

Nilpotent Orbits and Cayley Transform

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§1. INTRODUCTION.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its Cartan decomposition. Let $\mathfrak{g}_\mathbb{C}$, $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{p}_\mathbb{C}$ be the complexifications of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Put $G = \text{Int } \mathfrak{g}$, $G_\mathbb{C} = \text{Int } \mathfrak{g}_\mathbb{C}$ and let $K_\mathbb{C}$ be the analytic subgroup of $G_\mathbb{C}$ corresponding to $\mathfrak{k}_\mathbb{C}$. Then $K_\mathbb{C}$ acts on the vector space $\mathfrak{p}_\mathbb{C}$. If $\underline{N}(\mathfrak{p}_\mathbb{C})$ denotes the totality of the nilpotent elements of $\mathfrak{p}_\mathbb{C}$, $K_\mathbb{C}$ also acts on $\underline{N}(\mathfrak{p}_\mathbb{C})$. On the other hand, if $\underline{N}(\mathfrak{g})$ denotes the totality of the nilpotent elements of $\mathfrak{g}$, G acts on $\underline{N}(\mathfrak{g})$. Then B. Kostant proposed the following.

CONJECTURE (cf. [K]): Does there exist a bijective correspondence between the set of $K_\mathbb{C}$ -orbits of $\underline{N}(\mathfrak{p}_\mathbb{C})$ and that of G -orbits of $\underline{N}(\mathfrak{g})$.

It is easy to generalize this conjecture to the case of the nilpotent variety of the tangent space of a semisimple symmetric space. The purpose of this note is to formulate this generalization and explain the outline of its proof. For the details, refer to [S].

§2. EXAMPLE.

First we give a typical example.

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Take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and put $\theta(X) = -^tX$ for each $X \in \mathfrak{g}$. Then θ is a Cartan involution of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. In this case

$$\underline{N}(\mathfrak{g}) = \left\{ \begin{pmatrix} t & x \\ y & -t \end{pmatrix} ; t, x, y \in \mathbb{R}, t^2 + xy = 0 \right\}$$

and there are three G -orbits of $\underline{N}(\mathfrak{g})$ and we can choose the representatives as follows:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand,

$$\underline{N}(\mathfrak{p}_c) = \left\{ \begin{pmatrix} t & x \\ x & -t \end{pmatrix} ; t, x \in \mathbb{C}, t^2 + x^2 = 0 \right\}.$$

There are three K_c -orbits of $\underline{N}(\mathfrak{p}_c)$ and we can choose the representatives of them as follows:

$$\frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For any $X \in \underline{N}(\mathfrak{g})$, it is easy to show that there exist $A, Y \in \mathfrak{g}$ such that

$$(I) \quad [A, X] = 2X, \quad [A, Y] = -2Y, \quad [X, Y] = A.$$

For such a triple (A, X, Y) , we define

$$(II) \quad A^d = i(X-Y), \quad X^d = \frac{1}{2}(X+Y+iA), \quad Y^d = \frac{1}{2}(X+Y-iA).$$

Then

$$(III) \quad [A^d, X^d] = 2X^d, \quad [A^d, Y^d] = -2Y^d, \quad [X^d, Y^d] = A^d.$$

At this stage, we assume that

$$(IV) \quad \theta(X) = -Y, \quad \theta(A) = -A.$$

Then it follows that

$$(V) \quad \theta(A^d) = A^d, \quad \theta(X^d) = -X^d, \quad \theta(Y^d) = -Y^d.$$

This means that $A^d \in \underline{k}_c$, $X^d, Y^d \in \underline{p}_c$.

Now take a G -orbit $\underline{O}$ of $\underline{N}(g)$. Then one can show the following by direct calculation:

CLAIM. There exists a triple (A, X, Y) such that $X \in \underline{O}$ and that this satisfies the conditions (I) and (IV). Define A^d, X^d and Y^d by (II). Then the K_c -orbits of X^d and Y^d only depend on $\underline{O}$.

Take a G -orbit $\underline{O}$ of $\underline{N}(g)$. Using the notation in CLAIM, we define the K_c -orbit of X^d in $\underline{N}(\underline{p}_c)$. Then CLAIM implies that this map defines a required bijection.

For example, we take $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. If we put $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the triple (A, X, Y) satisfies (I), (IV). Then we find that $X^d = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ and $Y^d = \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$.

§3. A GENERALIZATION.

