $p$  is the real parameter. The parameter  $p$  is often chosen as  $p = 2$  or  $p = 3$  [21] [22] [23].

Figures 5.4, 5.5 and 5.6 show each BER curve with the parameter  $(P, L) = (16, 2), (8, 4)$  and  $(8, 8)$ . In these figures,  $E_b$  denotes the average energy per bit at the receiver which is need for a reliable recovery of the information. In the Rapp model, we set the parameters  $p = 2$  and  $r = \sqrt{P_{av}} 10^{\frac{1}{10}}$  ( $r = 2$  [dB]). The simulation environment is the same to one in calculating PAPR. As seen in Figs. 5.1-5.3, we have found that our methods discussed in Section V and VII have lower PAPR than ones of the phase random method and the neighborhood method. From Figs. 5.4-5.6, with nearly the same search complexity, BER of our methods discussed in Section V and VII is lower than ones of the two kinds of methods, the phase random method and the neighborhood method. From the above results, we conclude that our methods can achieve low PAPR more efficiently than the phase random method and the neighborhood method.

From the above two kinds of the results, our methods can achieve lower PAPR and BER than ones of the phase random method, the neighborhood method and conventional OFDM signals.

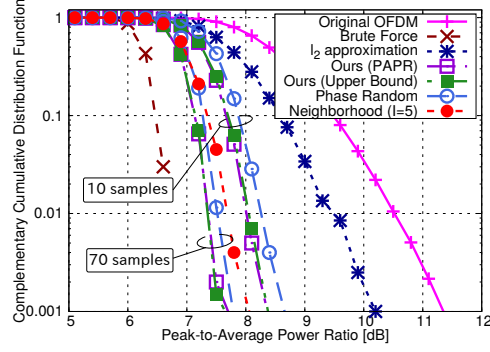


Figure 5.1: PAPR with the parameters  $(P, L) = (16, 2)$

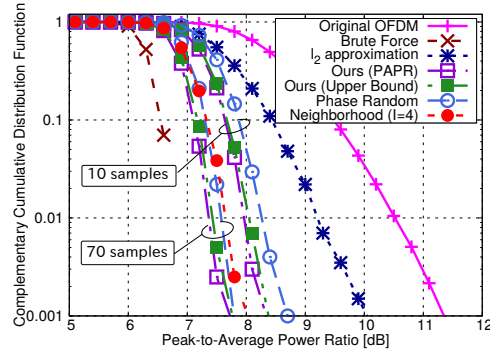


Figure 5.2: PAPR with the parameters  $(P, L) = (8, 4)$

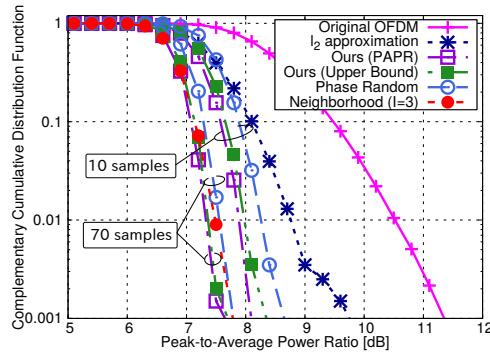


Figure 5.3: PAPR with the parameters  $(P, L) = (8, 8)$

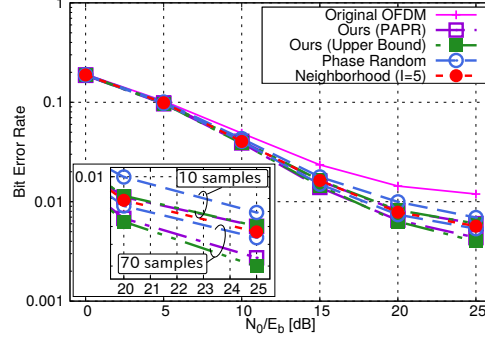


Figure 5.4: BER with the parameters  $(P, L) = (16, 2)$

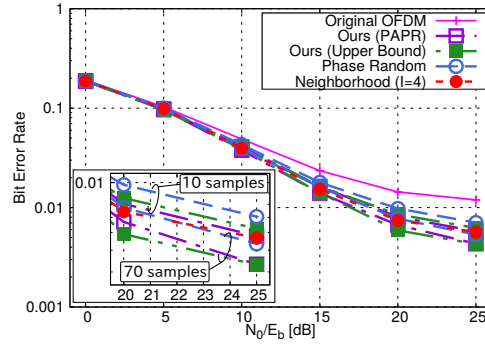


Figure 5.5: BER with the parameters  $(P, L) = (8, 4)$

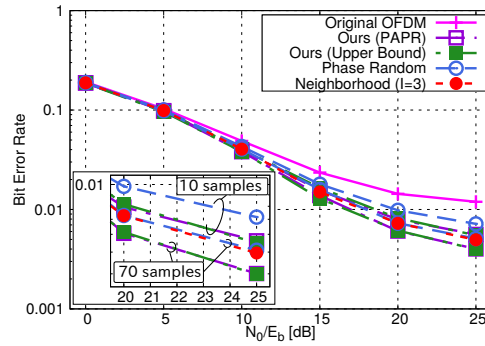


Figure 5.6: BER with the parameters  $(P, L) = (8, 8)$

## IX Summary and Discussions

In this chapter, we have discussed how to obtain a suitable vector for partial transmit sequence techniques and have proposed two kinds of randomization methods with semidefinite relaxation techniques. Further, we have shown the relation between our methods and the phase random method. Then, in our numerical results, we have shown their PAPR curves and that our methods can achieve lower PAPR than ones with the phase random method and the neighborhood method. Moreover, our numerical results have implied that randomization methods can achieve lower PAPR if a more suitable covariance matrix is obtained.

A remaining issue is to explore how to obtain a suitable covariance matrix for a randomization method. As seen in the discussions in Section V, it takes time to solve the optimization problem numerically to obtain the covariance matrix. Therefore, one of necessities to address computational complexity is to obtain the explicit form of a suitable covariance matrix. After giving such an explicit way, we expect an ideal method for obtaining low PAPR.

### 5.A Proof of Convexity of Equation (5.52)

In this appendix, we prove that the function defined in Eq. (5.52)

$$\|\hat{\rho}\|_{l_2}^2 = \frac{K}{2} \sum_{k=1}^K \left\{ (\mathbf{b}^* A^* V^* G_k V A \mathbf{b})^2 + (\mathbf{b}^* A^* \hat{V}^* G_k \hat{V} A \mathbf{b})^2 \right\},$$

is convex with respect to  $\mathbf{b}$ .

First, it follows that the matrices  $A^* V^* G_k V A$  and  $A^* \hat{V}^* G_k \hat{V} A$  are positive semidefinite matrices since they are the Gram matrices. To prove the convexity of the above function, it is sufficient to prove that each term of the above function is convex since the sum of convex functions is convex. Therefore, we prove

$$\begin{aligned} & \gamma (\mathbf{b}_1^* G \mathbf{b}_1)^2 + (1 - \gamma) (\mathbf{b}_2^* G \mathbf{b}_2)^2 \\ & \geq ((\gamma \mathbf{b}_1 + (1 - \gamma) \mathbf{b}_2)^* G (\gamma \mathbf{b}_1 + (1 - \gamma) \mathbf{b}_2))^2, \end{aligned} \quad (5.61)$$

where  $\gamma \in [0, 1]$ ,  $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{C}^P$  and  $G$  is a positive semidefinite matrix corresponding to either  $A^* V^* G_n V A$  or  $A^* \hat{V}^* G_n \hat{V} A$ .

Let us prove the convexity. Since  $x^2$  is a convex and non-decreasing function for  $x \geq 0$  and  $\mathbf{b}^* G \mathbf{b}$  is convex and non-negative, the following inequalities are satisfied

$$\begin{aligned} & ((\gamma \mathbf{b}_1 + (1 - \gamma) \mathbf{b}_2)^* G (\gamma \mathbf{b}_1 + (1 - \gamma) \mathbf{b}_2))^2 \\ & \leq (\gamma \mathbf{b}_1^* G \mathbf{b}_1 + (1 - \gamma) \mathbf{b}_2^* G \mathbf{b}_2)^2. \\ & \leq \gamma (\mathbf{b}_1^* G \mathbf{b}_1)^2 + (1 - \gamma) (\mathbf{b}_2^* G \mathbf{b}_2)^2. \end{aligned} \quad (5.62)$$

Applying the above inequalities to each term of Eq. (5.52), and the sum of convex functions is convex, we have that  $\|\hat{\rho}\|_{l_2}^2$  in Eq. (5.52) is convex. Another proof has been obtained with the Theorem 5.1 written in [99].

## 5.B Proof of Relations in Equation (5.57)

In this appendix, we prove the relations written in Eq. (5.57)

$$\begin{aligned}
& \sum_{k=1}^K \left\{ \text{Tr} (A^* V^* G_k V A X^*)^2 + \text{Tr} (A^* \hat{V}^* G_k \hat{V} A X^*)^2 \right\} \\
& \leq \sum_{k=1}^K \mathbb{E} \left\{ (\boldsymbol{\xi}^* A^* V^* G_k V A \boldsymbol{\xi})^2 + (\boldsymbol{\xi}^* A^* \hat{V}^* G_k \hat{V} A \boldsymbol{\xi})^2 \right\} \\
& \leq 3 \sum_{k=1}^K \left\{ \text{Tr} (A^* V^* G_k V A X^*)^2 + \text{Tr} (A^* \hat{V}^* G_k \hat{V} A X^*)^2 \right\}
\end{aligned}$$

for  $\boldsymbol{\xi} \sim \mathcal{CN}(\mathbf{0}, X^*)$ .

A proof that the first inequality holds is given as follows. From the Cauchy-Schwarz inequality, it holds that

$$\begin{aligned}
& \sum_{k=1}^K \left\{ \text{Tr} (A^* V^* G_k V A X^*)^2 + \text{Tr} (A^* \hat{V}^* G_k \hat{V} A X^*)^2 \right\} \\
& \leq \sum_{k=1}^K \mathbb{E} \left\{ (\boldsymbol{\xi}^* A^* V^* G_k V A \boldsymbol{\xi})^2 + (\boldsymbol{\xi}^* A^* \hat{V}^* G_k \hat{V} A \boldsymbol{\xi})^2 \right\}.
\end{aligned} \tag{5.63}$$

