

Elliptic Ruijsenaars operators and elliptic hypergeometric integrals

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ABSTRACT

We study a family of mutually commutative difference operators associated with root systems and discuss their simultaneous eigenvector in a special case. For root systems with rank n , we construct $3n$ commutative difference operators, which are a generalization of elliptic Ruijsenaars operators. Especially, for the BC_1 root system, we construct an explicit simultaneous eigenvector of these operators described in terms of elliptic hypergeometric integrals.

1 Introduction

In [10] Ruijsenaars introduced the operators acting on the space of meromorphic functions which are defined by

$$Y_n = \sum_{\substack{I \subset \{1, \dots, l\} \\ |I|=n}} \left(\prod_{\substack{j \in I \\ k \in I^c}} \frac{\sigma(x_j - x_k + \mu; \omega_1, \omega_2)}{\sigma(x_j - x_k; \omega_1, \omega_2)} \right) \prod_{j \in I} \tau_j(\omega_3), \quad (1.1)$$

where $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ are arbitrary such that $\omega_1/\omega_2 \notin \mathbb{R}$, and $\omega_3, \mu \in \mathbb{C} \setminus \{0\}$ and the action of $\tau_j(\omega)$ is defined by $(\tau_j(\omega)f)(x_1, \dots, x_j, \dots, x_n) = f(x_1, \dots, x_j - \omega, \dots, x_n)$. He showed that these operators are mutually commutative. The first result of this article is a generalization of the elliptic Ruijsenaars operators. We define

$$Y_n^{(p)} = \sum_{\substack{I \subset \{1, \dots, l\} \\ |I|=n}} \left(\prod_{\substack{j \in I \\ k \in I^c}} e^{\nu_p(x_j - x_k)} \frac{\sigma(x_j - x_k + \mu_p; \omega_q, \omega_r)}{\sigma(x_j - x_k; \omega_q, \omega_r)} \right) \prod_{j \in I} \tau_j(\omega_p), \quad (1.2)$$

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where $p, q, r \in \mathbb{Z}/3\mathbb{Z}$ are distinct, and for $p \in \mathbb{Z}/3\mathbb{Z}$, $\omega_p \in \mathbb{C} \setminus \{0\}$ are such that $\omega_p/\omega_q \notin \mathbb{R}$ if $p \neq q$. Put $\eta_{pq} = 2\zeta(\omega_p/2; \omega_p, \omega_q)$ with Weierstrass' zeta function ζ and $a_r = \eta_{pq}\omega_q - \eta_{qp}\omega_p$, then $a_r = \pm 2\pi i$. If ν_p, μ_p satisfy three equations $\nu_p\omega_q + \mu_p\eta_{qr} = \nu_q\omega_p + \mu_q\eta_{pr}$ for distinct $p, q, r \in \mathbb{Z}/3\mathbb{Z}$, then all $Y_n^{(p)}$ are shown to be mutually commutative. For instance, these equations are solved by

$$\nu_1 = \frac{a_1(\nu_3\omega_1 + \mu_3\eta_{12}) - (a_2\mu_2 + a_3\mu_3)\eta_{32}}{a_1\omega_3}, \quad (1.3)$$

$$\nu_2 = \frac{\nu_3\omega_2 + \mu_3\eta_{21} - \mu_2\eta_{31}}{\omega_3}, \quad (1.4)$$

$$\mu_1 = \frac{a_2\mu_2 + a_3\mu_3}{a_1}, \quad (1.5)$$

where ν_3, μ_2, μ_3 are regarded as free parameters. Although the discussion above is for the A -type root system, the construction can be applied to arbitrary root systems.

The second result is a construction of a simultaneous eigenvector of the elliptic Ruijsenaars operators of type BC_1 . So far, some class of eigenvectors was discussed in [11], where certain transcendental equations should be solved. We first obtain an explicit meromorphic eigenvector described in terms of the elliptic hypergeometric integral.

