

HYPERSPACES AND WHITNEY MAPS

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Throughout this note, the word compactum means a compact metric space. A connected compactum is a continuum. A Peano continuum is a locally connected continuum. If x and y are points of a metric space, $d(x,y)$ denotes the distance from x to y . For any subsets A and B of a metric space, let $d(A,B)=\inf \{d(a,b) \mid a \in A, b \in B\}$. Also, let $d_H(A,B)=\max \{ \sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A) \}$. d_H is called the Hausdorff metric. The hyperspaces of a continuum are the spaces $2^X = \{A \subset X \mid A \text{ is compact and nonempty}\}$ and $C(X) = \{A \in 2^X \mid A \text{ is connected}\}$ which are metrized with the Hausdorff metric d_H . Let $F_1(X) = \{\{x\} \mid x \in X\}$. A Whitney map for a hyperspace H of a continuum X is a continuous function $w: H \rightarrow [0, w(X)]$ such that $w(\{x\})=0$ for each $\{x\} \in F_1(X)$, and if $A, B \in H$ and $A \subsetneq B$, then $w(A) < w(B)$ (see [9]). The notion of Whitney map is an important and convenient tool for hyperspace theory. If w is a Whitney map for H and $t \in [0, w(X))$, then $w^{-1}(t)$ is called a Whitney level. Whitney levels are coverings of X which, as t gets closed to zero, converge to $w^{-1}(0) = F_1(X) \cong X$. It is of interest to obtain information about the structure of Whitney levels and determine those properties which are preserved by the convergence of positive Whitney levels to zero level. In [1] and [8], Curtis, Schori and West proved that for any Peano continuum (locally connected continuum) X , 2^X is a Hilbert cube $Q = \prod_{i=1}^{\infty} [-1,1]$ and if X contains no free arc, $C(X)$ is a Hilbert cube Q . Recently, Goodykoontz and Nadler introduced the notion "admissible Whitney map" and they proved the following

Theorem (Goodykoontz and Nadler). Let X be a Peano continuum and let w be an admissible ^{Whitney} map for $H=2^X$ or $C(X)$. If $H=C(X)$, assume that X contains no free arc. Then for any $t \in (0, w(X))$, $w^{-1}(t)$ is a Hilbert cube and w is an open map.

Let X be a continuum. A Whitney map w for $H=2^X$ or $C(X)$ is an admissible Whitney map for H [2] if there is a homotopy $h: H \times [0,1] \rightarrow H$ satisfying the following conditions:

- (1) $h(A,1)=A$, $h(A,0) \in F_1(X)$ for each $A \in H$, and
- (2) if $w(h(A,t)) > 0$ for some $A \in H$ and $t \in (0,1]$, then $w(h(A,s)) < w(h(A,t))$ for each $0 \leq s < t \leq 1$.

Moreover, in [4] we proved the following

Theorem [4]. Under the same hypotheses as in above theorem, the restriction $w|_{w^{-1}((0,w(X))): w^{-1}((0,w(X))) \rightarrow (0,w(X))}$ of w to $w^{-1}((0,w(X)))$ is a trivial bundle map with Hilbert cube fibers. If X is the Hilbert cube Q , there is a Whitney map w for H such that $w|_{w^{-1}[0,w(X))}$ is a trivial bundle map with Hilbert cube fibers. Also, if X is the n -sphere ($n \geq 1$), then there is a Whitney map w for $H=2^{S^n}$ ($n \geq 1$) or $C(S^n)$ ($n \geq 2$) such that for some $t \in (0,w(X))$, $w|_{w^{-1}((0,t])}$ is a trivial bundle map with $S^n \times Q$ fibers.

Also, in [5] we showed the following

Theorem [5]. Let P_i be a 1 or 2 dimensional connected polytope for each $i=1,2,\dots,n$. Then there is a Whitney map w for $H=2^{\prod_{i=1}^n P_i}$ or $C(\prod_{i=1}^n P_i)$ ($n \geq 2$) such that for some $t \in (0,w(\prod_{i=1}^n P_i))$,

$w|w^{-1}((0,t])$ is a trivial bundle map with $\prod_{i=1}^n P_i \times Q$ fibers.

In relation to above theorems, we have the following

Proposition [5]. Let X be a compact ANR but not AR.

Let $H=2^X$ or $C(X)$. If $H=C(X)$, assume that X contains no free arc.

If w is any Whitney map for H , there is a point $t_0 \in (0, w(X))$ such that $w|w^{-1}((0, t_0])$ is not a trivial bundle map.

