

On Telgárski's formula

Shinpei Oka

Faculty of Education, Kagawa University

The following formula due to R. Telgársky ([3]) is indispensable in dealing with products of scattered spaces.

Theorem 1. Let X, Y be scattered spaces with $\text{leng}(X) = \alpha$ and $\text{leng}(Y) = \beta$. Then

- (1) $(X \times Y)^{(\sigma)} = \bigcup_{\tau \oplus v = \sigma} X^{(\tau)} \times Y^{(v)}$ for every σ .
- (2) $(X \times Y)_{(\sigma)} = \bigcup_{\tau \oplus v = \sigma} X_{(\tau)} \times Y_{(v)}$ for every σ .
- (3) $\text{leng}(X \times Y) = \sup \{ \tau \oplus v + 1 \mid \tau < \alpha \text{ and } v < \beta \}$.

The symbol $\oplus$ means Hessenberg's sum defined as follows :

Definition 1. Let $\alpha > 0, \beta > 0$ be ordinals. Using Cantor's normal form, represent α, β uniquely as

$$\alpha = \omega^{\gamma_1} n_1 + \omega^{\gamma_2} n_2 + \cdots + \omega^{\gamma_k} n_k, \quad \beta = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_k} m_k,$$

$\gamma_1 > \gamma_2 > \cdots > \gamma_k$, $0 \leq n_i < \omega$, $0 \leq m_i < \omega$, so that $n_i = 0 = m_i$ does not occur. Define

$$\alpha \oplus \beta = \sum_{i=1}^k \omega^{\gamma_i} (n_i + m_i).$$

Also define $\alpha \oplus 0 = 0 \oplus \alpha = \alpha$ for every α .

Hessenberg's sum is certainly convenient for describing the derivatives $(X \times Y)^{(\sigma)}$ but not for describing $\text{leng}(X \times Y)$. With an emphasis on the length of product spaces, we define a binary operation π as follows :

Definition 2. Let $\alpha > 0, \beta > 0$ be ordinals. Represent α, β as in Definition 1. Put

$$l = \min \{ \max \{ i \mid n_i \neq 0 \}, \max \{ j \mid m_j \neq 0 \} \}$$

and define

$$\pi(\alpha, \beta) = \begin{cases} \sum_{i=1}^l \omega^{\gamma_i} (n_i + m_i) & \text{if } l < k \\ (\sum_{i=1}^{k-1} \omega^{\gamma_i} (n_i + m_i)) + \omega^{\gamma_k} (n_k + m_k - 1) & \text{if } l = k, \end{cases}$$

where $l = k$ is, of course, equivalent to $n_k \neq 0 \neq m_k$.

For convenience, define $\pi(\alpha, 0) = \pi(0, \alpha) = 0$ for every ordinal α .

It is to be noted that, unlike Hessenberg's sum, the operation π is a continuous operation with respect to the order topology.

Now we can restate Telgársky's formula as follows :

Theorem 2. Let X, Y be scattered spaces with $\text{len}(X) = \alpha$, $\text{len}(Y) = \beta$. Then

- (1) $\text{len}(X \times Y) = \pi(\alpha, \beta)$.
- (2) $(X \times Y)_{(\sigma)} = \cup \{X_{(\tau)} \times Y_{(v)} \mid \pi(\tau + 1, v + 1) = \sigma + 1\}$ for every ordinal σ .
- (3) $(X \times Y)^{(\sigma)} = \cup \{X^{(\tau)} \times Y^{(v)} \mid \pi(\tau + 1, v + 1) = \sigma + 1\}$ for every ordinal σ .

We write simply $\pi(\alpha, \beta) = \alpha * \beta$.

Proposition 1. $\alpha * \beta = \beta * \alpha$. $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

Definition 3. A factorization $\alpha = \beta * \gamma$ of an ordinal α is called *trivial* if one of β, γ is 1 (and the other is α). An ordinal α is called a *prime* ordinal if $\alpha > 1$ and it does not admit a non-trivial factorization.

As far as I know, the following notion was first defined and used by K. Borsuk ([1] also see [2]).

Definition 4. A factorization $X \approx Y \times Z$ of a space X is called *trivial* if one of Y, Z is a one point space (and the other is homeomorphic to X). A space X is called a *prime* space (or simply a *prime*) if $|X| > 1$ and it does not admit a non-trivial factorization.

By the definition of the operation π we have

Proposition 2. An ordinal α is a prime ordinal if and only if $\alpha = \omega^\gamma + 1$ with γ an ordinal.