Acknowledgement: The author expresses his sincere gratitude to Jun'ichi Shiraishi, Yuji Yamada, Takashi Takebe and Michitomo Nishizawa who kindly allowed him to speak about his research. Also thanks are due to Masatoshi Noumi, Yasuhiko Yamada, Yasuhiro Ohta and Tetsu Masuda for fruitful discussions and helpful comments.

2 Affine Root Systems

We summarize some facts about affine root systems and affine Weyl groups [1, 2, 4, 5]. In this article, we will omit $A_{2l}^{(2)}$ -type root system because of simplicity, though it is straightforward. The notation and symbols are a little different from those in the previous papers [6, 7] in order to generalize the results.

Let Δ be the irreducible reduced finite root system of type X_l in a complex vector space V with $\dim V = l$ and the inner product $\langle \cdot, \cdot \rangle$, $I = \{1, \dots, l\}$ a set of indices, $\Pi = \{\alpha_i \mid i \in I\} \subset V$ the set of simple roots, $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subset V$ the set of simple coroots, Q and $Q^\vee$ the root and coroot lattices, P and $P^\vee$ the weight and coweight lattices, $\{\Lambda_i \mid i \in I\}$ and $\{\Lambda_i^\vee \mid i \in I\}$ the fundamental weights and fundamental coweights such that $\langle \alpha_i, \Lambda_j^\vee \rangle = \langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. Then we have

$$Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \subset P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \subset V, \quad (2.1)$$

$$Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee \subset P^\vee = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i^\vee \subset V. \quad (2.2)$$

The inner product $\langle \cdot, \cdot \rangle$ is normalized such that $\langle \alpha, \alpha \rangle = 2$ for the longer roots α . Let Δ_+ and Δ_- be the set of positive roots and negative roots respectively.

Let $\Delta_s \subset \Delta$ be the set of the shorter roots and $\Delta_l \subset \Delta$ the set of the longer roots. Let r be the ratio of the square lengths of the longer roots and the shorter roots. Fix parameters γ_α for $\alpha \in \Delta$, such that in nontwisted cases all $\gamma_\alpha = 1$, and in twisted cases, $\gamma_\alpha = r$ if $\alpha \in \Delta_l$, $\gamma_\alpha = 1$ otherwise. Let $\widehat{V} = V \oplus \mathbb{C}\delta$ with $\langle \alpha_i, \delta \rangle = \langle \delta, \delta \rangle = 0$ and its linear extension. Then the associated affine root system $\widehat{\Delta} \subset \widehat{V}$ is written as

$$\widehat{\Delta} = \{\alpha + n\gamma_\alpha\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\}. \quad (2.3)$$

Let $\widehat{\Delta}_+$ and $\widehat{\Delta}_-$ be the set of the positive affine roots and the negative affine roots respectively. We denote by $\bar{v}$ for $v \in \widehat{V}$ the natural projection on V .

For $\alpha \in \widehat{\Delta}$, let s_α be a reflection defined by

$$s_\alpha(v) := v - \langle \alpha, v \rangle \alpha^\vee, \quad v \in V, \quad (2.4)$$

where $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. The Weyl group W is generated by the fundamental reflections $\{s_i := s_{\alpha_i} \mid i \in I\}$ on V and the affine Weyl group $\widehat{W}$ is generated by $\{s_i \mid i \in \widehat{I}\}$, where $\widehat{I} = I \cup \{0\}$ and $\alpha_0 = \delta - \theta$ with θ the highest root in nontwisted cases and the highest short root in twisted cases.