Example [5]. Let $X=S$ be the unit *circle*. Let $A \in H=2^X$ or $C(X)$. For each $n \geq 2$, let $F_n(A) = \{K \subset A \mid K \neq \emptyset \text{ and the cardinality of } K \text{ is } \leq n\}$, define $\lambda_n: F_n(A) \rightarrow [0, \infty)$ by letting $\lambda_n(\{a_1, a_2, \dots, a_n\}) = \min\{d(a_i, a_j) \mid i \neq j\}$ for each $\{a_i\} \in F_n(A)$, where d is the arc length metric for S . Also, let $w_n(A) = \sup \lambda_n(F_n(A))$ and let $w(A) = \sum_{n=2}^{\infty} w_n(A)/2^{n-1}$ for each $A \in H$. Then w is a Whitney map for H . Then $w|w^{-1}((0, \pi/2)) : w^{-1}((0, \pi/2)) \rightarrow (0, \pi/2)$ is a trivial bundle map with $S \times Q$ fibers, but $w|w^{-1}((0, \pi/2])$ is not a trivial bundle map. In fact, $w|w^{-1}((0, \pi/2])$ is not an open map.

Example [5]. There is a Whitney map $\overset{(w)}{\underbrace{\quad}}$ for $H=2^{[0,1]}$ such that for every $t \in (0, w([0,1]))$, $w|w^{-1}((0, t])$ is not a trivial bundle map.

Question [5]. Is it true that if P is a n -dimensional ($n \geq 3$) polytope, there is a Whitney map w for $H=2^P$ or $C(P)$ such that for some $t \in (0, w(P))$, $w|w^{-1}((0, t])$ is a trivial bundle map with $P \times Q$ fibers? (If $H=C(P)$, assume that P contains no free arc).

As an application of hyperspace theory, we obtain the following

Theorem [6]. If a compactum X has a scalene metric, then X is an absolute retract. Moreover, if a locally compact space X has a locally scalene metric, then X is an absolute neighborhood retract.

A metric d defined in a space X is a scalene metric [6] if x_1 and x_2 are different two points of X , then there is a point x_0 of X such that for each point x of X either $d(x, x_1) > d(x, x_0)$ or $d(x, x_2) > d(x, x_0)$ holds. This notion is a generalization of norm of a linear space. A metric d defined in a space X is a locally scalene metric [6] if for each point $x \in X$ there is a neighborhood U of x in X such that $d|_{U \times U}$ is a scalene metric.

Remark [6]. Every 1-dimensional AR has a scalene metric but there is a 2-dimensional AR which does not admit a scalene metric.

Question [6]. Is it true that every locally compact polytope has a locally scalene metric ?

Question [6]. Is it true that every compact strongly convex metric space admits a scalene metric ? Is it true that every compact scalene metric space admits a strongly convex metric ?

Proposition [6]. If d is a scalene metric and convex, then d is a strongly convex metric.

References

- [1] D. W. Curtis and R. M. Schori, Hyperspaces of Peano continua are Hilbert cubes, *Fund. Math.* 101 (1978), 19-38.
- [2] J. T. Goodykoontz and S. B. Nadler, Whitney levels in hyperspaces of certain Peano continua, *Trans. Amer. Math. Soc.* 274 (1982), 671-694.
- [3] J. L. Kelley, Hyperspaces of a continuum, *Trans. Amer. Math. Soc.* 52 (1942), 22-36.
- [4] H. Kato, Concerning hyperspaces of certain Peano continua and strong regularity of Whitney maps, to appear in *Pacific J. Math.*
- [5] _____, On hyperspaces and Whitney maps of 2-dimensional polytopes, preprint.
- [6] _____, Scalene metric spaces and absolute retracts, preprint.
- [7] S. B. Nadler, Hyperspaces of sets, *Pure and Appl. Math.* 49 Dekker, New York, 1978.
- [8] R. M. Schori and J. E. West, The hyperspace of the closed unit interval is a Hilbert cube, *Trans. Amer. Math. Soc.* 213 (1975), 217-235.
- [9] H. Whitney, Regular families of curves I, *Proc. Nat. Acad. Sci. U.S.A.* (1932), 275-278.
- [10] M. Wojdyslawski, Retracts absolus et hyperespaces des continus, *Fund. Math.* 32 (1939), 184-192.
- [11] _____, Sur la contractilite des hyperespaces des continus localement connexes, *Fund. Math.* 30 (1938), 247-252.