

# Parametrization by fixed-points multipliers of the polynomials with degree $n$

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## 1 Introduction

Let  $\text{Poly}_n(\mathbb{C})$  be the the polynomials from the Riemann sphere,  $\widehat{\mathbb{C}}$ , to itself, with degree  $n$ , and  $\mathbb{M}_n$ , called moduli space, the quotient space of  $\text{Poly}_n(\mathbb{C})$  under the action of the affine transformation group,  $\mathcal{A}(\mathbb{C})$ .

We parametrize  $\mathbb{M}_n$  by using multipliers of fixed points, and define a natural map  $\Psi$  from  $\mathbb{M}_n$  to  $\mathbb{C}^{n-1}$ . A new coordinate system is called multiplier coordinates. Exhibiting the moduli space of a higher degree under this system deserves particular attention. For example, in study of geometry and topology of  $\text{Poly}_n(\mathbb{C})$  from a viewpoint of complex dynamical systems, we make use of this system in order to express singular part, and dynamical loci as algebraic curves or surfaces([NF99], [NF00] ).

The subject of this paper is surjectivity-problem of the map  $\Psi$  from  $\mathbb{M}_n$  to  $\mathbb{C}^{n-1}$ : a problem of characterization of exceptinal part,  $\mathcal{E}_n(= \mathbb{C}^{n-1} \setminus \mathbb{M}_n)$  .

The initiator of the use of multiplier coordinates is J. Milnor ([Mil93]), to the case of the quadratic rational maps.

## 2 Polynomials of degree $n$

### 2.1 Polynomial maps and conjugacy

Let  $\widehat{\mathbb{C}}$  be the Riemann sphere, and  $\text{Poly}_n(\mathbb{C})$  be the space of all polynomial maps of degree  $n$  from  $\widehat{\mathbb{C}}$  to itself:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad (a_n \neq 0).$$

The group  $\mathcal{A}(\mathbb{C})$  of all affine transformations acts on  $\text{Poly}_n(\mathbb{C})$  by conjugation:

$$g \circ p \circ g^{-1} \in \text{Poly}_n(\mathbb{C}) \quad \text{for} \quad g \in \mathcal{A}(\mathbb{C}), p \in \text{Poly}_n(\mathbb{C}).$$

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Two maps  $p_1, p_2 \in \text{Poly}_n(\mathbb{C})$  are **holomorphically conjugate** if and only if there exists  $g \in \mathfrak{A}(\mathbb{C})$  with  $g \circ p_1 \circ g^{-1} = p_2$ .

Under this conjugacy of the action of  $\mathfrak{A}(\mathbb{C})$ , any map in  $\text{Poly}_n(\mathbb{C})$  is conjugate to a “monic” and “centered” map, i.e.,

$$p(z) = z^n + c_{n-2}z^{n-2} + c_{n-3}z^{n-3} \cdots + c_0.$$

We remark that this  $p$  is determined up to the action of the group  $G(n-1)$  of  $(n-1)$ -st roots of unity, where each  $\eta \in G(n-1)$  acts on  $p \in \text{Poly}_n(\mathbb{C})$  by the transformation  $p(z) \mapsto p(\eta z)/\eta$ .

Every polynomial map from  $\widehat{\mathbb{C}}$  to itself is conjugate under an affine change of variable to a monic centered one, and this is uniquely determined up to conjugacy under the action of the group  $G(n-1)$  of  $(n-1)$ -st roots of unity.

For example, in the case of  $n = 3$ , the following two monic and centered polynomials belong to the same conjugacy class:

$$z^3 + az + c, \quad z^3 + az - c.$$

In the case of  $n = 4$  the following three monic and centered polynomials belong to the same conjugacy class:

$$z^4 + az^2 + bz + c$$

$$z^4 + a\omega z^2 + bz + c\omega^2$$

$$z^4 + a\omega^2 z^2 + bz + c\omega$$

where  $\omega$  is a third root of unity.

## 2.2 Moduli space of polynomial maps

The quotient space of  $\text{Poly}_n(\mathbb{C})$  under the action  $\mathfrak{A}(\mathbb{C})$  will be denoted by  $\mathbb{M}_n$ , and called the **moduli space** of holomorphic conjugacy classes  $\langle p \rangle$  of polynomial maps  $p$  of degree  $n$ .

Let  $\mathcal{P}_1(n)$  be the affine space of all monic centered polynomials of degree  $n$

$$p(z) = z^n + c_{n-2}z^{n-2} + c_{n-3}z^{n-3} \cdots + c_0,$$

with coefficients-coordinate  $(c_0, c_1, \dots, c_{n-2})$ .