The defining relations are given by $s_i^2 = id$ and the Coxeter relations:

$$(s_i s_j)^{m_{ij}} = id, \quad \text{for } i \neq j \in \widehat{I}, \quad (2.5)$$

where $m_{ij} = 2$ if α_i and α_j are disconnected in the Dynkin diagram and $m_{ij} = 3, 4, 6$ if 1, 2, 3 lines respectively connect α_i and α_j . For $\mu \in V$, we define endomorphisms τ_μ of the vector space V by

$$\tau_\mu(\lambda) := \lambda - \langle \lambda, \mu \rangle \delta. \quad (2.6)$$

Let $M := \mathbb{Z}(W \cdot \theta^\vee) \subset V$. For an arbitrary lattice L , we denote by T_L the corresponding group of translations of L . Then one sees that $\widehat{W}$ is the semidirect product $\widehat{W} = W \ltimes T_M$. Let $\widetilde{M} := \{\lambda \in V \mid \alpha \in \Delta, \langle \alpha, \lambda \rangle \in \gamma_\alpha \mathbb{Z}\}$. The extended affine Weyl group $\widetilde{W}$ is defined by the semidirect product $\widetilde{W} := W \ltimes T_{\widetilde{M}}$. Let Ω be the subgroup of $\widetilde{W}$ which stabilizes the affine Weyl chamber C . Then one sees that $\widetilde{W}$ is isomorphic to the semidirect product $\widehat{W} \rtimes \Omega$. Here are the explicit description of $\widetilde{M}$ and its canonical basis $\{\lambda_i \mid i \in I\}$:

$$\widetilde{M} = \begin{cases} P^\vee, & \text{nontwisted case,} \\ P, & \text{twisted case,} \end{cases} \quad \lambda_i = \begin{cases} \Lambda_i^\vee, & \text{nontwisted case,} \\ \Lambda_i, & \text{twisted case.} \end{cases} \quad (2.7)$$

We also use $\widetilde{M}_- := \oplus_{i \in I} \mathbb{Z}_{\leq 0} \lambda_i$.

The length $\ell(w)$ of $w \in \widetilde{W}$ is defined by the length ℓ of a reduced decomposition:

$$w = s_{i_1} \dots s_{i_\ell} \omega, \quad i_k \in \widehat{I}, \omega \in \Omega. \quad (2.8)$$

It is equivalent to the number of the negative roots made positive by $\hat{w}$:

$$\ell(\hat{w}) := |\Delta_{\hat{w}}|, \quad \Delta_{\hat{w}} := \widehat{\Delta}_+ \cap \hat{w} \widehat{\Delta}_-. \quad (2.9)$$

The set $\Delta_{\hat{w}}$ is explicitly described as $\Delta_{\hat{w}} = \{\alpha^{(1)} = \alpha_{i_1}, \alpha^{(2)} = s_{i_1}(\alpha_{i_2}), \dots, \alpha^{(\ell)} = w s_{i_\ell}(\alpha_{i_\ell})\}$. By definition, $\Delta_{\hat{w}}$ is independent of reduced expressions. One sees that $\Omega = \{\omega \in \widehat{W} \mid \ell(\omega) = 0\}$. A weight $\lambda \in \widetilde{M}$ is said to be minuscule if $\Delta_{\tau_{-\lambda}} \subset \Delta_+$.

In the following, we use the constants

$$\rho_x := \sum_{i \in I} x_{\alpha_i} \Lambda_i = \frac{1}{2} \sum_{\alpha \in \Delta_+} x_\alpha \alpha, \quad (2.10)$$

$$h_x^\vee := \langle \rho_x, \theta \rangle + x_{\alpha_0}, \quad (2.11)$$

where $x_\alpha \in \mathbb{C}$ which depends only on the length of roots,

We shall define the root algebras after Cherednik [2, 7].