Let $\underline{g}$ be a semisimple Lie algebra of the non-compact type and let σ be its involution. Then we obtain the direct sum $\underline{g} = \underline{h} + \underline{q}$, where $\underline{h}$ and $\underline{q}$ are the 1- and (-1)-eigenspaces of σ , respectively. The pair $(\underline{g}, \underline{h})$ is called a semisimple symmetric pair.

It is known that there is a Cartan involution θ of $\underline{g}$ commuting with σ . Let $\underline{g} = \underline{k} + \underline{p}$ be the corresponding Cartan involution. Since $\theta\sigma$ is also an involution of $\underline{g}$, we obtain the direct sum $\underline{g} = \underline{h}^a + \underline{q}^a$, where $\underline{h}^a$ and $\underline{q}^a$ are the 1- and (-1)-eigenspaces of $\theta\sigma$, respectively. Putting $\underline{q}^a = \underline{g}$, we obtain a symmetric pair $(\underline{g}^a, \underline{h}^a) (= (\underline{g}, \underline{h})^a)$. This is called

the associated symmetric pair.

Let $\underline{g}_c$ be the complexification of $\underline{g}$ and we extend θ, σ to $\underline{g}_c$ as complex linear involutions. Put

$$\underline{g}^d = \underline{k} \cap \underline{h} + i(\underline{k} \cap \underline{g}) + i(\underline{p} \cap \underline{h}) + \underline{p} \cap \underline{g}.$$

Then $\underline{g}^d$ defines a real form of $\underline{g}_c$. Since θ is an involution of $\underline{g}^d$, we obtain the direct sum $\underline{g}^d = \underline{h}^d + \underline{g}^d$ for θ and a symmetric pair $(\underline{g}^d, \underline{h}^d) (= (\underline{g}, \underline{h})^d)$, which is called dual to $(\underline{g}, \underline{h})$.

The following diagram holds:

$$\begin{array}{ccccc} (\underline{g}, \underline{h}) & \xleftrightarrow{\text{associated}} & (\underline{g}, \underline{h})^a & \xleftrightarrow{\text{dual}} & (\underline{g}, \underline{h})^{ad} \\ \updownarrow \text{dual} & & & & \updownarrow \text{associated} \\ (\underline{g}, \underline{h})^d & \xleftrightarrow{\text{associated}} & (\underline{g}, \underline{h})^{da} & \xleftrightarrow{\text{dual}} & (\underline{g}, \underline{h})^{dad} \end{array}$$

Let $\underline{N}(\underline{g})$ be the totality of the nilpotent elements of $\underline{g}$ and let H be the analytic subgroup of $\text{Int } \underline{g}$ corresponding to $\underline{h}$. Since H acts on $\underline{N}(\underline{g})$, we denote by $[\underline{N}(\underline{g})]$ the set of H -orbits of $\underline{N}(\underline{g})$. It is known that $[\underline{N}(\underline{g})]$ is a finite set. Similarly, we define $[\underline{N}(\underline{g}^a)]$ and $[\underline{N}(\underline{g}^d)]$ for the pairs $(\underline{g}^a, \underline{g}^a)$ and $(\underline{g}^d, \underline{h}^d)$, respectively.

§4. THE MAIN THEOREM.

Let $(\underline{g}, \underline{h})$ be a symmetric pair as above.

Lemma 1. For any $X \in \underline{N}(\underline{g})$, there exist $A \in \underline{h}$ and $Y \in \underline{g}$ such that $[A, X] = 2X$, $[A, Y] = -2Y$, $[X, Y] = A$.

Definition 2. A triple (A, X, Y) satisfying the condition of Lemma 1 is called a normal S-triple.

Definition 3. A triple (A, X, Y) is called a strictly normal S-triple if (A, X, Y) is a normal S-triple such that $\theta(A) = -A$, $\theta(X) = -Y$.

Lemma 4. If (A, X, Y) is a normal S-triple, there is $h \in H$ such that $(h \cdot A, h \cdot X, h \cdot Y)$ is a strictly normal S-triple.

Lemma 5. Let (A_i, X_i, Y_i) ($i = 1, 2$) be strictly normal S-triples. If X_1 and X_2 are H -conjugate, there is $k \in H \cap K$ such that $(k \cdot A_1, k \cdot X_1, k \cdot Y_1) = (A_2, X_2, Y_2)$.

For the details of the proof of the above lemmas, refer to [S].