Then, a proof that the last inequality holds is given as follows. Similar to Appendix A, it is sufficient to prove

$$\mathbb{E} \left\{ (\boldsymbol{\xi}^* G \boldsymbol{\xi})^2 \right\} \leq 3 \text{Tr} (G X^*), \tag{5.64}$$

where  $G$  is a Hermitian and positive semidefinite matrix and  $\boldsymbol{\xi} \sim \mathcal{CN}(\mathbf{0}, X^*)$ . In [17], it has been shown that  $\mathcal{T}(\mathbf{z}) \sim \mathcal{N}(\mathcal{T}(\boldsymbol{\mu}), \frac{1}{2}\mathcal{T}(\Sigma))$  if  $\mathbf{z} \sim \mathcal{CN}(\boldsymbol{\mu}, \Sigma)$ . Note that the matrices  $\mathcal{T}(X^*)$  and  $\mathcal{T}(G)$  are symmetric and positive semidefinite since  $G$  and  $X^*$  are Hermitian and positive semidefinite [94]. From this result, it follows that  $\mathcal{T}(\boldsymbol{\xi}) \sim \mathcal{N}(\mathcal{T}(\mathbf{0}), \frac{1}{2}\mathcal{T}(X^*))$ . With this and discussions in [94], the left hand side of Eq. (5.64) is rewritten as

$$\begin{aligned}
& \mathbb{E} \left\{ (\boldsymbol{\xi}^* G \boldsymbol{\xi})^2 \right\} \\
& = \mathbb{E} \left\{ (\mathcal{T}(\boldsymbol{\xi})^\top \mathcal{T}(G) \mathcal{T}(\boldsymbol{\xi}))^2 \right\} \\
& = \sum_{i,j,k,l} \hat{g}_{i,j} \hat{g}_{k,l} \mathbb{E} \{ \hat{\xi}_i \hat{\xi}_j \hat{\xi}_k \hat{\xi}_l \},
\end{aligned} \tag{5.65}$$

where  $\hat{g}_{i,j}$  and  $\hat{\xi}_k$  are the  $(i,j)$ -th element of  $\mathcal{T}(G)$  and  $k$ -th element of  $\mathcal{T}(\boldsymbol{\xi})$ , respectively. In [138], for  $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ , the fourth moment about the mean has been derived as

$$\begin{aligned}
& \mathbb{E} \{ (x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k)(x_l - \mu_l) \} \\
& = \sigma_{i,j} \sigma_{k,l} + \sigma_{i,k} \sigma_{j,l} + \sigma_{i,l} \sigma_{j,k},
\end{aligned} \tag{5.66}$$

where  $x_i$ ,  $\mu_i$  and  $\sigma_{i,j}$  are the  $i$ -th element of  $\mathbf{x}$ , the  $i$ -th element of  $\boldsymbol{\mu}$  and the  $(i,j)$ -th element of the real valued-covariance matrix  $\Sigma$ , respectively. With Eq.

(5.66), Eq. (5.64) is rewritten as

$$\begin{aligned}
& \mathbb{E} \left\{ (\boldsymbol{\xi}^* G \boldsymbol{\xi})^2 \right\} \\
&= \frac{1}{4} \left\{ \sum_{i,j,k,l} \hat{g}_{i,j} \hat{g}_{k,l} (\hat{x}_{i,j}^* \hat{x}_{k,l}^* + \hat{x}_{i,k}^* \hat{x}_{j,l}^* + \hat{x}_{i,l}^* \hat{x}_{j,k}^*) \right\} \\
&= \frac{1}{4} \left\{ \text{Tr}(\mathcal{T}(G) \mathcal{T}(X^*))^2 + 2 \text{Tr}(\mathcal{T}(G) \mathcal{T}(X^*) \mathcal{T}(G) \mathcal{T}(X^*)) \right\},
\end{aligned} \tag{5.67}$$

where  $\hat{x}_{i,j}^*$  is the  $(i,j)$ -th element of  $\mathcal{T}(X^*)$ . In deriving the above second equality, we have used the property that the matrices  $\mathcal{T}(G)$  and  $\mathcal{T}(X^*)$  are symmetric. Let  $V$  be a matrix such that  $VV^\top = \mathcal{T}(G)$ , where such a  $V$  can be found since  $\mathcal{T}(G)$  is a positive semidefinite. With this decomposition, the relations

$$\begin{aligned}
& \text{Tr}(\mathcal{T}(G) \mathcal{T}(X^*) \mathcal{T}(G) \mathcal{T}(X^*)) \\
&= \text{Tr}(VV^\top \mathcal{T}(X^*) VV^\top \mathcal{T}(X^*)) \\
&= \text{Tr}(V^\top \mathcal{T}(X^*) V \cdot V^\top \mathcal{T}(X^*) V) \\
&\leq \text{Tr}(V^\top \mathcal{T}(X^*) V)^2 \\
&= \text{Tr}(\mathcal{T}(X^*) \mathcal{T}(G))^2
\end{aligned} \tag{5.68}$$

are obtained. In the above relations, we have used the properties that the matrix  $V^\top \mathcal{T}(X^*) V$  is positive semidefinite and  $\text{Tr}(XX) \leq \text{Tr}(X)^2$  for any positive semidefinite matrix  $X$ . In [95], it has been shown that  $\text{Tr}(\mathcal{T}(X) \mathcal{T}(Y)) = 2 \text{Tr}(XY)$  for positive semidefinite matrices  $X$  and  $Y$ . Combining this result and Eqs. (5.67) (5.68), we arrive at the relation

$$\mathbb{E} \left\{ (\boldsymbol{\xi}^* G \boldsymbol{\xi})^2 \right\} \leq 3 \text{Tr}(GX^*)^2. \tag{5.69}$$

This is the desired result.

We have proven Eq. (5.64) for general  $L$ . For  $L = 2$  and  $L = 4$ , we have the same expressions of Eq. (5.64).



## Chapter 6

# Moment-Based Upper Bound on PAPR

### I Introduction

As seen in the previous chapter, in OFDM systems, to reduce PAPR is one of significant tasks since it is known that in-band distortion and out-of-band distortion are caused by a large input power. These distortions are caused by the non-linearity of amplifiers with respect to an input power for a large power regime [9].

By definition, PAPR depends on a given codeword. However, PAPR is often regarded as a random variable since a codeword can also be regarded as a random variable. To investigate performances of PAPR-reducing methods, the complementary cumulative distribution function (CCDF) of PAPR is often evaluated [105] [106] [107] [111] [112]. Therefore, it is demanded to obtain the form of the CCDF. When each codeword is randomly and independently chosen from a given distribution and the central limit theorem can be applied, approximate forms of the CCDF have been obtained [102] [103]. These results are based on that an OFDM signal can be regarded as a Gaussian process. Furthermore, it has been proven that usual coded-OFDM signals can be regarded as Gaussian processes [26]. On the other hand, in the case where the central limit theorem cannot be applied, an approximate form has not been obtained.

In the case where the central limit theorem cannot be applied, the upper bounds of the CCDF of PAPR have been obtained [121] [104] [11]. It is expected to achieve lower PAPR as the upper bound decreases. Further, classes of error correction codes achieving low PAPR have been obtained [121]. To obtain upper bounds, some assumptions are often required. One of usual assumptions is about modulation schemes. Thus, with a given modulation scheme, methods to reduce PAPR have been discussed.

However, there is a case where codewords do not belong to a popular modulation scheme. For example, after applying an iterative clipping and filtering method [111] [139], it is unclear what modulation scheme each symbol in codewords belongs to. In such a situation, known PAPR bounds could not be valid. Therefore, it is demanded to obtain a more generalized bound under no assumption about a modulation scheme.

In this chapter, we derive an upper bound of the CCDF of PAPR with no assumption about a modulation scheme. Our bound is written in terms of fourth moments of codewords. As a similar bound, it has been proven that there is a bound which is written in terms of moments in BPSK systems [104]. Therefore, our result can be regarded as a generalization of such an existing result.

To reduce PAPR, we apply the technique which has been developed in Independent Component Analysis (ICA) [140]. The main idea of ICA is to find a suitable unitary matrix to reduce the kurtosis, which is a statistical quantity written in terms of fourth moments. From this idea used in ICA, it is expected that our bound can be reduced with unitary matrices since our bound is also written in terms of fourth moments of codewords. The known methods, a Partial Transmit Sequence (PTS) technique and a Selective Mapping (SLM) method are to modulate the phase of each symbol to reduce PAPR. Therefore, these known methods are to transform codewords with diagonal-unitary matrices and our method can be regarded as a generalization of these methods.

## II OFDM System and PAPR

In this section, we show the OFDM system model and the definition of PAPR. First, a complex baseband OFDM signal is written as

$$s(t) = \sum_{k=1}^K A_k \exp\left(2\pi j \frac{k-1}{T} t\right), \quad 0 \leq t < T, \quad (6.1)$$

where  $A_k$  is a transmitted symbol,  $K$  is the number of symbols,  $j$  is the unit imaginary number, and  $T$  is a duration of symbols. With Eq. (6.1), a radio frequency (RF) OFDM signal is written as

$$\begin{aligned} \zeta(t) &= \text{Re}\{s(t) \exp(2\pi j f_c t)\} \\ &= \text{Re}\left\{\sum_{k=1}^K A_k \exp\left(2\pi j \left(\frac{k-1}{T} + f_c\right) t\right)\right\}, \end{aligned} \quad (6.2)$$

where  $\text{Re}\{z\}$  is the real part of  $z$ , and  $f_c$  is a carrier frequency. With RF signals, PAPR is defined as [121] [122]

$$\begin{aligned} &\text{PAPR}(\mathbf{c}) \\ &= \max_{0 \leq t < T} \frac{\left| \text{Re}\left\{\sum_{k=1}^K A_k \exp\left(2\pi j \left(\frac{k-1}{T} + f_c\right) t\right)\right\} \right|^2}{P_{\text{av}}}, \end{aligned} \quad (6.3)$$

where  $\mathbf{c} = (A_1, A_2, \dots, A_K)^\top \in \mathcal{C}$  is a codeword,  $\mathbf{x}^\top$  is the transpose of  $\mathbf{x}$ ,  $\mathcal{C}$  is the set of codewords,  $P_{\text{av}}$  corresponds to the average power of signals,  $P_{\text{av}} = \sum_{k=1}^K \text{E}\{|A_k|^2\}$ , and  $\text{E}\{X\}$  is the average of  $X$ . On the other hand, with baseband signals, Peak-to-Mean Envelope Power Ratio (PMEPR) is defined as [121] [122]

$$\text{PMEPR}(\mathbf{c}) = \max_{0 \leq t < T} \frac{\left| \sum_{k=1}^K A_k \exp\left(2\pi j \frac{k-1}{T} t\right) \right|^2}{P_{\text{av}}}. \quad (6.4)$$

As seen in Eqs (6.3) and (6.4), PAPR and PMEPR are determined by the codeword  $\mathbf{c}$  and it is clear that  $\text{PAPR}(\mathbf{c}) \leq \text{PMEPR}(\mathbf{c})$  for any codeword  $\mathbf{c}$ . In [18], it has been proven that the following relation is established under some conditions described below

$$\left(1 - \frac{\pi^2 K^2}{2r^2}\right) \cdot \text{PMEPR}(\mathbf{c}) \leq \text{PAPR}(\mathbf{c}) \leq \text{PMEPR}(\mathbf{c}), \quad (6.5)$$

where  $r$  is an integer such that  $f_c = r/T$ . The conditions that Eq. (6.5) holds are  $K \ll r$  and  $\exp(2\pi j K/r) \approx 1$ . In addition to these, another relation has been shown in [11]. From Eq. (6.5), PAPR is approximately equivalent to PMEPR for sufficiently large  $f_c$ . Throughout this chapter, we assume that the carrier frequency  $f_c$  is sufficiently large, and we consider PMEPR instead of PAPR.

### III Bound of Peak-to-Average Power Ratio

In this section, we show the bound of a CCDF of PAPR. As seen in Section II, PAPR and PMEPR depend on a given codeword. Since codewords are regarded as random variables, PAPR and PMEPR are also regarded as random variables. In what follows, we merely write PAPR in formulas when PAPR is a random variable.