Definition 2.1. Root algebra $\mathcal{R}$ is generated by independent variables $\{R_\alpha \mid \alpha \in \widehat{\Delta}\}$ and $\{\tau_\lambda \mid \lambda \in \widetilde{M}\}$ with the following defining relations:

$$\underbrace{R_{w\alpha_i} R_{ws_i\alpha_j} R_{ws_i s_j \alpha_i} \cdots}_{m_{ij} \text{ factors}} = \underbrace{R_{w\alpha_j} R_{ws_j\alpha_i} R_{ws_j s_i \alpha_j} \cdots}_{m_{ij} \text{ factors}}, \quad \text{for } w \in \widetilde{W} \quad (2.12)$$

$$\tau_\lambda R_\alpha = R_{\tau_\lambda \alpha} \tau_\lambda, \quad (2.13)$$

$$\tau_\lambda \tau_{\lambda'} = \tau_{\lambda + \lambda'}. \quad (2.14)$$

Theorem 2.2 (Cherednik). 1. There exists a unique set $\{R_w \mid w \in \widetilde{W}\} \subset \mathcal{R}$ satisfying the relations:

$$R_{v\omega} = R_v {}^v R_\omega, \quad R_{s_i} = R_{\alpha_i} \quad (i \in \widehat{I}), \quad R_w = 1, \quad (2.15)$$

where $\omega \in \Omega$, $v, w \in \widetilde{W}$ and $\ell(v\omega) = \ell(v) + \ell(\omega)$, and ${}^v(R_{\alpha_1} \cdots R_{\alpha_i}) = R_{v\alpha_1} \cdots R_{v\alpha_i}$

2. We have the R -matrix for $w \in \widetilde{W}$ and its arbitrary reduced decomposition $w = s_{i_1} \cdots s_{i_\ell} \omega$ as

$$R_w = R_{\alpha^{(1)}} \cdots R_{\alpha^{(\ell)}}, \quad (2.16)$$

$$\alpha^{(1)} = \alpha_{i_1}, \quad \alpha^{(2)} = s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \alpha^{(\ell)} = ws_{i_\ell}(\alpha_{i_\ell}) \in \Delta_w.$$

Theorem 2.3. The subalgebra $\mathcal{S} \subset \mathcal{R}$ generated by $\{Y^\lambda := R_{\tau_\lambda} \tau_\lambda \mid \lambda \in \widetilde{M}_-\}$ forms a commutative algebra and is generated by $\{Y^{-\lambda_i} \mid i \in I\}$.

3 Representation and Difference Operators

Let $\gamma_\alpha^{(1)} = \gamma_\alpha$ and $\gamma_\alpha^{(2)}, \gamma_\alpha^{(3)}$ be taken similarly as $\gamma_\alpha^{(1)}$. Accordingly let $\widetilde{M}^{(1)} = \widetilde{M}$ and $\widetilde{M}^{(2)}, \widetilde{M}^{(3)}$ be taken similarly. Here $\gamma_\alpha^{(i)}$ and $\gamma_\alpha^{(j)}$ may differ if $i \neq j$. Let $\mathcal{M}$ be the set of meromorphic functions on V . To define the action of $\widetilde{W}$ on $\mathcal{M}$, it is sufficient to specify the action of s_i for $i \in I$ and τ_λ for $\lambda \in \widetilde{M}^{(1)}$. For $f \in \mathcal{M}$, we define

$$s_i(f)(v) = f(s_i(v)), \quad \tau_\lambda(f)(v) = \tau_\lambda^{(1)}(f)(v) = f(v - \omega_1 \lambda). \quad (3.1)$$

Fix $\xi^{(1)}, \zeta^{(1)} \in V$ and let $\mu_\alpha^{(1)}, \nu_\alpha^{(1)}$ be constants depending only on the length of roots. Then one can check that (3.1) and

$$(R_\alpha f)(v) := H_\alpha(\mu_\alpha^{(1)}, \nu_\alpha^{(1)})f(v) - H_\alpha(\langle \xi^{(1)}, \alpha^\vee \rangle, \langle \zeta^{(1)}, \alpha^\vee \rangle)f(s_\alpha v), \quad (3.2)$$

satisfy the defining relations of the root algebra [8], where $H_\alpha(\eta, \kappa)$ is the meromorphic function defined by