We are now going to formulate our main result. Let $\underline{Q}$ be an H -orbit of $\underline{N}(\underline{q})$. Then it follows from Lemmas 1 and 4 that there exist $X \in \underline{Q}$, $A \in \underline{h}$ and $Y \in \underline{q}$ such that (A, X, Y) is a strictly normal S-triple. Put

$$A^d = i(X - Y), \quad X^d = \frac{1}{2}(X + Y + iA), \quad Y^d = \frac{1}{2}(X + Y - iA).$$

Then (A^d, X^d, Y^d) is a strictly normal S-triple for the pair $(\underline{q}^d, \underline{h}^d)$. Moreover it follows from Lemma 5 that the H^d -orbits $H^d \cdot X^d$ and $H^d \cdot Y^d$ only depend on X . Noting this, we define maps

$$\Phi_{\pm} : [\underline{N}(\underline{q})] \rightarrow [\underline{N}(\underline{q}^d)]$$

by $\Phi_+(\underline{Q}) = H^d \cdot X^d$ and $\Phi_-(\underline{Q}) = H^d \cdot Y^d$. By a similar argument, we also define maps $\Phi_{\pm}^d : [\underline{N}(\underline{q}^d)] \rightarrow [\underline{N}(\underline{q})]$. Then we find that

$$\Phi_-^d(\Phi_+(H \cdot X)) = H \cdot X, \quad \Phi_+^d(\Phi_-(H \cdot X)) = H \cdot Y.$$

This, in particular, implies the bijectivity of $\Phi_{\pm}$.

Let (A, X, Y) be a strictly normal S-triple. Put

$$A' = X+Y, \quad X' = \frac{1}{2}(X-Y-A), \quad Y' = \frac{1}{2}(-X+Y-A).$$

Then it follows that (A', X', Y') is a strictly normal S-triple for the pair $(\underline{g}^a, \underline{h}^a)$. Noting this, we also obtain a bijection $[\underline{N}(\underline{g})] \rightarrow [\underline{N}(\underline{g}^a)]$ by an argument similar to the above one.

Hence we obtain the following theorem.

$$\begin{aligned} \text{THEOREM 5. } [\underline{N}(\underline{g})] &\simeq [\underline{N}(\underline{g}^a)] \simeq [\underline{N}(\underline{g}^d)] \simeq [\underline{N}(\underline{g}^{ad})] \\ &\simeq [\underline{N}(\underline{g}^{da})] \simeq [\underline{N}(\underline{g}^{ada})]. \end{aligned}$$

§5. PROOF OF THE CONJECTURE.

We return to the situation in §1. By definition, $(\underline{g}_c, \underline{k}_c)$ and $(\underline{g}_c, \underline{g})$ are symmetric pairs. Moreover, if we define an involution of $\underline{g} \oplus \underline{g}$ by $(X, Y) \rightarrow (Y, X)$, we find that $(\underline{g} \oplus \underline{g}, \underline{g})$ is also a symmetric pair. In this case, the following diagram holds:

$$\begin{array}{ccccc} (\underline{g} \oplus \underline{g}, \underline{g}) & \xleftarrow{\text{dual}} & (\underline{g}_c, \underline{k}_c) & \xleftarrow{\text{associated}} & (\underline{g}_c, \underline{g}) \\ \uparrow & & & & \uparrow \\ \text{associated} & & & & \text{dual} \end{array}$$

Let $[\underline{N}(\underline{g})]$ be the set of G -orbits of $\underline{N}(\underline{g})$ and let $[\underline{N}(\underline{p}_c)]$ be that of K_c -orbits of $\underline{N}(\underline{p})$. Then we find that the conjecture stated in §1 is a special case of THEOREM 5.

$$\text{Corollary 6. } [\underline{N}(\underline{p}_c)] \simeq [\underline{N}(\underline{g})]$$

Let $\tilde{K}_c$ and $\tilde{G}$ be the normalizers of $\underline{k}_c$ and $\underline{g}$ in G_c , respectively. Let $[\underline{N}(\underline{g})]_{\theta}$ be the set of $\tilde{G}$ -orbits of $\underline{N}(\underline{g})$ and

let $[\underline{N}(\underline{p}_c)]_\theta$ be that of $\tilde{K}_c$ -orbits of $\underline{N}(\underline{p}_c)$. Then an analogy of Corollary holds:

Theorem 7 (B. Kostant). $[\underline{N}(\underline{p}_c)]_\theta \simeq [\underline{N}(\underline{g})]_\theta$

Remark 8. Assume that $\underline{g}_c$ is simple of the classical type.

- (i) The G -orbital structure of $\underline{N}(\underline{g})$ is determined by Bourgoyne and Cushman ([BC]).
- (ii) D. King proves Corollary 6 in this case by using the classification ([K]).

References

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