

Location of the asymptotic profile for one-dimensional chemotaxis system

早稲田大学・政経学院 西原 健二 (Kenji Nishihara)¹

Faculty of Political Science and Economics,
Waseda University

1 Introduction

We consider the Cauchy problem for a one-dimensional model system of chemotaxis

$$(P) \quad \begin{cases} u_t = au_{xx} - (uv_x)_x, & (t, x) \in \mathbf{R}^+ \times \mathbf{R}^1 \\ v_t = bv_{xx} - v + u, & (t, x) \in \mathbf{R}^+ \times \mathbf{R}^1 \\ (u, v)(0, x) = (u_0, v_0)(x), & x \in \mathbf{R}^1 \end{cases} \quad (a, b > 0 : \text{constants}).$$

Our interest is in the asymptotic profile of solutions (u, v) as $t \rightarrow \infty$ when bounded solutions exist in the sense that

$$(1.1) \quad \sup_{t>0} (\|u(t, \cdot)\|_{L^q} + \|v(t, \cdot)\|_{L^q}) < +\infty \quad (q = 1, \infty).$$

By Nagai, Shukuinn and Umesako [2] and Nagai and Yamada [3], it has been showed that the bounded solution to (P) in $\mathbf{R}^N$ ($N \geq 1$) with $a = b = 1$ satisfies

$$(1.2) \quad \sup_{t>2} d(t; p) \|(u - M_0 G, v - M_0 G)(t, \cdot)\|_{L^p} < +\infty, \quad M_0 = \int_{\mathbf{R}^N} u_0(x) dx$$

$$\text{with } d(t; p) = \begin{cases} t^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{2}}(\log t)^{-1} & (N = 1) \\ t^{\frac{N}{2}(1-\frac{1}{p})+\frac{1}{2}} & (N \geq 2), \end{cases} \quad \text{where } G(t, x) = (4\pi t)^{-1/2} \exp(-|x|^2/4t).$$

Kato [1] has recently improved (1.2) for $N = 1$ as that the "logarithmic tail" in $d(t; p)$ can be deleted even for $a, b > 0$, not necessarily $a = b = 1$. More precisely, the second term of the asymptotics is given. If $W(t, x)$ is defined by the solution to

$$(1.3) \quad \begin{aligned} W_t &= W_{xx} - \frac{M_0^2}{2a} (G^2(a+t, x))_{xx}, \\ W(0, x) &= - \left(\int_{\mathbf{R}^1} x u_0(x) dx + \int_0^\infty \int_{\mathbf{R}^1} (uv_x)(t, x) dx dt \right) \frac{d}{dx} \delta(x), \end{aligned}$$

then it satisfies

$$(1.4) \quad \lim_{t \rightarrow \infty} t^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{2}} \|u(t, \cdot) - M_0 G(at, \cdot) - W(at, \cdot)\|_{L^p} = 0$$

¹ This work was supported in part by Grant-in-Aid for Scientific Research (C) 20540219 of Japan Society for the Promotion of Science.

with $\|W(t, \cdot)\|_{L^p} \leq CM_0^2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$ and $\|W(t, \cdot)\|_{L^\infty} \geq CM_0^2(1+t)^{-1}$, $t \geq 2$. The same estimate on v also holds. In the result, the logarithmic tail in (1.2) is deleted.

Here and after, let $a = 1$, $b > 0$ without loss of generality.

In this note we want to discuss the profile of solutions from the following point of view. The results above mentioned, of course, show that $M_0G(t, x)$ is an asymptotic profile of both u and v . However, we take the location of the profile into consideration. For example, when discrete statistical data are distributed by the Gauss distribution, the data are approximated by

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (\mu : \text{mean}, \sigma : \text{standard deviation}).$$

Here, the choice of both μ and σ is important. Suggested by this, we propose an asymptotic profile with the location μ_∞

$$(1.5) \quad M_0G(t+1, x - \mu_\infty), \quad \mu_\infty = \frac{1}{M_0} \left\{ \int_{-\infty}^{\infty} xu_0(x) dx + \int_0^\infty \int_{-\infty}^{\infty} (uv_x)(t, x) dx dt \right\}.$$

Then we have the following theorem.