First, we make the following assumptions

- the probability density of  $\mathbf{c}$ ,  $p(\mathbf{c})$  is given and fixed.
- the carrier frequency  $f_c$  is sufficiently large.
- For  $1 \leq k, l, m, n \leq K$ , the statistical quantity  $E\{A_k A_l \bar{A}_m \bar{A}_n\}$  exists, where  $\bar{z}$  is the conjugate of  $z$ .

The second assumption about a carrier frequency is often used [102]. As seen in Section II, PAPR is approximately equivalent to PMEPR if the carrier frequency  $f_c$  is sufficiently large. Thus, we consider PMEPR instead of PAPR. The last assumption has been used in [11]. We call the quantity  $E\{A_k A_l \bar{A}_m \bar{A}_n\}$  the fourth moment of  $A_k$ ,  $A_l$ ,  $\bar{A}_m$  and  $\bar{A}_n$ . For details about complex multivariate distributions and moments, we refer the reader to [141] [17] [142]. From the Cauchy-Schwarz inequality and this assumption, it can be proven that the average power  $P_{\text{av}}$  exists, that is,  $P_{\text{av}} < \infty$ .

Let us consider the PAPR with a given codeword  $\mathbf{c} = (A_1, A_2, \dots, A_K)^\top$ . In [28], the following relation has been proven

$$\max_t |s(t)|^2 \leq \rho(0) + 2 \sum_{i=1}^{K-1} |\rho(i)|, \quad (6.6)$$

where

$$\rho(i) = \sum_{k=1}^{K-i} A_k \bar{A}_{k+i}. \quad (6.7)$$

We let  $\rho(K)$  be 0. Note that the quantity  $\rho(0)$  is the power of a codeword and that the time  $t$  does not appear in the right hand side (r.h.s) of Eq. (6.6). It is

not straightforward to analyze Eq. (6.6) since the absolute-value terms appear in Eq. (6.6). To overcome this obstacle, we obtain the upper bound of r.h.s of Eq. (6.6). From the Cauchy-Schwarz inequality, we obtain the following relation

$$\begin{aligned} \max_t |s(t)|^2 &\leq \rho(0) + 2 \sum_{i=1}^{K-1} |\rho(i)| \\ &\leq \sqrt{2K-1} \sqrt{|\rho(0)|^2 + 2 \sum_{i=1}^{K-1} |\rho(i)|^2}. \end{aligned} \quad (6.8)$$

The above bound is rewritten as

$$\max_t |s(t)|^4 \leq (2K-1) \left\{ |\rho(0)|^2 + 2 \sum_{i=1}^{K-1} |\rho(i)|^2 \right\}. \quad (6.9)$$

The r.h.s of Eq. (6.9) is rewritten as

$$\begin{aligned} &(2K-1) \left\{ |\rho(0)|^2 + 2 \sum_{k=1}^{K-1} |\rho(k)|^2 \right\} \\ &= (2K-1) \left\{ \sum_{k=0}^{K-1} |\rho(k)|^2 + \sum_{k=0}^{K-1} |\rho(K-k)|^2 \right\} \\ &= \frac{2K-1}{2} \left\{ \sum_{k=0}^{K-1} |\rho(k) + \overline{\rho(K-k)}|^2 \right. \\ &\quad \left. + \sum_{k=0}^{K-1} |\rho(k) - \overline{\rho(K-k)}|^2 \right\}. \end{aligned} \quad (6.10)$$

From the above equations, the r.h.s of Eq. (6.9) is written with periodic correlation terms and odd periodic correlation terms. These terms are written as

$$\begin{aligned} \rho(k) + \overline{\rho(K-k)} &= \mathbf{c}^* B_{1,1}^{(k)} \mathbf{c}, \\ \rho(k) - \overline{\rho(K-k)} &= \mathbf{c}^* B_{-1,1}^{(k)} \mathbf{c}, \end{aligned} \quad (6.11)$$

where  $\mathbf{z}^*$  is the conjugate transpose of  $\mathbf{z}$ , the matrices  $B_{1,1}^{(k)}$  and  $B_{-1,1}^{(k)}$  are

$$B_{1,1}^{(k)} = \begin{pmatrix} O & I_k \\ I_{K-k} & O \end{pmatrix}, \quad B_{-1,1}^{(k)} = \begin{pmatrix} O & -I_k \\ I_{K-k} & O \end{pmatrix}. \quad (6.12)$$

Since these matrices are regular, they can be transformed to diagonal matrices. From this general discussion, these matrices are decomposed with the eigenvalue decomposition as [136]

$$B_{1,1}^{(k)} = V^* D^{(k)} V \quad B_{-1,1}^{(k)} = \hat{V}^* \hat{D}^{(k)} \hat{V}, \quad (6.13)$$

where  $V$  and  $\hat{V}$  are unitary matrices whose  $(m, n)$ -th elements are

$$\begin{aligned} V_{m,n} &= \frac{1}{\sqrt{K}} \exp \left( -2\pi j \frac{mn}{K} \right), \\ \hat{V}_{m,n} &= \frac{1}{\sqrt{K}} \exp \left( -2\pi j n \left( \frac{m}{K} + \frac{1}{2K} \right) \right), \end{aligned} \quad (6.14)$$

and  $D^{(k)}$  and  $\hat{D}^{(k)}$  are diagonal matrices whose  $n$ -th diagonal elements are

$$\begin{aligned} D_n^{(k)} &= \exp\left(-2\pi j k \frac{n}{K}\right), \\ \hat{D}_n^{(k)} &= \exp\left(-2\pi j k \left(\frac{n}{K} + \frac{1}{2K}\right)\right). \end{aligned} \quad (6.15)$$

With these expressions, Eq. (6.9) is written as

$$\max_t |s(t)|^4 \leq \frac{K(2K-1)}{2} \left\{ \sum_{k=1}^K |\alpha_k|^4 + \sum_{k=1}^K |\beta_k|^4 \right\}, \quad (6.16)$$

where  $\alpha_k$  and  $\beta_k$  are the  $k$ -th element of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  written as  $\boldsymbol{\alpha} = V\mathbf{c}$  and  $\boldsymbol{\beta} = \hat{V}\mathbf{c}$ , respectively. With the codeword  $\mathbf{c}$ , the above inequality is written as

$$\begin{aligned} &\max_t |s(t)|^4 \\ &\leq \frac{K(2K-1)}{2} \sum_{k=1}^K \left\{ (\mathbf{c}^* V^* G_k V \mathbf{c})^2 + (\mathbf{c}^* \hat{V}^* G_k \hat{V} \mathbf{c})^2 \right\}, \end{aligned} \quad (6.17)$$

where  $G_k$  is a matrix whose  $(k, k)$ -th element is unity and the other elements are zero. Note that  $G_k^* G_k = G_k$ . For the later convenience, we set  $C_k = V^* G_k V$  and  $\hat{C}_k = \hat{V}^* G_k \hat{V}$ , respectively. Note that the matrices  $C_k$  and  $\hat{C}_k$  are positive semidefinite Hermitian matrices since  $C_k$  and  $\hat{C}_k$  are the Gram matrices. From Eq. (6.17), with a given codeword  $\mathbf{c}$ , the bound of the squared PAPR is obtained as

$$\begin{aligned} \text{PAPR}(\mathbf{c})^2 &\leq \frac{\max_t |s(t)|^4}{P_{\text{av}}^2} \\ &\leq \frac{K(2K-1)}{2P_{\text{av}}^2} \sum_{k=1}^K \left\{ (\mathbf{c}^* C_k \mathbf{c})^2 + (\mathbf{c}^* \hat{C}_k \mathbf{c})^2 \right\}. \end{aligned} \quad (6.18)$$

In the above relations, the first inequality is obtained from the result that  $\text{PAPR}(\mathbf{c}) \leq \text{PMEPR}(\mathbf{c})$ .

From the above discussions, we have arrived at the bound of PAPR with a given codeword  $\mathbf{c}$ . From this bound, we can obtain the bound of the CCDF of PAPR as follows. Let  $\Pr(\text{PAPR} > \gamma)$  be the CCDF of PAPR, where  $\gamma$  is positive. Then, the following relations are obtained

$$\begin{aligned} &\Pr(\text{PAPR} > \gamma) \\ &= \Pr(\text{PAPR}^2 > \gamma^2) \\ &\leq \Pr(\max_t |s(t)|^4 > P_{\text{av}}^2 \gamma^2) \\ &\leq \frac{\mathbb{E} \{ \max_t |s(t)|^4 \}}{P_{\text{av}}^2 \gamma^2} \\ &\leq \frac{K(2K-1)}{2P_{\text{av}}^2 \gamma^2} \sum_{k=1}^K \mathbb{E} \left\{ (\mathbf{c}^* C_k \mathbf{c})^2 + (\mathbf{c}^* \hat{C}_k \mathbf{c})^2 \right\}. \end{aligned} \quad (6.19)$$

In the course of deriving Eq. (6.19), the first equation has been obtained from the fact that PAPR is positive. The first inequality has been obtained from Eq.

(6.4) and the fact that  $\text{PAPR}(\mathbf{c}) \leq \text{PMEPR}(\mathbf{c})$  for any codeword  $\mathbf{c}$  (see Section II). The second inequality has been obtained with the Markov inequality [88]. The last inequality has been obtained from Eq. (6.17).

As seen in Eq. (6.19), the bound of the CCDF is written in terms of the fourth moments of codewords and the bound does not depend on a modulation scheme. Further, if each codeword  $\mathbf{c}$  is randomly and uniformly chosen from the set of codewords  $\mathcal{C}$  and the number of codewords is finite, then Eq. (6.19) is written as [11]

$$\begin{aligned} & \Pr(\text{PAPR} > \gamma) \\ & \leq \frac{1}{|\mathcal{C}|} \frac{K(2K-1)}{2P_{\text{av}}^2 \gamma^2} \sum_{\mathbf{c} \in \mathcal{C}} \sum_{k=1}^K \left\{ (\mathbf{c}^* C_k \mathbf{c})^2 + (\mathbf{c}^* \hat{C}_k \mathbf{c})^2 \right\}, \end{aligned} \quad (6.20)$$

where  $|\mathcal{C}|$  is the number of components in  $\mathcal{C}$ .

## IV Reducing PAPR with Unitary Matrix

In Section III, we have obtained the bound of the CCDF of PAPR. From Eq. (6.19), the bound is written in terms of the fourth moments of codewords. It is expected that PAPR decreases as the bound decreases. In this section, we propose a method to reduce the bound with unitary matrices. Our technique can be seen in ICA [140] [36] since the main idea of ICA is to reduce the kurtosis, which is written in terms of the fourth moment.

In known methods, it has been proposed to modulate the phase of each symbol to reduce PAPR, and these methods are to transform a codeword with a diagonal-unitary matrix. Thus, our technique can be regarded as an extension of these methods.

In addition to the assumptions made in Section III, we make the following assumptions to introduce a technique to reduce our bound

- the number of components in codewords,  $|\mathcal{C}|$  is finite, that is,  $|\mathcal{C}| = M < \infty$ .
- each codeword is chosen with equal probability from  $\mathcal{C}$ .

Under the above assumptions, we propose a method to reduce PAPR with unitary matrices. The main idea of our method is to find unitary matrices which make our bound small. Through our technique, the average power  $P_{\text{av}}$  and SNR are preserved.