$$H_\alpha(\eta, \kappa)(v) := e^{\kappa\alpha(v)} \frac{\sigma(\mu_\alpha^{(1)}; \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3) \sigma(\alpha(v) + \eta; \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)}{\sigma(\eta; \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3) \sigma(\alpha(v); \gamma_\alpha^{(2)}\omega_2, \gamma_\alpha^{(3)}\omega_3)}, \quad (3.3)$$

and a root α acts on V as an affine linear functional $\alpha(v) = \langle \alpha, v \rangle + n\omega_1$ for $v \in V$ and $\alpha = \alpha' + n\delta$, $\alpha' \in \Delta$. Then we have the following theorems [6, 7]

Theorem 3.1. *Let $\mathcal{V} := \mathcal{M}^W$, the W -invariant subspace of $\mathcal{M}$ and let $\xi^{(1)} = -\rho_{\mu^{(1)}}$, $\zeta^{(1)} = -\rho_{\nu^{(1)}}$. Then $Y_\lambda^{(1)} := Y^\lambda \in \text{End}_{\mathbb{C}} \mathcal{V}$.*

Theorem 3.2. *Let $(-\lambda) \in \widetilde{M}^{(1)}$ be minuscule. Then we have*

$$Y^\lambda|_{\mathcal{V}} = \frac{1}{|W_\lambda|} \sum_{w \in W} w \left(\prod_{\substack{\alpha \in \Delta_+ \\ \langle \lambda, \alpha \rangle = -\gamma_\alpha}} H_\alpha(\mu_\alpha^{(1)}, \nu_\alpha^{(1)}) \tau_\lambda^{(1)} \right) \Big|_{\mathcal{V}}, \quad (3.4)$$

where W_λ is the stabilizer of λ in W .

4 Commutativity in Case of Minusculer Weights

Let $Y_\lambda^{(2)}$ be the operator obtained by changing the role of the indices 1 and 2 in the construction of $Y_\lambda^{(1)}$, and $Y_\lambda^{(3)}$ be obtained in the same manner. Then one sees that $Y_\lambda^{(j)}$ for $\lambda \in \widetilde{M}^{(j)}$ are commutative [7]. In the following, we assume that $\mu_\alpha^{(j)}\eta_{kl} + \nu_\alpha^{(j)}\omega_k = \mu_\alpha^{(k)}\eta_{jl} + \nu_\alpha^{(k)}\omega_j$ where $j \neq k \neq l \neq j$. We demonstrate that $Y_\lambda^{(j)}$ and $Y_\nu^{(k)}$ are also commutative if $-\lambda$ and $-\nu$ are minuscule.

Lemma 4.1 (Proposition 7.7 in [7]). *Let x_α be constants dependent only on the length of roots. Then we have*

$$- \sum_{\alpha \in \Delta_{r_\lambda}} x_\alpha \bar{\alpha} = h_x^\vee \lambda. \quad (4.1)$$

Theorem 4.2. Let $j, k \in \mathbb{Z}/3\mathbb{Z}$ and $-\lambda \in \widetilde{M}^{(j)}, -\nu \in \widetilde{M}^{(k)}$ be minuscule and Then $Y_\lambda^{(j)}$ and $Y_\nu^{(k)}$ are commutative.

Proof. If $j = k$, then the statement follows from Theorem 6.3 in [7]. So it is sufficient to show the case $j \neq k$ and the commutativity of each term. Let

$$F_{j,\lambda} := \prod_{\substack{\alpha \in \Delta_+ \\ \langle \lambda, \alpha \rangle = -\gamma_\alpha}} e^{\nu_\alpha^{(j)} \langle \alpha, v \rangle} \frac{\sigma(\langle \alpha, v \rangle + \mu_\alpha^{(j)}; \gamma_\alpha^{(k)} \omega_k, \gamma_\alpha^{(l)} \omega_l)}{\sigma(\langle \alpha, v \rangle; \gamma_\alpha^{(k)} \omega_k, \gamma_\alpha^{(l)} \omega_l)}, \quad (4.2)$$