Theorem 1 *Let $N = 1$, and suppose that $u_0, v_0, v_{0x} \in L^1 \cap \mathcal{B}$ with*

$$(1.6) \quad (1 + |x|^2)u_0(x) \in L_x^1 \quad \text{with} \quad M_0 = \int_{\mathbf{R}^1} u_0(x) dx \neq 0.$$

Then the bounded solution (u, v) to (P) satisfies for $1 \leq p \leq \infty$ and $t \geq 0$

$$(1.7) \quad \begin{aligned} & \|u(t, \cdot) - M_0G(t+1, \cdot - \mu_\infty) + W_1(t, \cdot; \mu_\infty)\|_{L^p} \\ & \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1} (\log(2+t))^2, \end{aligned}$$

where μ_∞ by (1.5) is well-defined and the second term W_1 of asymptotics is given by

$$(1.8) \quad W_1(t, x; \mu_\infty) = \int_0^t \int_{-\infty}^{\infty} G(t-s, \cdot - y) M_0^2 (GG_x)_x(s+1, y - \mu_\infty) dy ds.$$

The same estimate on v as (1.7) also holds. Moreover, W_1 is estimated from above and below:

$$(1.9) \quad \begin{aligned} & \|W_1(t, \cdot; \mu_\infty)\|_{L^p} \leq CM_0^2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad t \geq 0, \\ & \|W_1(t, \cdot; \mu_\infty)\|_{L^p} \geq C^{-1}M_0^2t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad t \geq t_0 > 0. \end{aligned}$$

In Theorem 1.1 we apply (1.8)-(1.9) to (1.7) and have the following behaviors from above and below.

Corollary 1 *Under the assumptions in Theorem 1.1, for $1 \leq p \leq \infty$ there hold that*

$$\|u(t, \cdot) - M_0G(t+1, \cdot - \mu_\infty), v(t, \cdot) - M_0G(t+1, \cdot - \mu_\infty)\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

for $t \geq 0$, and that, for $t \geq t_1 > 0$

$$\|u(t, \cdot) - M_0G(t+1, \cdot - \mu_\infty), v(t, \cdot) - M_0G(t+1, \cdot - \mu_\infty)\|_{L^p} \geq C^{-1}M_0^2t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.$$

2 Location of the profile

Integrating $(P)_1$ (first equation of (P)) over $(0, t) \times \mathbf{R}^1$, we have

$$(2.1) \quad \int_{-\infty}^{\infty} u(t, x) dx = \int_{-\infty}^{\infty} u_0(x) dx = M_0.$$

For v , by integration of $(P)_2$,

$$(2.2) \quad \int_{-\infty}^{\infty} v(t, x) dx = e^{-t} \int_{-\infty}^{\infty} v_0(x) dx + M_0(1 - e^{-t}) \rightarrow M_0 \quad (t \rightarrow \infty).$$

Hence, taking the location into consideration, we define the profile by

$$(2.3) \quad \phi(t, x) := M_0 G(t + 1, x - \mu(t)),$$

and choose $\mu(t)$ as $\int_{-\infty}^{\infty} \int_{-\infty}^x (u - \phi)(t, y) dy dx = 0$. Since ϕ satisfies

$$(2.4) \quad \partial_t \phi = \phi_{xx} - \frac{d\mu}{dt}(t) \cdot \phi_x(t, x),$$

$u - \phi$ does

$$(2.5) \quad \partial_t(u - \phi) = (u - \phi)_{xx} + \mu'(t)\phi_x - (uv_x)_x.$$