To introduce our method, we define subsets of codewords. First, from the above assumption, we can divide the codewords  $\mathcal{C}$  into  $N$  disjoint subsets which satisfy

$$\mathcal{C} = \bigcup_{n=1}^N \mathcal{C}_n, \quad \mathcal{C}_m \cap \mathcal{C}_n = \emptyset \quad \text{for } m \neq n. \quad (6.21)$$

Since the number of components in  $\mathcal{C}$  is finite, each number of components in  $\mathcal{C}_n$  is also finite. For each subset  $\mathcal{C}_n$ , we define a unitary matrix  $W_n$ .

The scheme of our method is described as follows. First, let the transmitter and the receiver know the unitary matrices  $\{W_n\}_{n=1}^N$ . At the transmitter side, each codeword  $\mathbf{c} \in \mathcal{C}_i$  is modulated to  $W_i \mathbf{c}$  with the unitary matrix  $W_i$ . Then,

the transmitter sends the number  $i$  and  $W_i \mathbf{c}$ . At the receiver side, the symbol  $\mathbf{y}$  and the number  $i$  are received. Then, the receiver estimates the codeword  $\hat{\mathbf{c}}$  as  $\hat{\mathbf{c}} = W_i^* \mathbf{y}$ . It is clear that  $\hat{\mathbf{c}} = \mathbf{c}$  if  $\mathbf{y} = W_i \mathbf{c}$ . With the above scheme, the bound in Eq. (6.20) is written as

$$\begin{aligned} & \Pr(\text{PAPR} > \gamma) \\ & \leq \frac{1}{M} \frac{K(2K-1)}{2P_{\text{av}}^2 \gamma^2} \\ & \quad \cdot \sum_{n=1}^N \sum_{\mathbf{c} \in \mathcal{C}_n} \sum_{k=1}^K \left\{ (\mathbf{c}^* W_n^* C_k W_n \mathbf{c})^2 + (\mathbf{c}^* W_n^* \hat{C}_k W_n \mathbf{c})^2 \right\}. \end{aligned} \quad (6.22)$$

In known methods, a PTS technique and a SLM method, one diagonal unitary matrix corresponds to one codeword. By contrast, in our methods, one unitary matrix corresponds to one set of codewords. This is the main difference between our method and the known methods.

Let us consider the case where the channel is a Gaussian channel and the codeword  $\mathbf{c} \in \mathcal{C}_i$  is sent. In such a situation, the received symbol  $\mathbf{y}$  is written as

$$\mathbf{y} = W_i \mathbf{c} + \mathbf{n}, \quad (6.23)$$

where  $\mathbf{n}$  is a noise vector whose components follow the complex Gaussian distribution independently. Then, the estimated codeword is written as

$$\hat{\mathbf{c}} = \mathbf{c} + W_i^* \mathbf{n}. \quad (6.24)$$

From the above equation, SNR is preserved through our method since the matrix  $W_i$  is unitary.

We have shown the main idea of our method. The remained problem is how to find  $W_n$  which achieves low PAPR for  $n = 1, 2, \dots, N$ . In our method, unitary matrices  $W_n$  are given as the solutions which make our bound in Eq. (6.22) small. To analyze our bound, we define

$$\begin{aligned} & f(\{W_n\}_{n=1}^N) \\ & = \sum_{n=1}^N \sum_{\mathbf{c} \in \mathcal{C}_n} \sum_{k=1}^K \left\{ (\mathbf{c}^* W_n^* C_k W_n \mathbf{c})^2 + (\mathbf{c}^* W_n^* \hat{C}_k W_n \mathbf{c})^2 \right\}. \end{aligned} \quad (6.25)$$

Note that the variables of the function  $f$  is the unitary matrices  $\{W_n\}_{n=1}^N$  and that  $f$  is a real function. To find the  $\{W_n\}_{n=1}^N$  achieving low PAPR, we minimize  $f(\{W_n\}_{n=1}^N)$  under the condition that  $\{W_n\}_{n=1}^N$  is unitary. To minimize  $f$ , its gradient is necessary. However, in general, expressions involving complex conjugate or conjugate transpose do not satisfy the Cauchy-Riemann equations [143]. Thus, the function  $f$  may not be differentiable. To avoid this, the generalized complex gradient of  $f$  is defined as [144]

$$\frac{\partial f}{\partial W_n} = \frac{\partial f}{\partial \text{Re}\{W_n\}} + j \frac{\partial f}{\partial \text{Im}\{W_n\}}, \quad (6.26)$$

where  $\text{Re}\{Z\}$  and  $\text{Im}\{Z\}$  are the real part and imaginary part of the matrix  $Z$ , respectively. With this definition, the gradient of  $f$  with respect to  $W_n$  is

calculated as

$$\begin{aligned} & \frac{\partial f}{\partial W_n} \\ & = 4 \sum_{\mathbf{c} \in \mathcal{C}_n} \sum_{k=1}^K \left\{ (\mathbf{c}^* W_n^* C_k W_n \mathbf{c}) C_k + (\mathbf{c}^* W_n^* \hat{C}_k W_n \mathbf{c}) \hat{C}_k \right\} W_n \mathbf{c} \mathbf{c}^*. \end{aligned} \quad (6.27)$$

With the above equation, we propose the following gradient descent algorithm at the  $l$ -th iteration

$$W_n^{(l+1)} \leftarrow W_n^{(l)} - \epsilon \frac{\partial f}{\partial W_n^{(l)}}, \quad (6.28)$$

where  $W_n^{(l)}$  is the matrix obtained at the  $l$ -th iteration and  $\epsilon$  is a positive parameter. With the above iteration, we can obtain the matrix  $W_n^{(l+1)}$  from  $W_n^{(l)}$ . Since the matrix  $W_n^{(l+1)}$  is not always unitary, we have to project  $W_n^{(l+1)}$  onto the set of unitary matrices. One method is to use the Gram-Schmidt process [145] [36] [144]. First, we decompose the matrix  $W_n^{(l+1)}$  as  $W_n^{(l+1)} = (\mathbf{w}_1, \dots, \mathbf{w}_K)^\top$  and update  $\mathbf{w}_1 \leftarrow \mathbf{w}_1 / \|\mathbf{w}_1\|_2$ , where  $\|\mathbf{z}\|_2$  is the  $l_2$  norm of  $\mathbf{z}$ . Then, the following steps are iterated for  $k = 2, \dots, K$

1.  $\mathbf{w}_k \leftarrow \mathbf{w}_k - \sum_{i=1}^{k-1} \mathbf{w}_k^* \mathbf{w}_i \mathbf{w}_i$ .
2.  $\mathbf{w}_k \leftarrow \mathbf{w}_k / \|\mathbf{w}_k\|_2$ .

Finally, the projected matrix is obtained as  $W_n^{(l+1)} = (\mathbf{w}_1, \dots, \mathbf{w}_K)^\top$ .

With the above iterations, we can obtain a unitary matrix. However, it is unclear what order to choose and to normalize vectors. To avoid this ambiguity, a symmetric decorrelation technique has been proposed [145] [36] [146]. A symmetric decorrelation technique is to normalize  $W_i^{(l+1)}$  as

$$W_n^{(l+1)} \leftarrow \left( W_n^{(l+1)} \left( W_n^{(l+1)} \right)^* \right)^{-1/2} W_n^{(l+1)}, \quad (6.29)$$

where  $(ZZ^*)^{-1/2}$  is obtained from the eigenvalue decomposition of  $ZZ^* = F\Lambda F^*$  as  $F\Lambda^{-1/2}F^*$  with  $F$  being a unitary matrix,  $\Lambda$  and  $\Lambda^{-1/2}$  being diagonal positive matrices written as  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_K)$  and  $\Lambda^{-1/2} = \text{diag}(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \dots, \lambda_K^{-1/2})$ , respectively. With the above projection, we can obtain the unitary matrix  $W_n^{(l+1)}$ . The algorithm of our method is summarized in Algorithm 3.

## V Numerical Results

In this Section, we show the performance of our proposed method. As seen in Section II, we assume that the carrier frequency  $f_c$  in Eqs (6.2) and (6.3) is sufficiently large and then PAPR is approximately equivalent to PMEPR. Thus, we measure PMEPR instead of PAPR. We set the parameters as  $K = 128$  and  $M = 2000$ . To measure PMEPR, we choose oversampling parameter  $J = 16$  [18] [6]. The modulation scheme is 16-QAM. All symbols are generated independently from the 16QAM set and then we obtain the set of codewords  $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_M\}$ . In each  $N$ , we set the subsets of codewords as

$$\mathcal{C}_n = \left\{ \mathbf{c}_{\frac{M}{N}(n-1)+1}^M, \mathbf{c}_{\frac{M}{N}(n-1)+2}^M, \dots, \mathbf{c}_{\frac{Mn}{N}}^M \right\} \quad (6.30)$$



---

**Algorithm 3:** How to Find Unitary Matrix in Our Method

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- 1 Set the initial unitary matrix  $W_n^{(1)}$  for  $n = 1, 2, \dots, N$  and the iteration count  $l = 1$ .
- 2 For  $n = 1, 2, \dots, N$ , calculate  $\frac{\partial f}{\partial W_n^{(l)}}$  and obtain the matrix  $W_n^{(l+1)}$  as

$$W_n^{(l+1)} \leftarrow W_n^{(l)} - \epsilon \frac{\partial f}{\partial W_n^{(l)}}.$$

- 3 Project  $W_n^{(l+1)}$  onto the set of unitary matrices for  $n = 1, 2, \dots, N$  with Gram-Schmidt process or Eq. (6.29).
  - 4 Let  $\|W\|$  be the norm of the matrix  $W$ . If  $\|W_n^{(l+1)} - W_n^{(l)}\| \approx 0$  for  $n = 1, 2, \dots, N$  or the iteration count  $l$  gets sufficiently large, then stop. Otherwise, set  $l \leftarrow l + 1$  and go to step 2.
- 

for  $n = 1, 2, \dots, N$ . Thus, each subset of codewords is randomly obtained from the original 16QAM set and the number of components in  $\mathcal{C}_n$  is  $M/N$  for  $n = 1, 2, \dots, N$ . As initial points, we set  $W_n^{(1)} = E$ , where  $E$  is identity matrix for  $n = 1, 2, \dots, N$ . The gradient parameter  $\epsilon$  is set as  $\epsilon = N/M \cdot 1/K^2$ . In Algorithm 3, we have used the symmetric decorrelation technique described in Eq. (6.29).

Figure 6.1 shows PAPR in our method with the parameter  $N = 50$ . Each curve in the figure corresponds to the iteration times. As seen in this figure, PAPR gets small as the iteration time increases. This result shows that we can obtain the unitary matrices which achieve lower PAPR as the iteration time increases. Since our method is to reduce the bound of PAPR in Eq. (6.22), this result implies that decreasing the our bound may lead to decrease PAPR. We conclude that our bound closely related to the CCDF of PAPR.