where $j \neq l \neq k$. Since for $u, w \in W$,

$$\tau_{uv}^{(k)} \left(e^{\nu_\alpha^{(j)} \langle w\alpha, v \rangle} \frac{\sigma(\langle w\alpha, v \rangle + \mu_\alpha^{(j)}; \gamma_\alpha^{(k)} \omega_k, \gamma_\alpha^{(l)} \omega_l)}{\sigma(\langle w\alpha, v \rangle; \gamma_\alpha^{(k)} \omega_k, \gamma_\alpha^{(l)} \omega_l)} \right) \quad (4.3)$$

$$\begin{aligned} &= e^{\nu_\alpha^{(j)} \langle w\alpha, v - \omega_k u \nu \rangle} \frac{\sigma(\langle w\alpha, v - \omega_k u \nu \rangle + \mu_\alpha^{(j)}; \gamma_\alpha^{(k)} \omega_k, \gamma_\alpha^{(l)} \omega_l)}{\sigma(\langle w\alpha, v - \omega_k u \nu \rangle; \gamma_\alpha^{(k)} \omega_k, \gamma_\alpha^{(l)} \omega_l)} \\ &= e^{-(\mu_\alpha^{(j)} \eta_{kl} + \nu_\alpha^{(j)} \omega_k) \langle w\alpha, u \nu \rangle} e^{\nu_\alpha^{(j)} \langle w\alpha, v \rangle} \frac{\sigma(\langle w\alpha, v \rangle + \mu_\alpha^{(j)}; \gamma_\alpha^{(k)} \omega_k, \gamma_\alpha^{(l)} \omega_l)}{\sigma(\langle w\alpha, v \rangle; \gamma_\alpha^{(k)} \omega_k, \gamma_\alpha^{(l)} \omega_l)}, \end{aligned} \quad (4.4)$$

we have

$$\tau_{uv}^{(k)}(F_{j,w\lambda}) = e^{A \langle w\lambda, u \nu \rangle} (F_{j,w\lambda}), \quad (4.5)$$

where by Lemma 4.1

$$- \sum_{\substack{\alpha \in \Delta_+ \\ \langle \lambda, \alpha \rangle = -\gamma_\alpha}} (\mu_\alpha^{(j)} \eta_{kl} + \nu_\alpha^{(j)} \omega_k) \alpha = A\lambda. \quad (4.6)$$

Thus

$$\begin{aligned} F_{k,u\nu} \tau_{uv}^{(k)} F_{j,w\lambda} \tau_{w\lambda}^{(j)} &= e^{A \langle w\lambda, u \nu \rangle} F_{k,u\nu} F_{j,w\lambda} \tau_{uv}^{(k)} \tau_{w\lambda}^{(j)} \\ &= F_{j,w\lambda} \tau_{w\lambda}^{(j)} F_{k,u\nu} \tau_{uv}^{(k)}. \end{aligned} \quad (4.7)$$

□

For minuscule weights $-\lambda$, the periodicity of the coefficients is easily obtained since the explicit forms of the operators Y^λ are calculated. However Y^λ for general λ is complicated and the proof of the commutativity requires a further investigation and is omitted here.

5 BC_1 -type Operators and Eigenvector

Generally, it is very difficult to construct explicit eigenvectors of the elliptic Ruijsenaars operators. However we can construct a simultaneous eigenvector in the BC_1 root system. In this root system, there are three mutually commutative operators.