By (2.1) we can integrate (2.5) in x twice to get

$$(2.6) \quad \frac{d}{dt} \int_{-\infty}^{\infty} \int_{-\infty}^x (u - \phi)(t, y) dy dx = M_0 \mu'(t) - \int_{-\infty}^{\infty} (uv_x)(t, x) dx,$$

and hence

$$(2.7) \quad \begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^x (u - \phi)(t, y) dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^x (u_0(y) - \phi(0, y)) dy dx + M_0(\mu(t) - \mu(0)) - \int_0^t \int_{-\infty}^{\infty} (uv_x)(s, x) dx ds \\ &= - \int_{-\infty}^{\infty} x u_0(x) dx + M_0 \mu(t) - \int_0^t \int_{-\infty}^{\infty} (uv_x)(s, x) dx ds, \end{aligned}$$

because

$$\int_{-\infty}^{\infty} x \phi(0, x) dx = \int_{-\infty}^{\infty} x \cdot M_0 G(1, x - \mu(0)) dx = M_0 \mu_0, \quad \mu_0 = \mu(0).$$

We now define $\mu(t)$ by

$$(2.8) \quad \mu(t) = \frac{1}{M_0} \left\{ \int_{-\infty}^{\infty} x u_0(x) dx + \int_0^t \int_{-\infty}^{\infty} (uv_x)(s, x) dx ds \right\}.$$

Therefore, we can define

$$(2.9) \quad U(t, x) := \int_{-\infty}^x \int_{-\infty}^y (u - \phi)(t, z) dz dy \quad \text{or} \quad u = \phi + U_{xx},$$

which satisfies

$$(2.10) \quad \begin{cases} U_t = U_{xx} + \int_{-\infty}^x [\mu'(t)\phi(t, y) - (uv_x)(t, y)] dy \\ U(0, x) := U_0(x) = \int_{-\infty}^x \int_{-\infty}^y (u_0(z) - M_0 G(1, z - \mu_0)) dz dy. \end{cases}$$

To show Theorem 1.1, we need to estimate

$$(2.11) \quad \begin{aligned} (u - \phi)(t, x) &= \int_{-\infty}^{\infty} G_{xx}(t, x - y) U_0(y) dy \\ &+ \int_0^t \int_{-\infty}^{\infty} G_{xx}(t - s, x - y) \int_{-\infty}^y [\mu'(s)\phi(s, z) - (uv_x)(s, z)] dz dy ds. \end{aligned}$$

Here we note that $U_0 \in L^1 \cap \mathcal{B}$ by (1.6) and that

$$(2.12) \quad \int_{-\infty}^{\infty} [\mu'(t)\phi(t, z) - (uv_x)(t, z)] dz = 0.$$

3 Proof of Theorem 1.1

We only sketch the proof, whose details are given in [4]. Known estimates on the solution (u, v) to (P) in Nagai and Yamada [3] and Kato [1] are the followings.

Lemma 3.1 *For $1 \leq p \leq \infty$ and $t \geq 0$, the bounded solution (u, v) to (P) satisfies*

$$(3.1) \quad \|u(t, \cdot) - M_0 G(t + 1, \cdot), v(t, \cdot) - M_0 G(t + 1, \cdot)\|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1 - \frac{1}{p}) - \frac{1}{2}},$$

$$(3.2) \quad \|v_x(t, \cdot) - M_0 G_x(t + 1, \cdot)\|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1 - \frac{1}{p}) - 1} \log(2 + t),$$

$$(3.3) \quad \|(u - v)(t, \cdot)\|_{L^p} \leq C(1 + t)^{-\frac{1}{2}(1 - \frac{1}{p}) - 1} \log(2 + t).$$

By (3.1)-(3.3) we have the properties of $\mu(t)$.