Figure 6.2 shows PAPR in our method with the parameter  $N = 100$ . Similar to the result with the parameter  $N = 50$ , our method achieves lower PAPR as the iteration time increases. However, from Fig. 6.1 and Fig. 6.2, our method with the parameter  $N = 100$  achieves lower PAPR than one of our methods with  $N = 50$  at each iteration. The reason may be explained as follows. In our simulations, for  $\mathbf{c} \in \mathcal{C}_n$ , codewords are randomly and independently generated from 16QAM set and the average of  $\mathbf{c}$  is 0. Then, the average of the quantity  $\mathbf{c}\mathbf{c}^*$  is

$$E\{\mathbf{c}\mathbf{c}^*\} = E, \quad (6.31)$$

where  $E$  is the identity matrix. From the Cauchy-Schwarz inequality [147], the lower bound of the bound in Eq. (6.19) can be written as

$$\begin{aligned}
& \frac{K(2K-1)}{2P_{\text{av}}^2\gamma^2} \sum_{k=1}^K \mathbb{E} \left\{ (\mathbf{c}^* C_k \mathbf{c})^2 + (\mathbf{c}^* \hat{C}_k \mathbf{c})^2 \right\} \\
& \geq \frac{K(2K-1)}{2P_{\text{av}}^2\gamma^2} \sum_{k=1}^K \left( \text{Tr}(\mathbb{E}\{C_k \mathbf{c} \mathbf{c}^*\})^2 + \text{Tr}(\mathbb{E}\{\hat{C}_k \mathbf{c} \mathbf{c}^*\})^2 \right) \\
& = \frac{K(2K-1)}{2P_{\text{av}}^2\gamma^2} \sum_{k=1}^K \left( \text{Tr}(C_k)^2 + \text{Tr}(\hat{C}_k)^2 \right) \\
& = \frac{K^2(2K-1)}{P_{\text{av}}^2\gamma^2},
\end{aligned} \tag{6.32}$$

where  $\text{Tr}(X)$  is the trace of  $X$ . In the above inequalities, we have used  $\text{Tr}(C_k) = \text{Tr}(\hat{C}_k) = 1$  and Eq. (6.31). It is clear that the above lower bound is invariant under the action  $\mathbf{c} \mapsto W\mathbf{c}$ , where  $W$  is a unitary matrix. Let us consider the situations of the simulations with  $N = 50$  and  $N = 100$  and define the sample mean for each subset of codewords as

$$g(\mathcal{C}_n) = \frac{1}{|\mathcal{C}_n|} \sum_{\mathbf{c} \in \mathcal{C}_n} \mathbf{c} \mathbf{c}^*. \tag{6.33}$$

Here,  $|\mathcal{C}_n|$  is the number of components in the set  $\mathcal{C}_n$ , and we have assumed that  $\mathbf{c}$  is randomly and independently chosen and that the quantity the average of  $\mathbf{c}$  equals to  $\mathbf{0}$ . Then, by the Law of Large Numbers, the quantity  $g(\mathcal{C}_n)$  may be closer to the identity matrix as the number of components in  $\mathcal{C}_n$  increases. From these discussions, if each subset  $\mathcal{C}_n$  is randomly chosen from  $\mathcal{C}$  and the number of components in  $\mathcal{C}_n$  increases, then the quantity  $g(\mathcal{C}_n)$  is nearly equivalent to the identity matrix. In such a situation, the lower bound in Eq. (6.32) may be tight. For these reasons, since each number of components in the subsets with  $N = 50$  is larger than one with  $N = 100$ , our method with the parameter  $N = 100$  achieves lower PAPR than one of our methods with  $N = 50$  at each iteration.

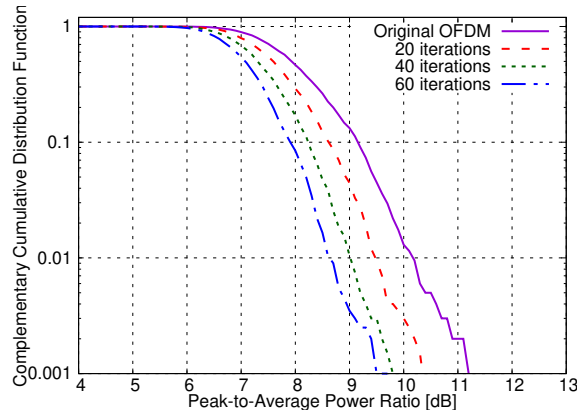


Figure 6.1: PAPR in each iteration with  $N = 50$

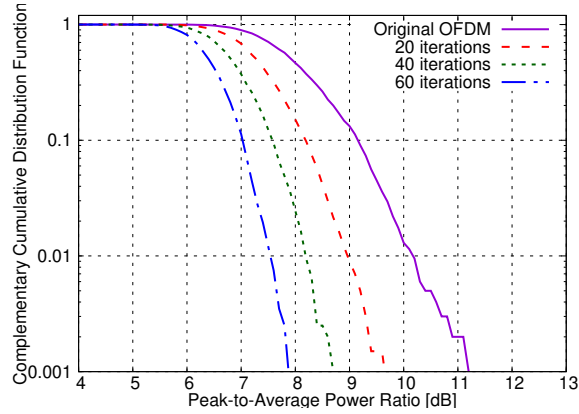


Figure 6.2: PAPR in each iteration with  $N = 100$

## VI Summary and Discussions

In this chapter, we have shown the bound of CCDF of PAPR and our proposed method to reduce PAPR. The main idea of our method is to transform each subset of codewords with the unitary matrix to reduce the bound of CCDF of PAPR. Further, the unitary matrices are obtained with the gradient method and the projecting method.

As seen in Section V, it may not be straightforward to reduce PAPR with our method when the quantity  $g(\mathcal{C}_n)$  is nearly equivalent to the identity matrix. This obstacle may be overcome when we choose efficiently the subsets of codewords  $\mathcal{C}_n$ . Therefore, one of the remained issues is to explore how to obtain the subsets of codewords  $\mathcal{C}_n$ . Further, it is necessary to explore other methods to reduce our bound.

## Chapter 7

# Conclusion

In this thesis, we have focused on SNR, SINR, and SNDR in communication systems. As seen in Chapter 1, these quantities relate to BER, which is the most important index not only in communication systems but also in information theory. In CDMA systems, SINR is evaluated as the performance index. On the other hand, in OFDM systems, SNDR relates to PAPR and the CCDF of PAPR is often evaluated as the performance index. We have discussed how to increase SNR, SINR and SN-DR in the both communication systems.

## I CDMA System

In CDMA systems, we have discussed how to obtain spreading sequences to achieve large SINR since SINR is written in terms of spreading sequences. In Chapter 2, we have discussed the way to express two kinds of the correlation terms: the periodic correlation term and the odd periodic correlation term. These two correlation terms play important roles in the analysis of SINR. In Chapter 3, we have defined the Weyl sequence class and considered sequences belonging to the class. It is known that sequences in this class have low cross-correlation. We have discussed the optimal parameters in the Weyl class and obtained the optimal solutions from the convex optimization problem. Further, we have considered the situation where the number of users is not fixed and shown the way to assign the parameters. In Chapter 4, we have evaluated Rician fading effects and obtained SINR formula which is different from one in [13]. To obtain this formula, we have used the result obtained in Chapter 2. From this formula, we have shown the relation between SINR and mean-square correlations, which are often used as a performance index instead of SINR. Further, in the situation where there is no fading effects and all but the sequence of user  $i$ ,  $\mathbf{s}_i$ , are fixed, we have derived an optimal sequence for the user  $i$  in a sense of SINR. With this sequence, the maximum SINR and capacity have been shown. To deal with a general problem, we have applied semidefinite relaxation technique to the optimization problem and we have obtained the multiplicative convex form. Note that this problem is written with the bi-convex form and that this problem can be solve with the ADMM technique.

## II OFDM System

On the other hand, in OFDM systems, we have discussed the way to reduce PAPR to achieve large SNDR since SNDR relates to PAPR (see Chapter 1). In Chapter 5, we have considered the PTS technique, which is one of the techniques to reduce PAPR. One of the problems in the PTS technique is how to find the suitable vector. To solve this problem, we have proposed the randomization method with Gaussian distribution whose covariance matrix is obtained from the relaxed optimization problem. Further, we have shown that the phase random method corresponds to our randomization method whose covariance matrix is identity. Since our problem depends on the oversampling parameter  $J$ , we have to choose  $J$ . From [18], it is known that sufficiently large  $J$  is necessary. Then, however, the number of constraints gets large and the problem gets complicated. To overcome this obstacle, we have considered the upper bound of PAPR and derived another optimization problem which is independent of  $J$ . To derive this upper bound, we have used the result obtained in Chapter 2. In Chapter 6, we have considered the upper bound of CCDF of PAPR since CCDF of PAPR is often evaluated as the performance index. Our bounds can be derived without any assumptions about the modulation scheme. Thus, our bound can be applied to any modulation schemes. From our bound, we have proposed the PAPR-reduction technique with unitary matrices. This method is an expansion of known methods, the PTS technique and selective mapping technique. Note that the result obtained chapter 2 has also been used to derive our bound of CCDF.

## III Correlation and Communication System

As seen in the results discussed here, correlation plays important roles in the both communication systems, CDMA systems and OFDM systems. It is well known that the performance of CDMA systems relate to correlation of spreading sequences. This thesis has shown that correlation plays an important role in analysis of CCDF of PAPR with OFDM systems. This result has been explicitly obtained in Chapter 6 (It has been originally pointed out that PAPR relates to correlation of symbols in [28]). Thus, this thesis has shown that SNR relates to correlation in the both communication systems.

One of advantages in focusing on correlation is that correlation can be written in quadratic forms, which has been obtained in Chapter 2. With these forms, it has been shown that the upper bound of CCDF of PAPR is written in terms of fourth moments of symbols. Thus, we conclude that focusing on correlation is an useful way to analyze communication system from the perspective of statistics.

## IV Future Works and Remained Issues

In this thesis, we have considered the two kinds of quantities, SNR and PAPR, and obtained optimization problems. Unfortunately, it is not straightforward to obtain the explicit form of the solutions even in convex problems. Thus, one of remained issues is to obtain the explicit forms of solutions. If we obtain the

explicit forms, an algorithm can be written in terms of explicit forms. Such an algorithm would develop communication systems.

Possible future works are as follows. In CDMA systems, it has been demanded to derive optimal sequences for all the users in a general situation. In Chapter 3, we have derived the optimal sequence for a certain user in the situation where there is no fading effect and the sequences of the other users are fixed. This optimal sequence is derived as the eigenvector corresponding to the minimum eigenvalue of the matrix  $\Sigma$ . It is known that the optimal sequence in chip-synchronized CDMA systems can be derived in the same way [60]. Similarly, optimal sequences in asynchronous CDMA systems for all the users may be derived in the nearly same way. In OFDM systems, in a general situation, explicit form of CCDF of PAPR has not been obtained. Further, its approximate form has not been obtained. In Chapter 4, we have shown the upper bound of CCDF. However, in the double logarithmic scale, our bound is linear with respect to the threshold. Thus, to analyze PAPR further, it is demanded to obtain a tighter bound or an approximate form. Such a bound and an approximate form help us to understand the mechanism for reducing PAPR.

If the above works are achieved, communication systems would be developed. As seen in Chapter 1, communication systems are necessary for our lives and large SNR means that data can be sent at high speed. The results in this thesis may lead to realize high-speed communication systems.

## Appendix A

# Universal OFDM System and Peak-to-Average Power Ratio

### I Introduction

As seen in Chapter 5 and 6, Although Orthogonal Frequency Division Multiplexing (OFDM) systems have been widely used for broadband multicarrier communication since, OFDM systems have disadvantages. One disadvantage is that OFDM systems have large PAPR [148]. In Chapter 5 and 6, we have discussed how to reduce PAPR. Another disadvantage is that OFDM signals have large side-lobes [101]. This problem leads to leakage of signal powers among the bands of different users. Therefore, there are some improved and proposed systems to solve this problem. One of the systems is a Universal-Filtered OFDM (UF-OFDM) system [149]. In UF-OFDM systems, band-pass filters are used to reduce side-lobes. Therefore, methods to design efficient filters are demanded. There are some investigations about improving UF-OFDM systems [150] [151].