First we give the explicit forms of these operators. Let $\omega_1, \omega_2, \mu_0, \dots, \mu_6 \in \mathbb{C}$ such that $\Im \omega_1, \omega_2 > 0$ and $2\mu_0 + \mu_1 + \dots + \mu_6 + \omega_1 + \omega_2 = 0$. Then the commutative operators are given by

$$Y^{(1)} = e^{2\pi i x} \frac{\vartheta(x - \mu_0 + \omega_1; \omega_2) \prod_{j=0}^6 \vartheta(x - \mu_j; \omega_2)}{\vartheta(2x; \omega_2) \vartheta(2x + \omega_1; \omega_2)} (\tau(\omega_1) - 1), \quad (5.1)$$

$+ (x \leftrightarrow -x)$

$$Y^{(2)} = e^{2\pi i x} \frac{\vartheta(x - \mu_0 + \omega_2; \omega_1) \prod_{j=0}^6 \vartheta(x - \mu_j; \omega_1)}{\vartheta(2x; \omega_1) \vartheta(2x + \omega_2; \omega_1)} (\tau(\omega_2) - 1), \quad (5.2)$$

$+ (x \leftrightarrow -x)$

$$Y^{(3)} = \tau(1) + \tau(-1) - 2, \quad (5.3)$$

where ϑ denotes the Jacobi odd theta function. For these commutative difference operators we can construct an explicit simultaneous eigenvector:

Theorem 5.1. *Let*

$$\begin{aligned} \phi(c) = & \frac{\prod_{i=1}^7 \Gamma(pqc_0/c_i; p, q)}{\Gamma(pqc_0; p, q) \prod_{1 \leq i < j \leq 7} \Gamma(pqc_0/c_i c_j; p, q)} \\ & \times \int_C \frac{\Gamma(pq/(pqc_0)^{1/2} z^{\pm 1}; p, q) \prod_{j=1}^7 \Gamma((pqc_0)^{1/2}/c_j z^{\pm 1}; p, q)}{\Gamma(z^2, z^{-2}; p, q)} \frac{dz}{z}, \end{aligned} \quad (5.4)$$

with

$$\begin{aligned} c_0 &= e^{2\pi i(\mu_0 - \mu_1 - \omega_1 - \omega_2)}, & c_1 &= e^{2\pi i(\mu_0 + \mu_2)}, & c_2 &= e^{2\pi i(\mu_0 + \mu_3)}, \\ c_3 &= e^{2\pi i(\mu_0 + \mu_4)}, & c_4 &= e^{2\pi i(\mu_0 + \mu_5)}, & c_5 &= e^{2\pi i(\mu_0 + \mu_6)}, \\ c_6 &= e^{2\pi i(-x - \mu_1)}, & c_7 &= e^{2\pi i(x - \mu_1)}, \\ p &= e^{2\pi i\omega_1}, & q &= e^{2\pi i\omega_2}, \end{aligned}$$

and C a closed circle taken appropriately, and $\Gamma(z; p, q)$ elliptic Gamma function defined by

$$\Gamma(z; p, q) = \frac{(pqz^{-1}; p, q)_\infty}{(z; p, q)_\infty}.$$

Then $\phi(c)$ is a simultaneous eigenvector of $Y^{(j)}$ with the eigenvalues

$$\begin{aligned} E^{(1)} &= e^{-2\pi i\mu_0} \prod_{i=1}^6 \vartheta(\mu_0 + \mu_i; \omega_2), & E^{(3)} &= 0, \\ E^{(2)} &= e^{-2\pi i\mu_0} \prod_{i=1}^6 \vartheta(\mu_0 + \mu_i; \omega_1). \end{aligned}$$

sketch of proof. One can show the following contiguous relations directly or by use of the result in [9]:

$$\begin{aligned} \frac{\langle c_i \rangle \prod_{k \neq i, j} \langle pqc_0 / c_j c_k \rangle}{\langle pqc_0 / c_j, p^2 q^2 c_0 / c_j \rangle} \phi^+(c_j^-) - \frac{\langle c_j \rangle \prod_{k \neq i, j} \langle pqc_0 / c_i c_k \rangle}{\langle pqc_0 / c_i, p^2 q^2 c_0 / c_i \rangle} \phi^+(c_i^-) \\ = \frac{\langle c_i / c_j \rangle \prod_{k \neq i, j} \langle pqc_0 / c_k \rangle}{\langle pqc_0, p^2 q^2 c_0 \rangle} \phi, \end{aligned} \quad (5.5)$$

$$\begin{aligned} \phi - \phi(c_j^-, c_i^+) \\ = \frac{\langle pqc_i / c_j \rangle \langle pqc_0 / c_i c_j \rangle \langle pqc_0 \rangle \langle p^2 q^2 c_0 \rangle}{\langle pqc_0 / c_j \rangle \langle p^2 q^2 c_0 / c_j \rangle \langle c_0 / c_i \rangle \langle pqc_0 / c_i \rangle} \prod_{k \neq i, j} \frac{\langle c_k \rangle}{\langle pqc_0 / c_k \rangle} \phi^+(c_j^-), \end{aligned} \quad (5.6)$$

where $\langle \cdot \rangle$ is the multiplicative form of $\vartheta(\cdot; p)$, $\phi^+ = \phi(q^2 c_0, qc_1, \dots, qc_7)$ and $\phi(c_j^\pm) = \phi(c_0, c_1, \dots, q^{\pm 1} c_j, \dots, c_7)$. Combining these relations, we see that $\phi(c)$ is an eigenvector of $Y^{(1)}$ and thus is an eigenvector of $Y^{(2)}$ due to the symmetry of p and q , and that by the definition of c_6 and c_7 , $\phi(c)$ is an eigenvector of $Y^{(3)}$. $\square$

6 Special Cases

In this section, we clarify the relation between $\phi(c)$ and the elliptic hypergeometric series ${}_{10}E_9$ introduced by Frenkel and Turaev [3].

Theorem 6.1. *If $c_i = p^{-M}q^{-N}$, then*

$$\phi(c) = \tilde{\phi}(c_0, \dots, q^{-N}, \dots, c_7; p) \times \tilde{\phi}(c_0, \dots, p^{-M}, \dots, c_7; q), \quad (6.1)$$

where

$$\tilde{\phi}(c_0, \dots, c_7; r) = \sum_{k=0}^{\infty} \frac{\langle (pq)^{2k} c_0; r \rangle}{\langle c_0; r \rangle} \frac{\langle c_0; r \rangle_k}{\langle pq; r \rangle_k} \prod_{i=1}^7 \frac{\langle c_i; r \rangle_k}{\langle pq c_0 / c_i; r \rangle_k}, \quad (6.2)$$

and $\langle u; r \rangle_k = \langle u; r \rangle \cdots \langle u(pq)^{k-1}; r \rangle$.

sketch of proof. We see that both the right hand side of (6.1) and the elliptic hypergeometric integral (5.4) satisfy the contiguous relation (5.6), with the same initial condition. $\square$

Now we state the relation between $\phi(c)$ and ${}_{10}E_9$ which is defined by

$${}_{10}E_9(b_0; b_1, \dots, b_7; r) = \sum_{k=0}^{\infty} \frac{\langle q^{2k} b_0; r \rangle}{\langle b_0; r \rangle} \frac{\langle b_0; r \rangle_k}{\langle r; r \rangle_k} \prod_{i=1}^7 \frac{\langle b_i; r \rangle_k}{\langle r b_0 / b_i; r \rangle_k},$$

where one of b_i should be r^{-N} and $\langle u; r \rangle_k = \langle u; r \rangle \cdots \langle ur^{k-1}; r \rangle$. The following is shown by a direct calculation.

Theorem 6.2.

$$\tilde{\phi}(c_0, \dots, c_7; p) = {}_{10}E_9(pc_0; c_1, \dots, pc_k, \dots, c_7; p),$$

which is independent of the choice of k .

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