Lemma 3.2 *The location $\mu(t)$ by (2.8) satisfies for $t \geq 0$*

$$(3.4) \quad |\mu'(t)| \leq C(1 + t)^{-\frac{3}{2}} \log(2 + t),$$

which implies that $\mu(\infty) = \mu_\infty$ is well-defined, and

$$(3.5) \quad |\mu(t) - \mu_\infty| \leq C(1 + t)^{-\frac{1}{2}} \log(2 + t).$$

Proof. By (3.1)-(3.2), (3.4) follows from

$$\begin{aligned} |\mu'(t)| &\leq \frac{1}{M_0} (\|(u - M_0 G)(t)\|_{L^1} \|v_x(t)\|_{L^\infty} + \|M_0 G(t)\|_{L^1} \|(v_x - M_0 G_x)(t)\|_{L^\infty}) \\ &\leq C(1+t)^{-\frac{1}{2}} \log(2+t). \end{aligned}$$

Hence (3.5) follows easily. $\square$

By the mean value theorem we have

$$\|\phi(t, \cdot) - M_0 G(t+1, \cdot - \mu_\infty)\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1} \log(2+t)$$

and

$$(3.6) \quad \|W_1(t, \cdot; \mu_\infty) - W_1(t, \cdot; 0)\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}.$$

Hence, to show (1.7), it is enough to prove the following proposition, which is a main estimate in this note. The same estimate on v is derived by (3.3).

Proposition 3.1 *Under the conditions in Theorem 1.1 it holds*

$$(3.7) \quad \|u(t, \cdot) - \phi(t, \cdot) + W_1(t, \cdot; 0)\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1} (\log(2+t))^2.$$

Proof. By (2.11) and (1.8)

$$\begin{aligned} &(u - \phi)(t, x) + W_1(t, x; 0) \\ &= \int G_{xx}(t, x - y) U_0(y) dy + \int_0^t \int G_{xx}(t - s, x - y) \times \\ &\quad \times \int_{-\infty}^y [\mu'(s) \phi(s, z) - (uv_x)(s, z) + M_0^2 (GG_x)(s+1, z)] dz dy ds \\ &=: I_0 + I_1. \end{aligned}$$

By $U_0 \in L^1 \cap \mathcal{B}$ it is easy to see that

$$\|I_0\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}$$

for $t \geq 0$. For $0 \leq t \leq 1$, $\|I_1\|_{L^p} \leq C$ easily. For $t \geq 1$ we set

$$I_1 = \int_0^{t/2} + \int_{t/2}^t =: I_{11} + I_{12}.$$

By (1.6) we note that

$$(3.8) \quad \|u(t, \cdot)\|_{L^{1,1}} \leq C(1+t)^{\frac{1}{2}}, \quad \|u(t, \cdot) - M_0 G(t+1, \cdot)\|_{L^{1,1}} \leq C,$$

where $L^{p,m} = \{f \in L^p; \|f\|_{L^{p,m}} := \|(1 + |\cdot|)^m f\|_{L^p} < +\infty\}$ (These are shown by applying the method in [2]). Therefore, by (2.12) and (3.8)

$$\begin{aligned}
& \|I_{11}\|_{L^p} \\
& \leq C \int_0^{t/2} (t-s)^{-\frac{1}{2}(1-\frac{1}{p})-1} \left\| \int_{-\infty}^x \{\mu'(s)\phi(s, z) \right. \\
& \quad \left. - [(u - M_0 G)v_x + M_0 G(v_x - M_0 G_x)](s, z)\} dz \right\|_{L_x^1} ds \\
& \leq C t^{-\frac{1}{2}(1-\frac{1}{p})-1} \int_0^{t/2} [\|\mu'(s)\| \|G(s, \cdot - \mu(s))\|_{L^{1,1}} + \|(u - M_0 G)(s)\|_{L^{1,1}} \|v_x(s)\|_{L^\infty} \\
& \quad + \|G(s)\|_{L^{1,1}} \|(v_x - M_0 G_x)(s)\|_{L^\infty}] ds \\
& \leq C t^{-\frac{1}{2}(1-\frac{1}{p})-1} \int_0^{t/2} [(1+s)^{-\frac{3}{2}} \log(2+s) \cdot (1+s)^{\frac{1}{2}} \\
& \quad + (1+s)^{-1} + (1+s)^{\frac{1}{2}} (1+s)^{-\frac{3}{2}} \log(2+s)] ds \\
& \leq C t^{-\frac{1}{2}(1-\frac{1}{p})-1} (\log(2+t))^2.
\end{aligned}$$