In this appendix, we show a method to design filters for UF-OFDM systems. As a signal processing technique for recovering symbols, we consider Zero Forcing equalization. Then, we derive the sufficient condition for increase of the Signal-to-Noise Ratio (SNR). From this condition, we obtain the optimization problem to increase the SNR and reduce side-lobes. Therefore, the filter for UF-OFDM systems is obtained as a solution of the optimization problem. Further, we evaluate the PAPR with UF-OFDM systems and show the relation between PAPR and SNR. Then, we show the numerical results about power spectrum density, Bit Error Rate (BER) and PAPR with filters obtained from the optimization problem.

## Mathematical Notation

In what follows, we use the Fourier transformation as a discrete Fourier transformation

$$\mathcal{F}_{\mathbf{x}}(\omega) = \sum_n x_n \exp(-jn\omega), \quad (\text{A.1})$$

where  $\mathbf{x} \in \mathbb{C}^p$ ,  $x_n$  is the  $n$ -th element of  $\mathbf{x}$ ,  $j$  is the unit imaginary number and  $p$  is a positive integer. For convenience, we write the discrete Fourier transformation of  $\mathbf{x}$  as its capital, that is,

$$X(\omega) = \mathcal{F}_{\mathbf{x}}(\omega). \quad (\text{A.2})$$

Further, we write the complex Gaussian distribution whose average and variance are  $\mu$  and  $\sigma^2$  as  $\mathcal{N}(\mu, \sigma^2)$ . Note that their real part and imaginary part obey the Gaussian distribution whose variance is  $\sigma^2/2$ , respectively.

## II UF-OFDM Model

In this section, we fix our model used thorough this appendix and mathematical symbols that will be used in the following sections. We consider Zero Padding (ZP) UF-OFDM systems. For more details of this model, we refer the reader to [152]. Let us define  $A_k$  as a symbol transmitted by the  $k$ -th carrier. We assume that each  $A_k$  is independent and  $\mathbb{E}\{A_i A_k^*\} = \delta_{ik}$ , where  $\mathbb{E}\{X\}$  is the average of  $X$ ,  $z^*$  is the complex conjugate of  $z$  and

$$\delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}. \quad (\text{A.3})$$

Then, a discrete OFDM signal  $x_n$  is written as

$$x_n = \sum_{k \in J} \frac{A_k}{\sqrt{M}} \exp\left(2\pi j \frac{kn}{M}\right) \Pi\left(\frac{n}{M} - \frac{1}{2}\right) \quad (n = 0, 1, \dots, M-1), \quad (\text{A.4})$$

where  $J \subseteq \{0, 1, \dots, M-1\}$  is the set of the numbers of used carriers and  $\Pi(x)$  is defined as [12]

$$\Pi(x) = \begin{cases} 1 & -\frac{1}{2} \leq x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}. \quad (\text{A.5})$$

Note that the duration of symbols is  $M$ . In particular, we denote by  $K$  the number of elements in the set  $J$ . Equation (A.4) is implemented with the Inverse Fast Fourier Transformation (IFFT). In UF-OFDM systems, discrete OFDM signals are filtered to reduce their side-lobes. A filter  $\mathbf{f} \in \mathbb{C}^N$  is written as

$$\mathbf{f} = [f_0 \quad f_1 \quad \cdots \quad f_{N-1}]^\top, \quad (\text{A.6})$$

where  $\mathbf{x}^\top$  is the transpose of  $\mathbf{x}$ . We assume that  $f_0 \neq 0$  and  $f_{N-1} \neq 0$ . This assumption is equivalent to that the convolved signal has the length  $M+N-1$ . When the signal  $x_n$  is filtered by  $\mathbf{f}$ , the output signal  $y_n$  is written as

$$y_n = \sum_{m=0}^{N-1} f_m \sum_{k \in J} \frac{A_k}{\sqrt{M}} \exp\left(2\pi j \frac{k(n-m)}{M}\right) \Pi\left(\frac{n-m}{M} - \frac{1}{2}\right) \quad (\text{A.7})$$



for  $n = 0, 1, \dots, M + N - 2$ . Note that the duration of symbols becomes  $M$  to  $M + N - 1$  due to the filtering. Equation (A.7) is often written as a linear system with a Toeplitz matrix and a Discrete Fourier Transformation (DFT) matrix.

In our model, we apply a zero padding technique to avoid intersymbol interference. Let us define  $D$  as the length of zero paddings. Then, the transmitted signal  $\tilde{y}_n$  is written as

$$\tilde{y}_n = \begin{cases} y_n & 0 \leq n < M + N - 1 \\ 0 & M + N - 1 \leq n < M + N + D - 1 \end{cases} \quad (\text{A.8})$$

In wireless communication systems, fading effects should be considered. In OFDM systems, fading effects are expressed as a discrete convolution. Let us define  $\mathbf{h} \in \mathbb{C}^L$  as

$$\mathbf{h} = [h_0 \quad h_1 \quad \dots \quad h_{L-1}]^\top, \quad (\text{A.9})$$

where  $L$  is the length of the fading  $\mathbf{h}$ . Further, we assume that  $L - 1 \leq D$ . This assumption is equivalent to that there we have the knowledge about the length of the fading effects. This assumption is often used [8]. Then, the received signal  $r_n$  is written as

$$r_n = \sum_{l=0}^{L-1} h_l \tilde{y}_{n-l} + v_n \quad (n = 0, 1, \dots, M + N + D - 2), \quad (\text{A.10})$$

where  $v_n$  is additive white Gaussian noise (AWGN).

To estimate symbols, zero padding techniques are used. Then, the length of  $r_n$  is extends to  $2M$ . Therefore, the zero padded signal  $\tilde{r}_n$  is written as

$$\tilde{r}_n = \begin{cases} r_n & 0 \leq n < M + N + D - 1 \\ 0 & M + N + D - 1 \leq n < 2M \end{cases}. \quad (\text{A.11})$$

From Eqs. (A.10) and (A.11), their Fourier transformations are written as

$$R(\omega) = \tilde{R}(\omega) = F(\omega)H(\omega)X(\omega) + V(\omega). \quad (\text{A.12})$$

In particular,  $X(2\pi \frac{k}{M})$  ( $k \in J$ ) is written as

$$X\left(2\pi \frac{k}{M}\right) = \sqrt{M}A_k. \quad (\text{A.13})$$

Therefore, Eq. (A.12) is rewritten as

$$\frac{1}{\sqrt{M}}R\left(2\pi \frac{k}{M}\right) = A_k F\left(2\pi \frac{k}{M}\right) H\left(2\pi \frac{k}{M}\right) + \frac{1}{\sqrt{M}}V\left(2\pi \frac{k}{M}\right). \quad (\text{A.14})$$

The quantity  $\tilde{R}(2\pi \frac{k}{M})$  can be obtained with the  $2M$ -Fast Fourier Transformation and a down sampling technique [152]. We can calculate and obtain  $F(\omega)$  since we know  $\mathbf{f}$ . With some techniques, for example pilot symbols,  $H(\omega)$  can be estimated and obtained. Therefore, from Eq. (A.14), the symbol  $A_k$  can be recovered with multiplication of the inverse of  $H(\omega)$ . Then, when the  $H(\omega)$  is given and fixed, the SNR about the symbol  $A_k$  is written as

$$\text{SNR}_k = \sqrt{\frac{M}{M + N + D - 1}} \frac{|F(2\pi \frac{k}{M}) H(2\pi \frac{k}{M})|}{\sigma_n}, \quad (\text{A.15})$$

where  $\sigma_n$  satisfies  $E\{v_n^* v_n\} = \sigma_n^2$ . From Eq. (A.15), SNR gets higher as  $F(2\pi\frac{k}{M})$  gets larger. From this result, the large  $F(2\pi\frac{k}{M})$  is demanded for high SNR.

Note that UF-OFDM systems are different from OFDM systems in a sense of SNR. In UF-OFDM systems, the duration of a symbol becomes longer than original signals with zero padding techniques and filtering. Then, the duration of a symbol with UF-OFDM system is longer than that with conventional OFDM systems. As seen in Eq. (A.15), the SNR in UF-OFDM systems has the term  $\sqrt{\frac{M}{M+N+D-1}}$ , which is related to the length of filters  $N$  and do not appear in conventional OFDM systems. From this result, UF-OFDM systems are different from OFDM systems in a sense of SNR.

### III Optimization Problem for Designing Filter

In this section, we show the criteria for designing filters. We focus on Bit Error Rate (BER) and side-lobes. For convenience, let us define the  $n$ -th element of  $\mathbf{g}$  as

$$g_n = \sum_m f_{n+m} f_m^*. \quad (\text{A.16})$$

Note that  $\mathbf{g}$  is written as

$$\mathbf{g} = [g_{-N+1} \quad g_{-N+2} \quad \cdots \quad g_{N-1}]^\top \quad (\text{A.17})$$

It is known that the Fourier transformation of  $\mathbf{g}$  is written as

$$\mathcal{F}_{\mathbf{g}}(\omega) = \sum_{n=-N+1}^{N-1} g_n \exp(-jn\omega) = |F(\omega)|^2. \quad (\text{A.18})$$

From Eq. (A.18),  $|F(\omega)|^2$  is expressed as the inner product of the two vectors,  $\mathbf{g}$  and the vector whose  $n$ -th element is  $\exp(-jn\omega)$ . Further, when the vector  $\mathbf{g}$  is given, we can obtain the original vector  $\mathbf{f}$  with a spectral factorization method [153].

#### III.I Condition

First, we consider the condition that filters have to satisfy. We assume that the average power of signals is conserved through the filter convolution. This assumption is written as

$$\frac{1}{M} \sum_{n=0}^{M-1} E\{|x_n|^2\} = \frac{1}{M+N-1} \sum_{n=0}^{M+N-1} E\{|y_n|^2\}, \quad (\text{A.19})$$

where  $E\{X\}$  is the average of  $X$ . With  $\mathbf{g}$ , this condition is written as [151]

$$K(M+N-1) = \mathbf{b}_J^\top \mathbf{g}, \quad (\text{A.20})$$

where  $K$  is the number of the elements of  $J$  and  $\mathbf{b}_J$  is the vector written as

$$\begin{aligned} \mathbf{b}_J &= [b_{J,-N+1} \quad b_{J,-N+2} \quad \cdots \quad b_{J,N-1}], \\ b_{J,n} &= \sum_{k \in J} (M - |n|) \exp\left(-2\pi j \frac{kn}{M}\right). \end{aligned} \quad (\text{A.21})$$

Note that Eq. (A.20) is invariant under the following transformations:

- each carrier  $k \rightarrow k + s$  and
- the  $n$ -th element of the filter  $f_n \rightarrow f_n \exp\left(2\pi j \frac{sn}{M}\right)$

for any real value  $s$ . Therefore, when  $J$  consists of  $K$  consecutive integers<sup>1</sup>, we can assume that the frequencies of carriers are symmetric around  $\omega = 0$ , where  $\omega$  is an angular frequency. This assumption is realized with the operation  $k \rightarrow k + s$ . Then, it is sufficient to design the filter whose power spectrum is symmetric around  $\omega = 0$ . Therefore, we treat  $\mathbf{f} \in \mathbb{R}^N$ , and rewrite  $\mathbf{g}$  as

$$\mathbf{g} = [g_0 \quad g_1 \quad \cdots \quad g_{N-1}]^\top \quad (\text{A.22})$$

since  $\{g_n\}$  is symmetric around  $n = 0$ .