Then we have an  $(n-1)$ -to-one canonical projection  $\Phi$  from  $\mathcal{P}_1(n)$  onto  $\mathbb{M}_n$ .

Hence the affine space  $\mathcal{P}_1(n)$  is regarded as an  $(n-1)$ -sheeted covering space of  $\mathbb{M}_n$ . Thus we can use  $\mathcal{P}_1(n)$  as a coordinate space for the moduli space  $\mathbb{M}_n$ , though it remains the ambiguity up to the group  $G(n-1)$ . This coordinate space has the advantages of being easy to be treated.

However, it would be also worthwhile to introduce another coordinate system having any merit different from  $\mathcal{P}_1(n)$ 's.

In fact, Milnor successfully introduced coordinates in the moduli space of the space of all quadratic rational maps using the elementary symmetric functions of the multipliers at the fixed points of a map ([Mil93]). To the case of  $\text{Poly}_n(\mathbb{C})$ , we try to explore an analogy.

## 2.3 Multiplier coordinates

Now we intend to explore another coordinate space for  $\mathbb{M}_n$ . For each  $p(z) \in \text{Poly}_n(\mathbb{C})$ , let  $z_1, \dots, z_n, z_{n+1} (= \infty)$  be the fixed points of  $p$  and  $\mu_i$  the multipliers of  $z_i$ ;  $\mu_i = p'(z_i)$  ( $1 \leq$

$i \leq n$ ), and  $\mu_{n+1} = 0$ . Consider the elementary symmetric functions of the  $n$  multipliers,

$$\begin{aligned}\sigma_{n,1} &= \mu_1 + \cdots + \mu_n, \\ \sigma_{n,2} &= \mu_1\mu_2 + \cdots + \mu_{n-1}\mu_n = \sum_{i=1}^{n-1} \mu_i \sum_{j>i}^n \mu_j, \\ &\dots \\ \sigma_{n,n} &= \mu_1\mu_2 \cdots \mu_n, \\ \sigma_{n,n+1} &= 0.\end{aligned}$$

Note that these are well defined on the moduli space  $\mathbb{M}_n$ , since  $\mu_i$ 's are invariant by affine conjugacy.

### 2.3.1 The holomorphic index fixed point formula

For an isolated fixed point  $f(x_0) = x_0$ ,  $x_0 \neq \infty$  we define the holomorphic index of  $f$  at  $x_0$  to be the residue

$$\iota(f, x_0) = \frac{1}{2\pi i} \oint \frac{1}{z - f(z)} dz$$

For the point at infinity, we define the residue of  $f$  at  $\infty$  to be equal to the residue of  $\phi \circ f \circ \phi$  at origin, where  $\phi(z) = \frac{1}{z}$ . The Fatou index theorem (see [Mil90]) is as follows:

For any rational map  $f : \mathbb{C} \rightarrow \mathbb{C}$  with  $f(z)$  not identically equal to  $z$ , we have the relation  $\sum_{f(z)=z} \iota(f, z) = 1$ . This theorem can be applied to these  $\mu_i$ 's ;  $\sum_{i=1}^n \frac{1}{1-\mu_i} + \frac{1}{1-0} = 1$ , provided  $\mu_i \neq 1$  ( $1 < i < n$ ). Arranging this equation for the form of elementary symmetric functions, we have

$$\gamma_0 + \gamma_1\sigma_{n,1} + \gamma_2\sigma_{n,2} + \cdots + \gamma_{n-1}\sigma_{n,n-1} = 0$$

where

$$\gamma_k = (-1)^k n \binom{n-1}{k} \bigg/ \binom{n}{k} = (-1)^k (n-k).$$

Note that  $\mu_i = 1$  ( $1 \leq i \leq n$ ) is allowable here. Then we have the following Linear Relation : •

For the cubic case ( $n = 3$ ), we have  $3 - 2\sigma_{3,1} + \sigma_{3,2} = 0$

• For the quartic case ( $n = 4$ ), we have  $4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0$

And in general the following linear relation holds:

**Theorem 1** Among  $\sigma_{n,i}$ 's, there is a linear relation

$$\sum_{k=0}^{n-1} (-1)^k (n-k) \sigma_{n,k} = 0, \quad (1)$$

where we put  $\sigma_{n,0} = 1$ .

In view of Theorem 1, we have the natural map  $\Psi$  from  $\mathbb{M}_n$  to  $\mathbb{C}^{n-1}$  corresponding to

$$\Psi(< p >) = (\sigma_{n,1}, \sigma_{n,2}, \dots, \sigma_{n,n-2}, \sigma_{n,n}).$$

We remark that  $\Psi(\mathbb{M}_n) \subset \mathbb{C}^{n-1}$ .