Here, we have denoted $\|u(s, y) - M_0 G(s+1, y)\|_{L_y^p} = \|(u - M_0 G)(s)\|_{L^p}$ etc. for simplicity. For I_{12} , by the integral by parts,

$$\begin{aligned}
I_{12} &= \int_{t/2}^t \int G(t-s, x-y) \mu'(s) M_0 G_x(s+1, y - \mu(s)) dy ds \\
&\quad + \int_{t/2}^t \int G_x(t-s, x-y) [(u - M_0 G)v_x + M_0 G(v_x - M_0 G_x)](s, y) dy ds \\
&=: I_{12}^1 + I_{12}^2,
\end{aligned}$$

Each part is estimated as follow:

$$\begin{aligned}
\|I_{12}^1\|_{L^p} &\leq C \int_{t/2}^t (t-s)^{-\frac{1}{2}(1-\frac{1}{p})} |\mu'(s)| \|G_x(s)\|_{L^1} ds \\
&\leq C \int_{t/2}^t (t-s)^{-\frac{1}{2}(1-\frac{1}{p})} (1+s)^{-\frac{3}{2}} \log(2+s) \cdot (1+s)^{-\frac{1}{2}} ds \\
&\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1} \log(2+t), \\
\|I_{12}^2\|_{L^1} &\leq C \int_{t/2}^t \|G_x(t-s)\|_{L^1} (\|(u - M_0 G)(s)\|_{L^1} \|v_x(s)\|_{L^\infty} \\
&\quad + \|G(s)\|_{L^1} \|(v_x - M_0 G_x)(s)\|_{L^\infty}) ds \\
&\leq C \int_{t/2}^t (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{3}{2}} \log(2+s) ds \\
&\leq C(1+t)^{-1} \log(2+t)
\end{aligned}$$

and

$$\begin{aligned}
\|I_{12}^2\|_{L^\infty} &\leq C \int_{t/2}^t \|G_x(t-s)\|_{L^2} (\|(u - M_0 G)(s)\|_{L^2} \|v_x(s)\|_{L^\infty} \\
&\quad + \|G(s)\|_{L^2} \|(v_x - M_0 G_x)(s)\|_{L^\infty}) ds \\
&\leq C \int_{t/2}^t (t-s)^{-\frac{1}{4}} (1+s)^{-\frac{7}{4}} \log(2+s) ds \\
&\leq C(1+t)^{-1} \log(2+t).
\end{aligned}$$

Combining all estimates, we obtain (3.6). $\square$

Completion of the proof of Theorem 1.1. We show (1.9). By an elementary calculation

$$\int_{-\infty}^{\infty} G(t-s, x-y) G^2(s+1, y) dy = \frac{G(t - \frac{s-1}{2}, x)}{\sqrt{8\pi(s+1)}}.$$

Hence, when $\mu_{\infty} = 0$,

$$(3.9) \quad W_1(t, x; 0) = \frac{M_0^2}{2} \int_0^t \frac{G_{xx}(t - \frac{s-1}{2}, x)}{\sqrt{8\pi(s+1)}} ds.$$

Similar representation to (3.9) is found in [1]. We claim, for $t \geq t_0 > 0$,

$$(3.10) \quad \int_0^{\sqrt{(t+1)/2}} \left| \int_0^t \frac{G_{xx}(t - \frac{s-1}{2}, x)}{\sqrt{s+1}} ds \right| dx \geq c(t+1)^{-\frac{1}{2}},$$

and, when $0 \leq x \leq \sqrt{(t+1)/2}$,

$$(3.11) \quad \left| \int_0^t \frac{G_{xx}(t - \frac{s-1}{2}, x)}{\sqrt{s+1}} ds \right| \geq c(t+1)^{-1}.$$