### III.II Optimization Problem

First, we divide the region  $[0, \pi)$  into three regions as follows

$$[0, \pi) = \Omega_p \cup \Omega_t \cup \Omega_s, \quad (\text{A.23})$$

where  $\Omega_p, \Omega_t$  and  $\Omega_s$  are the regions of the passband, the transition-band and stop-band, respectively. We define the set of frequencies of carriers as  $\Omega_c$ , which is the frequency shifted set of  $J$ .

In Section II, we have shown that the large  $F(\omega_c)$  ( $\omega_c \in \Omega_c$ ) is demanded for high SNR. As criteria of designing filters, high SNR and the rapid decay of side-lobes are demanded. Therefore, we obtain the following optimization problem

$$\begin{aligned} (P') \quad & \min \quad t_1 - \lambda t_2 \\ \text{s.t.} \quad & |F(\omega)|^2 \mathbb{E}\{|X(\omega)|^2\} \leq t_1 \quad (\omega \in \Omega_s) \\ & |F(\omega_c)|^2 \geq t_2 \quad (\omega_c \in \Omega_c), \\ & K(M + N - 1) = \mathbf{b}_c^\top \mathbf{g}, \\ & \mathcal{F}_{\mathbf{g}}(\omega) \geq 0, \quad \omega \in [0, \pi], \\ & t_1, t_2 \geq 0, \end{aligned}$$

where  $\lambda$  is the positive weight parameter,  $\mathbb{E}\{|X(\omega)|^2\}$  is written as

$$\begin{aligned} \mathbb{E}\{|X(\omega)|^2\} &= \frac{1}{M} \sum_{\omega_c \in \Omega_c} \alpha(\omega_c - \omega)^2, \\ \alpha(\omega) &= \begin{cases} M & \omega = 2n\pi \quad (n \in \mathbb{Z}) \\ \sin\left(\frac{M\omega}{2}\right) / \sin\left(\frac{\omega}{2}\right) & \text{otherwise} \end{cases} \end{aligned} \quad (\text{A.24})$$

and  $\mathbf{b}_c \in \mathbb{R}^N$  is the vector whose  $n$ -th element is defined as

$$b_{c,n} = \begin{cases} 2 \sum_{\omega_c \in \Omega_c} (M - |n|) \cos(n\omega_c) & n = 1, 2, \dots, N-1 \\ MN & n = 0 \end{cases}. \quad (\text{A.25})$$

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<sup>1</sup>For example,  $\{M-2, M-1, 0, 1, 2, 3\}$  is allowed.

In the above equations, we have used the assumption that each  $A_k$  is independent and  $E\{A_i A_k^*\} = \delta_{ik}$ , which has been made in Section II. In problem  $(P')$ , the variables are  $\mathbf{g}$ ,  $t_1$  and  $t_2$ . It is expected that the side-lobes get lower as the lambda becomes smaller. Besides, it is expected that the SNR gets larger as the lambda becomes larger.

In the problem  $(P')$ , the first constraints are about side-lobes. The second constraints are about desired signals, which have been discussed in Section II. The third constraints are about the conservation of the signal power. The fourth constraints are necessary to satisfy Eq. (A.16) [154]. Note that the vector  $\mathbf{b}_c$  is the same as  $\mathbf{b}_J$  defined in Eq. (A.21) when the set  $\Omega_c$  consists of the angular frequencies  $2\pi k/M$ , where  $k$  is the number carriers. In the problem  $(P')$ , it is not straightforward to handle the first and fourth constraints since  $\Omega_s$  and  $[0, \pi]$  are regions. To overcome these obstacles, we transform the problem.

One method to overcome the obstacles in the first constraints is to transform the constraints into discrete constraints [154] [155]. With this method, the first constraints in the problem  $(P')$  is replaced with

$$|F(\omega_i)|^2 E\{|X(\omega_i)|^2\} \leq t_1 \quad i = 1, 2, \dots, S, \quad (\text{A.26})$$

where  $\omega_i$  ( $i = 1, 2, \dots, S$ ) is the element in the region  $\Omega_s$ . In this appendix,  $S = 15N$  is chosen. Note that how to choose the number  $S$  is discussed in [154] [155] [156].

It is shown in [154] that the fourth constraint is satisfied if there is a symmetric matrix  $P$  which satisfies the following inequality

$$\begin{bmatrix} P - A^\top P A & C^\top - A^\top P B \\ C - B^\top P A & D + D^\top - B^\top P B \end{bmatrix} \succcurlyeq 0, \quad (\text{A.27})$$

where

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (\text{A.28})$$

$$C = [g_1 \quad g_2 \quad \cdots \quad g_{N-1}], \quad D = \frac{1}{2}g_0$$

and  $X \succcurlyeq 0$  means that  $X$  is a positive semidefinite matrix. With these methods, the problem  $(P')$  is rewritten as

$$\begin{aligned} (P) \quad & \min \quad t_1 - \lambda t_2 \\ \text{s.t.} \quad & |F(\omega_i)|^2 E\{|X(\omega_i)|^2\} \leq t_1 \quad (\omega_i \in \Omega_s, i = 1, 2, \dots, S) \\ & |F(\omega_c)|^2 \geq t_2 \quad (\omega_c \in \Omega_c), \\ & K(M + N - 1) = \mathbf{b}_c^\top \mathbf{g}, \\ & \begin{bmatrix} P - A^\top P A & C^\top - A^\top P B \\ C - B^\top P A & D + D^\top - B^\top P B \end{bmatrix} \succcurlyeq 0, \\ & t_1, t_2 \geq 0, P \in \mathbb{S}^{N-1}, \end{aligned}$$

where  $\mathbb{S}^n$  is the set of  $n \times n$  symmetric matrices. Note that the variables in the problem  $(P)$  are  $\mathbf{g}, P, t_1$  and  $t_2$ . Further, the problem  $(P)$  is convex.

## IV Peak-to-Average Power Ratio Analysis

In Section III, we have shown the criteria for designing filters and the optimization problem. In this section, we discuss the PAPR of UF-OFDM signals and show the trade-off between PAPR and BER.

We make the following assumptions:

1. Continuous OFDM signals are regarded as a Gaussian process when the time  $t$  is fixed.
2. Continuous UF-OFDM signals are regarded as a Gaussian process when the time  $t$  is fixed.
3. Continuous OFDM signals are perfectly reconstructed from their discrete signals that are sampled.

The first assumption is often used [102] [6]. This assumption is based on the central limit theorem. To analyse PAPR of UF-OFDM signals in a same way for OFDM signals, we make the second assumption. About the last assumption, we assume that the following formula is satisfied:

$$\exp\left(2\pi j \frac{k}{T} t\right) = \sum_{n=-\infty}^{\infty} \exp\left(2\pi j \frac{kn}{M}\right) \Pi\left(\frac{n}{M} - \frac{1}{2}\right) \text{sinc}\left(\pi \left(\frac{t}{\Delta t} - n\right)\right) \quad (\text{A.29})$$

for  $0 \leq t \leq T$ , where  $\text{sinc}(x)$  is the sinc function,  $T$  and  $\Delta t$  are the symbol duration and the sampling time which satisfy  $T = M\Delta t$ . This formula is based on the sampling theorem. Note that Eq. (A.29) is not always satisfied since OFDM signals are not strictly band-limited due to the rectangular pulse  $\Pi(t)$  (see Eq. (A.4)) [101]. Therefore, Eq. (A.29) is an approximation.

The definition of the PAPR  $\mathcal{P}$  is written as

$$\mathcal{P} = \frac{\max_{0 \leq t \leq T} |s(t)|^2}{P_{\text{av}}}, \quad (\text{A.30})$$

where  $s(t)$  is a baseband signal and  $P_{\text{av}}$  is the average power of baseband signals. Note that PAPR is a random variable since signals are regarded as random variables. In this appendix, it is called that the PAPR becomes high when the PAPR tends to have a large value.

From Eq. (A.19) and the previously made assumption that the average power of signals are conserved through the filter convolution, the average power of UF-OFDM signals is written as

$$P_{\text{av}} = \frac{K}{M}. \quad (\text{A.31})$$

Therefore, instead of  $y_n$ , we consider the following signal  $\bar{y}_n$ ,

$$\begin{aligned} \bar{y}_n &= \frac{y_n}{\sqrt{P_{\text{av}}}} \\ &= \sum_{m=0}^{N-1} f_m \sum_{k \in J} \frac{A_k}{\sqrt{K}} \exp\left(2\pi j \frac{k(m-n)}{M}\right) \Pi\left(\frac{m-n}{M} - \frac{1}{2}\right). \end{aligned} \quad (\text{A.32})$$

Let us consider the continuous UF-OFDM signal. We fix  $t \in [T(N-1)/M, T]$ . Note that the symbol duration of UF-OFDM signals is  $T \times (M+N-1)/M$ . From the sampling theorem, the continuous UF-OFDM signal is written as

$$\bar{y}(t) = \sum_{n=-\infty}^{\infty} \bar{y}_n \text{sinc} \left( \pi \left( \frac{t}{\Delta t} - n \right) \right). \quad (\text{A.33})$$

With Eq. (A.29), Eq. (A.33) is rewritten as

$$\begin{aligned} \bar{y}(t) &= \frac{1}{\sqrt{K}} \sum_{n=0}^{N-1} \sum_{k \in J} A_k f_n \exp \left( 2\pi j \frac{k}{T} \left( t - \frac{nT}{M} \right) \right) \\ &= \frac{1}{\sqrt{K}} \sum_{k \in J} A_k F \left( 2\pi \frac{k}{M} \right) \exp \left( 2\pi j \frac{k}{T} t \right) \end{aligned} \quad (\text{A.34})$$

for  $t \in [T(N-1)/M, T]$ . Note that  $\bar{y}(t)$  is a random value since  $A_k$  is a random value. The following formulas are satisfied:

$$\mathbb{E}\{\bar{y}(t)\} = 0, \quad \mathbb{E}\{|\bar{y}(t)|^2\} = \frac{1}{K} \sum_{k \in J} \left| F \left( 2\pi \frac{k}{M} \right) \right|^2. \quad (\text{A.35})$$

We denote by  $\bar{\sigma}^2 = \sum_{k \in J} \left| F \left( 2\pi \frac{k}{M} \right) \right|^2 / K$ .

In OFDM signals, the covariance of signals equals 1. Therefore, from assumptions 2 and 3, signals of UF-OFDM  $\bar{y}(t)$  and ones of OFDM  $x(t)$  obey  $\mathcal{N}(0, \bar{\sigma}^2)$  and  $\mathcal{N}(0, 1)$ , respectively. Therefore, their amplitudes obey the Rayleigh distributions whose scale parameters equal  $\bar{\sigma}/\sqrt{2}$  and  $1/\sqrt{2}$ , respectively. From this result, it is expected that the PAPR for UF-OFDM signals becomes higher as the parameter  $\bar{\sigma}$  gets larger. As seen in Section II, the parameter  $\bar{\sigma}$  relates to desired signals and large  $\bar{\sigma}$  is necessary for high SNR. Therefore, it is expected that the PAPR for UF-OFDM signals becomes higher as SNR gets higher.