The compact countable metric space X satisfying $\text{len}(X) = \alpha$ and $|X^{(\alpha-1)}| = |X_{(\alpha-1)}| = n$ is denoted by $MS(\alpha, n)$. By the uniqueness of $MS(\alpha, n)$ we have $MS(\alpha, n) \approx MS(\alpha, 1) \times MS(1, n)$ and, by Theorem 2, $MS(\alpha, 1) \times MS(\beta, 1) \approx MS(\alpha * \beta, 1)$. These facts are combined with Proposition 2 to give

Proposition 3. A compact countable metric space X is a prime if and only if $X \approx MS(1, p)$ with p a prime number or $X \approx MS(\omega^\gamma + 1, 1)$ with γ an ordinal.

Theorem 3. Every non-limit ordinal $\alpha > 1$ is factorized uniquely as

$$\alpha = \alpha_1 * \alpha_2 * \cdots * \alpha_j$$

into prime ordinals $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_j$. In full detail, if the normal form of α is

$$\alpha = \omega^{\zeta_1} n_1 + \omega^{\zeta_2} n_2 + \cdots + \omega^{\zeta_k} n_k$$

with $\zeta_k = 0$ then

$$\alpha = \overbrace{(\omega^{\zeta_1} + 1) * (\omega^{\zeta_1} + 1) * \dots * (\omega^{\zeta_1} + 1)}^{n_1} * \overbrace{(\omega^{\zeta_2} + 1) * (\omega^{\zeta_2} + 1) * \dots * (\omega^{\zeta_2} + 1)}^{n_2} * \dots * \overbrace{(\omega^{\zeta_{k-1}} + 1) * (\omega^{\zeta_{k-1}} + 1) * \dots * (\omega^{\zeta_{k-1}} + 1)}^{n_{k-1}} * \underbrace{2 * 2 * \dots * 2}_{n_{k-1}} \quad . \quad (\text{Do not confuse } n_{k-1} \text{ with } n_k - 1 .)$$

A translation of the theorem into product spaces is as follows :

Corollary 1. *Every compact countable metric space X with $|X| > 1$ is factorized uniquely as*

$$X \approx X_1 \times X_2 \times \dots \times X_j$$

into primes $X_1, X_2, \dots, X_j$. In further detail, if $X = MS(\alpha, n)$, $\alpha = \omega^{\zeta_1} n_1 + \omega^{\zeta_2} n_2 + \dots + \omega^{\zeta_k} n_k$ with $\zeta_k = 0$, then

$$X \approx \overbrace{MS(\omega^{\zeta_1} + 1, 1) \times MS(\omega^{\zeta_1} + 1, 1) \times \dots \times MS(\omega^{\zeta_1} + 1, 1)}^{n_1} \times \overbrace{MS(\omega^{\zeta_2} + 1, 1) \times MS(\omega^{\zeta_2} + 1, 1) \times \dots \times MS(\omega^{\zeta_2} + 1, 1)}^{n_2} \times \dots * \overbrace{MS(\omega^{\zeta_{k-1}} + 1, 1) \times MS(\omega^{\zeta_{k-1}} + 1, 1) \times \dots \times MS(\omega^{\zeta_{k-1}} + 1, 1)}^{n_{k-1}} \times \underbrace{MS(2, 1) \times MS(2, 1) \times \dots \times MS(2, 1)}_{n_{k-1}} \times MS(1, p_1) \times MS(1, p_2) \times \dots \times MS(1, p_r) \quad ,$$

where $n = p_1 p_2 \dots p_r$ is the usual factorization of the natural number n into primes.

Examples. Put, for example, $\alpha = \omega^2 + \omega 2 + 3$. Then

$$\omega^2 + \omega 2 + 3 = (\omega^2 + 1) * (\omega + 1) * (\omega + 1) * 2 * 2 .$$

Thus

$$MS(\omega^2 + \omega 2 + 3, 1) = MS(\omega^2 + 1, 1) \times MS(\omega + 1, 1) \times MS(\omega + 1, 1) \times MS(2, 1) \times MS(2, 1) \quad .$$

$$MS(\omega^2 + \omega 2 + 3, 6) = MS(\omega^2 + 1, 1) \times MS(\omega + 1, 1) \times MS(\omega + 1, 1) \times \\ MS(2, 1) \times MS(2, 1) \times MS(1, 3) \times MS(1, 2) .$$

References

- [1] K. Borsuk, *Sur la décomposition des polyèdres en produits cartésiens*, Fund. Math. **31** (1938), 137-148.
- [2] K. Borsuk, *On the decomposition of a locally connected compactum into Cartesian product of a curve and a manifold*, Fund. Math. **40** (1953), 140-159.
- [3] R. Telgársky, *Derivatives of Cartesian product and dispersed spaces*, Colloq. Math. **19** (1968), 59-66.