In fact, since $G_x(t, x) = -\frac{x}{2t}G(t, x)$ and $G_{xx}(t, x) = \frac{1}{2t}(\frac{x^2}{2t} - 1)G(t, x)$,

$$-G_{xx}(t - \frac{s-1}{2}, x) \geq \frac{G(t - \frac{s-1}{2}, x)}{4(t - \frac{s-1}{2})} > 0, \quad \text{for } 0 \leq x \leq \sqrt{(t+1)/2}.$$

Hence,

$$\begin{aligned} & \text{the left-hand side in (3.10)} \\ & \geq c \int_0^{\sqrt{(t+1)/2}} \int_0^t \frac{-G_{xx}(t - \frac{s-1}{2}, x)}{\sqrt{s+1}} ds dx = c \int_0^t \frac{-G_x(t - \frac{s-1}{2}, \sqrt{\frac{t+1}{2}})}{\sqrt{s+1}} ds \\ & \geq c(t+1)^{\frac{1}{2}} \int_0^t (s+1)^{-\frac{1}{2}} (t - \frac{s-1}{2})^{-\frac{3}{2}} ds \\ & \geq c(t+1)^{-\frac{1}{2}}, \quad t \geq t_0, \end{aligned}$$

and

$$\begin{aligned} & \text{the left-hand side in (3.11)} \\ & \geq c \int_0^t \frac{G(t - \frac{s-1}{2}, x)}{\sqrt{s+1}(t - \frac{s-1}{2})} ds \geq c \int_0^t \frac{G(t - \frac{s-1}{2}, \sqrt{\frac{t+1}{2}})}{\sqrt{s+1}(t - \frac{s-1}{2})} ds \\ & \geq c \int_0^t (s+1)^{-\frac{1}{2}} (t - \frac{s-1}{2})^{-\frac{3}{2}} ds \\ & \geq c(t+1)^{-1}, \quad t \geq t_0. \end{aligned}$$

By (3.10) and (3.11), for $1 \leq p < \infty$,

$$\begin{aligned} \|W_1(t, \cdot; 0)\|_{L^p} &\geq \left(\int_0^{\sqrt{(t+1)/2}} \left| \int_0^t \frac{G_{xx}(t - \frac{s-1}{2}, x)}{\sqrt{s+1}} ds \right|^p dx \right)^{\frac{1}{p}} \\ &\geq c \left(\int_0^{\sqrt{(t+1)/2}} (t+1)^{-(p-1)} \int_0^t \frac{-G_{xx}(t - \frac{s-1}{2}, x)}{\sqrt{s+1}} ds dx \right)^{\frac{1}{p}} \\ &\geq c(t+1)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad t \geq t_0. \end{aligned}$$

When $p = \infty$, it is easy to show

$$\|W_1(t, \cdot; 0)\|_{L^\infty} \geq |W_1(t, 0; 0)| \geq c(t+1)^{-1}, \quad t \geq t_0.$$

When $\mu_\infty \neq 0$, $W_1(t, x; \mu_\infty) = W_1(t, x; 0) + (W_1(t, x; \mu_\infty) - W_1(t, x; 0))$ and $W_1(t, x; \mu_\infty) - W_1(t, x; 0)$ decays faster by (3.6). Hence the estimate from below in (1.9) holds. The estimate from above is obtained easier by (3.9), which completes the proof of Theorem 1.1.

References

- [1] M. Kato, Sharp asymptotics for a parabolic system of chemotaxis in one space dimension, Osaka Univ. Research Report in Math. 07-03.
- [2] T. Nagai, R. Sukuinn and M. Umesako, Decay properties and asymptotic profiles of bounded solutions to a parabolic system of chemotaxis in $\mathbf{R}^n$, Funkcial. Ekvac. 46(2003), 383-407.
- [3] T. Nagai and T. Yamada, Large time behavior of bounded solutions to a parabolic system of chemotaxis in the whole space, J. Math. Anal. Appl. 336(2007), 704-726.
- [4] K. Nishihara, Asymptotic profile of solutions to a parabolic system of chemotaxis in one dimensional space, preprint(2008).