### 2.3.2 Characterization of exceptional set

To investigate whether this map  $\Psi$  is surjective or not is our main subject: a problem of characterization of the part of  $\mathbb{C}^{n-1} \setminus \Psi(\mathbb{M}_n)$ .

We call this set **exceptional set** and denote it by

$$\mathcal{E}_n = \mathbb{C}^{n-1} \setminus \Psi(\mathbb{M}_n).$$

Our main subject is as follows:

For a given  $(s_1, s_2, \dots, s_{n-2}, s_n) \in \mathbb{C}^{n-1}$ , we set  $s_{n-1}$  a solution of

$$\sum_{k=0}^{n-1} (-1)^k (n-k) s_k = 0, \quad s_0 = 1.$$

Then for the point  $(s_1, \dots, s_n) \in \mathbb{C}^{n-1}$ , we set a polynomial

$$m(z) = z^n + s_1 z^{n-1} + s_2 z^{n-2} + \dots + s_{n-1} z + s_n$$

Then we denote the roots of this polynomial by

$$\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n.$$

**Can we obtain a polynomial  $p(z) \in \mathcal{P}_1(n)$  whose multiplier-coordinate  $(\sigma_1, \dots, \sigma_n)$  is corresponding to  $(s_1, \dots, s_n)$  ?**

Namely can we find a polynomial satisfying that for fixed points  $z_i$

$$p(z_i) = z_i, \quad (i = 1, \dots, n) \text{ with } \mu_i = p'(z_i).$$

The case  $n = 3$  is nicely solved:  $\Psi$  is surjective. ([NF96], [FN97]. This fact is mentioned in [Mil93] without any details.)

We also solved this problem for the case  $n = 4$  ([NF96], [FN97]):

**Theorem 2**  $\Psi : \mathbb{M}_4 \longrightarrow \mathbb{C}^3$  is not surjective:

$$\begin{aligned} \mathcal{E}_4 &= \mathbb{C}^3 \setminus \Psi(\mathbb{M}_4) \\ &= (4, s, \frac{s^2}{4} - 2s + 4) \quad s \neq 4 \end{aligned}$$

As for the cases of general  $n$ , we expect analogous results.

Recently, we have a following result:

**Theorem 3** (M.FUJIMURA)

Let  $\Omega = \{\mu_i\}_{i=1, \dots, n}$  be the set of all roots of a polynomial  $m(z)$ . If  $\Omega$  satisfies one of the following cases (A), (B) and (C), then there exists a polynomial  $p(z) \in \mathcal{P}_1(n)$  such that

$$p(z_i) = z_i, \quad (i = 1, \dots, n) \text{ with } \mu_i = p'(z_i).$$

(A):

1. Any element of  $\Omega$  is not equal 1 :  $\mu_i \neq 1$ ,
2.  $\sum_i \frac{1}{b_i} = 0$ ,  $b_i = 1 - \mu_i$ ,

3. for any proper subset  $\omega$  of roots,  $\sum_{s \in \omega} \frac{1}{b_s} \neq 0$ ,

(B):

1. Let  $\Omega' = \{\mu_i\}_{i=1, \dots, m}$   $1 \leq m \leq n - 2$  be a subset of  $\Omega$  whose elements are not equal 1 :  $\mu_i \neq 1$ ,

2. for any subset  $\omega$  of  $\Omega'$  ,  $\sum_{s \in \omega} \frac{1}{b_s} \neq 0$ ,

(C):

1. Any element of  $\Omega$  is equal 1 :  $\mu_i = 1$ .

### 2.3.3 Examples

We shall show some examples for our inverse problem. By these examples show that the Fujimura's theorem only gives a sufficient condition for surjectivity.

- For a set  $\{\mu, 2 - \mu, \lambda, 2 - \lambda\}$ ,  $\mu \neq \lambda$ ,  $\mu \neq 1$  a corresponding polynomial exists in  $\mathcal{P}_1(4)$  .
- For a set  $\{\mu, 2 - \mu, \mu, 2 - \mu\}$   $\mu \neq 1$ , no corresponding polynomial exists  $\mathcal{P}_1(4)$ .
- For a set  $\{\mu, \mu, \mu, \lambda, \lambda\}$ ,  $\mu \neq 1$ ,  $5 - 2\mu - 3\lambda = 0$  a corresponding polynomial exists  $\mathcal{P}_1(5)$ .
- For a set  $\{\mu, \mu, \mu, 2 - \mu, \frac{3-\mu}{2}\}$ ,  $\mu \neq 1$ , no corresponding polynomial exists  $\mathcal{P}_1(5)$ .

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