## V Numerical Results

In this section, we show the BER, PAPR, and the power spectrum of filters obtained from the problem (P). In our simulations, the parameters, the size of IFFT  $M = 128$ , the length of filter  $N = 16$ , the size of the zero-padding  $D = 16$  and the length of fading channels  $L = 12$  are chosen. Further, we assume that each element of the fading effect  $h_n$  obeys  $\mathcal{N}(0, 1/L)$ . As carrier frequencies, we set  $J = \{4, 5, \dots, 19\}$ . With frequency shifts, we design filters on the region  $[0, \pi)$ . We set the regions,  $\Omega_p$ ,  $\Omega_t$  and  $\Omega_s$  as follows

$$\Omega_p = [0, 17\pi/256), \Omega_t = [17\pi/256, 17\pi/64), \Omega_s = [7\pi/64, \pi). \quad (\text{A.36})$$

With the above parameters, we obtain filters from the problem (P) for various  $\lambda$ . To solve the problem (P), we use the matlab package, CVX [137]. The filter  $\mathbf{f}$  is obtained from the solution of the problem (P),  $\mathbf{g}$  with spectral factorization methods [154]. We compare these filters with Dolph-Chebyshev filter whose side-lobe level is  $-45$  dB. In each simulation, the modulation scheme is the QPSK modulation. In each figure, “Our filters” means the filters obtained from the problem (P) with various  $\lambda$ , “D-C” means the Dolph-Chebyshev filter.

Figure A.1 shows the power spectrum of each filter. With filters obtained from the problem (P), the side-lobe decreases as the weight parameter  $\lambda$  gets smaller. In particular, with the filter whose parameter is  $\lambda = 0.0001$ , its side-lobe is nearly  $-90$  dB and lower than one of the Dolph-Chebyshev filter. It is known that side-lobe reductions in order of  $-70$  dB are mandatory for broadcasting systems [12]. We observe that the filter whose parameter is  $\lambda = 0.0001$  is appropriate for broadcasting systems. Further, the side-lobe tends to be increased by the PAPR reduction schemes [10]. Thus, the filter whose parameter is  $\lambda = 0.0001$  may be still appropriate after processes of PAPR reductions.

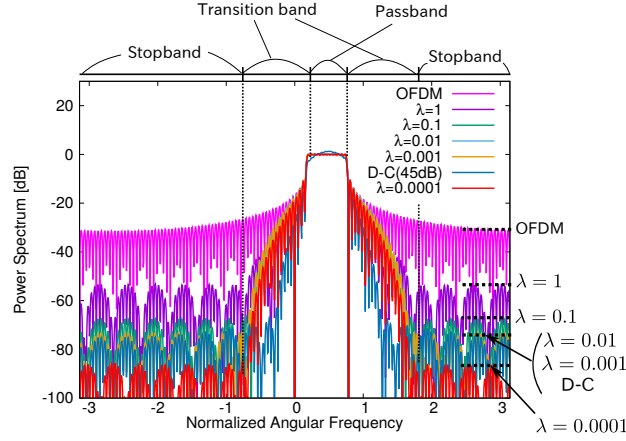


Figure A.1: Power Spectrum for UF-OFDM and OFDM systems

Figure A.2 shows the BER with various filters. We assume that the receiver has the perfect knowledge about the fading function  $H(\omega)$ . Then, the receiver can estimate the symbol  $A_k$  with the knowledge of  $H(\omega)$  (see Eq. (A.14)). In Fig. A.2,  $\sigma_s^2$  is the energy per bit before filtering and  $\sigma_n^2$  is the variance of the AWGN channel, which has been defined in Eq. (A.15). Note that the energy per bit of transmitted signals is  $\frac{M+N-1}{M}\sigma_s^2$ . The BER for our filters is lower than that for the Dolph-Chebyshev filter. In designing filters with the problem (P), we increase the power of all desired signals. However, in the Dolph-Chebyshev filter, the power of all desired signals is not equal and there are some desired signals whose power is low. Therefore, it is conceivable that this difference at each power spectrum causes the difference of BER.

Figure A.3 shows the PAPR with each filter. To compare equally, there is no zero paddings in UF-OFDM systems, that is, UF-OFDM signals are uniformly filled in the symbol duration. In calculating the PAPR, we set the  $M = 128$  points in each symbol duration and obtain each amplitude in the points. Then, the maximum amplitude is regarded as the maximum value in them. In OFDM signals, it is conceivable that the over-sampling factor  $M/K = 8$  is sufficiently large to measure the PAPR. As the over-sampling factor,  $M/K \geq 4$  is commonly used. The discussion about the over-sampling factor is described in [18].

In Section IV, we have discussed the PAPR of the filters obtained from the problem (P). As seen in Fig. A.3, the PAPR for the signals with filters obtained from the problem (P) is higher than that with the Dolph-Chebyshev filter and the OFDM signals. Since each BER of the filters obtained by the problem (P)

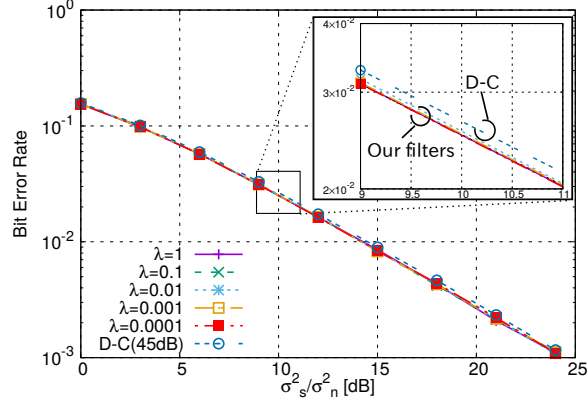


Figure A.2: Bit Error Rate of UF-OFDM with various filters

is nearly the same, it is conceivable that each  $\tilde{\sigma}$ , which is defined in Section IV is the same. Therefore, the PAPR for the filters obtained by the problem (P) is uniformly high and they are nearly the same. On the other hands, the PAPR for the Dolph-Chebyshev filter is lower than that for the filters obtained by the problem (P). From this result and the result about BER, there is a relation between BER and PAPR in UF-OFDM systems, that is, PAPR gets higher as the SNR gets larger.

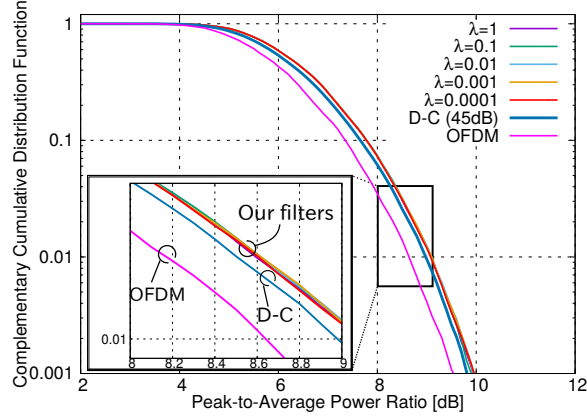


Figure A.3: Peak-to-Average Power Ratio for UF-OFDM and OFDM systems

## VI Summary and Discussion

In this appendix, we have shown the UF-OFDM model and the criteria of designing filters, which is related to side-lobes and SNR. Then, we have obtained the convex optimization problem and the filters as its solutions. As numerical results, we have shown their BER, power spectrum and PAPR and compared them with the Dolph-Chebyshev filter and conventional OFDM signals. From these results, the BER for our filters is lower than that for the Dolph-Chebyshev



filter. In particular, the side-lobe of the filter whose parameter  $\lambda = 0.0001$  is the lowest. This filter is appropriate for communication systems since the power of side-lobes is low. However, the signals of UF-OFDM systems have high PAPR than ones of conventional OFDM systems. It is necessary to consider this relation for communication systems.

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## List of Author's papers related to this thesis

### ■Journal Paper (referred)

- [3] H. Tsuda and K. Umeno, "Non-Linear Programming: Maximize SINR for Designing Spreading Sequence", IEEE Transactions on Communication, 66. 1 (2018): 278-289
- [2] H. Tsuda and K. Umeno, "Weyl Spreading Sequence Optimizing CDMA," IEICE on Communications, Vol.E101-B,No.3,pp.897-908, Mar. 2018
- [1] H. Tsuda and K. Umeno, "Orthogonal Basis Spreading Sequence for Optimal CDMA", JSIAM Letters, Vol. 8 (2016): 77-80.

### ■International Conference (referred)

- [4] H. Tsuda and K. Umeno, "Randomization Algorithm for Partial Transmit Sequence with Semidefinite Relaxation", VTC Fall 2018, Chicago, United States (2018)
- [3] H. Tsuda and K. Umeno, "Method to Design UF-OFDM Filter and Its Analysis", EuCNC 2018, Ljubljana, Slovenia(2018)
- [2] H. Tsuda and K. Umeno, "Semidefinite Relaxation based Randomization Approach for Partial Transmit Sequence", IEEE WCNC 2018, Barcelona, Spain (2018)
- [1] H. Tsuda and K. Umeno, "New Expression of SNR Formula", 2016 ICSGTEIS, Bali, Indonesia (2016).

### ■Submitted

- [3] H. Tsuda, "Optimal Sequence and SINR for Desired User in Asynchronous CDMA System", submitted to EuCNC 2019, arxiv:1902.06422.
- [2] H. Tsuda, "Moment-Based Bound on Peak-to-Average Power Ratio and Reduction with Unitary Matrix", submitted to WCNC 2019, arxiv:1902.06420.
- [1] H. Tsuda and K. Umeno, "Randomization Approaches for Reducing PAPR with Partial Transmit Sequences and Semidefinite Relaxation", submitted to IEEE Transactions on Vehicular Technology, arxiv:1805.04666.

## Figures

### ■ Chapter 3

Fig. 3.1-3.6 are copied from the following paper:

□ H. Tsuda and K. Umeno, “Weyl Spreading Sequence Optimizing CDMA,” IEICE on Communications, Vol.E101-B, No.3, pp.897-908, Mar. 2018

### ■ Chapter 4

Fig 4.1 and 4.2 are copied from the following paper:

□ H. Tsuda and K. Umeno, “Non-Linear Programming: Maximize SINR for Designing Spreading Sequence”, IEEE Transactions on Communication, 66. 1 (2018): 278-289

### ■ Chapter 5

Fig 5.1-5.6 are copied from the following paper:

□ H. Tsuda and K. Umeno, “Randomization Approaches for Reducing PAPR with Partial Transmit Sequences and Semidefinite Relaxation”, submitted to IEEE transactions on Vehicular Technology, arxiv:1805.04666.

This chapter is also partially based on the following paper:

□ H. Tsuda and K. Umeno, “Randomization Algorithm for Partial Transmit Sequence with Semidefinite Relaxation”, VTC Fall 2018, Chicago, United States (2018)

### ■ Chapter 6

Fig 6.1 and 6.2 are copied from the following paper:

□ H. Tsuda, “Moment-Based Bound on Peak-to-Average Power Ratio and Reduction with Unitary Matrix”, submitted to WCNC 2019, arxiv:1902.06420.