

LOCAL SUBGROUPS AND GROUP ALGEBRAS OF FINITE p -SOLVABLE GROUPS

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1. INTRODUCTION

Let k be an algebraically closed field of prime characteristic p , and let G and H be finite groups with Sylow p -subgroups P and Q , respectively. In representation theory of finite groups it seems important to consider a problem that if the two group algebras kG and kH are isomorphic as k -algebras, then which kind of properties of G can be heritable to H ?

In this talk we consider this problem for a property that $N_G(P)/P$ is abelian, where $N_G(P)$ is the normalizer of P in G . Namely, we want to know whether the property $N_G(P)/P$ is abelian implies that $N_H(Q)/Q$ is abelian under the case that $kG \cong kH$ as k -algebras. Here, actually, we consider the above problem for p -nilpotent groups and groups of p -length 1. It seems that this problem is difficult even if groups are p -nilpotent. For a p -nilpotent group G , we give some necessary conditions for $N_G(P)/P$ to be abelian, but they cannot be sufficient conditions since there exist trivial counter examples. For a group G of p -length 1, we give some necessary and sufficient conditions for $N_G(P)/P$ to be abelian, but they contain some group theoretic condition. It seems that the problem for groups of p -length 1 can be reduced to one for p -nilpotent groups.

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2. PRELIMINARY

Let H be a finite group, and let K be a finite group acting on H . Then $\text{Irr}(H)$ denotes the set of all irreducible ordinary characters of H , $\text{LIrr}(H)$ denotes the set of all linear ordinary characters of H , and $\text{Irr}_K(H)$ and $\text{LIrr}_K(H)$ denote the set of all K -invariant irreducible characters and the set of K -invariant linear characters of H , respectively. We fix a prime p and an algebraically closed field k of characteristic p , and $\text{IBr}(H)$ denotes the set of all irreducible p -Brauer characters of H . For a k -algebra A , $\text{IRR}(A)$ denotes the set of all non-isomorphic irreducible (simple) A -modules, and $\text{IRR}^0(A)$ denotes the set of all

non-isomorphic irreducible A -modules whose k -dimensions are not divisible by p . For the group algebra kG of a p -solvable group G and $S \in \text{IRR}(kG)$, it follows from [2, Theorem 2.1] that S is in $\text{IRR}^0(kG)$ if and only if the vertex of S is a Sylow p -subgroup of G . We write $[G, G]$ for the commutator subgroup of G and $|\text{IRR}(A)|$ for the number of elements of $\text{IRR}(A)$ for a k -algebra A . For other notation and terminology see the books of Isaacs [3] and Nagao and Tsushima [6]. Throughout this paper groups mean always finite groups.

First we introduce some results related to our problem.

Proposition 2.1. *Let G and H be finite groups, and let P and Q be Sylow p -subgroups of G and H , respectively. Assume that $kG \cong kH$ as k -algebras. Then*

- (1) *if G is p -nilpotent, then so is H ,*
- (2) *[Okuyama-Michler] if G is p -closed, then so is H ,*
- (3) *[Morita] if $G/O_{p',p}(G)$ is abelian, then so is $H/O_{p',p}(H)$,*
- (4) *[Navarro] if G is q -nilpotent, then so is H , for $p \neq q$,*
- (5) *if G is of p -length 1, then so is H .*

Proof. (1) Well known.

(2) Okuyama [9, Theorem 2] for $p = 2$, and Michler [4, Theorem 5.5] for $p \neq 2$. It should be noted that in his proof the classification of finite simple groups is used in the proof of Michler [4].

(3) Morita [5, Theorem 6].

(4) Navarro [7, Theorem].

(5) is proved essentially by almost the same argument in [9] and (2). It seems that the proof is unpublished, but we omit it here since we do not need this result for our argument. $\square$

Let A be a k -algebra. We say A is *primary* if $A/J(A)$ is a simple ring, and A is *quasi-primary* if $A/J(A)$ is a direct sum of isomorphic simple rings.

Theorem 2.2. [5, Theorem 6, 7] *A finite group G is p -nilpotent iff every block of the groups algebra kG is primary, and $G/O_{p',p}(G)$ is abelian iff every block of the groups algebra kG is quasi-primary.*

A block B of kG is quasi-primary if and only if all irreducible B -modules have the same dimensions.

We prepare one more easy group theoretic lemma.

Lemma 2.3. *Assume that G is a finite group of p -length 1 with Sylow p -subgroup P . Then*

- (1) $G = N_G(P)O_{p'}(G)$,
- (2) *if $N_G(P)/P$ is abelian, then so is $G/O_{p',p}(G)$.*

Proof. (1) By Frattini argument, we have $G = N_G(P)O_{p',p}(G)$. Now the result holds clearly.

(2) By (1), $G/O_{p',p}(G) \cong N_G(P)/(N_G(P) \cap O_{p',p}(G))$. Since $N_G(P) \cap O_{p',p}(G)$ contains P , there is an epimorphism from $N_G(P)/P$ to $G/O_{p',p}(G)$. $\square$

3. p -NILPOTENT CASE

Now we consider the condition that $N_G(P)/P$ is abelian for a finite group G with a Sylow p -subgroup P . Note that $N_G(P)/P \cong C_{O_{p'}(G)}(P)$ for a p -nilpotent group G with Sylow p -subgroup P . In this section, we use character theoretic descriptions.

Theorem 3.1. *Let H be a finite p' -group, and P a finite p -group acting on H . Assume that $C_H(P)$ is abelian, $\chi \in \text{Irr}_P(H)$, and $\phi \in \text{LIrr}_P(H)$ which is non-trivial. Then $\chi \neq \chi\phi$.*

Proof. Put $M = C_H(P)$. Then there exists the Glauberman correspondence $\pi : \text{Irr}_P(H) \rightarrow \text{Irr}(M)$ (See [3, §13]). By [3, Theorem 13.1(c)], $\pi(\chi\phi) = \pi(\chi)\phi_M$. Since M is abelian, $\pi(\chi)$ is linear. So if ϕ_M is non-trivial, then $\pi(\chi) \neq \pi(\chi\phi)$ and thus $\chi \neq \chi\phi$.

By [1, Exercise 8.8], $H = M[H, P]$. Since ϕ is P -invariant and linear, $[H, P]$ is contained in the kernel of ϕ . So if ϕ_M is trivial, then ϕ must be trivial. Now the result is proved. $\square$

Corollary 3.2. *Let G be a p -nilpotent group with a Sylow p -subgroup P . If $N_G(P)/P$ is abelian, then the number of linear characters of G divides the number of irreducible characters of G of degree d for any positive integer d with $p \nmid d$.*

Proof. Put $H = O_{p'}(G)$. Then $N_G(P)/P \cong C_H(P)$. Every P -invariant character of H is extendible to G and the number of its extensions is $|P : [P, P]|$. So $|\text{LIrr}(G)| = |\text{LIrr}_P(H)||P : [P, P]|$. Let $\chi \in \text{Irr}_P(H)$. Then, by Theorem 3.1, there are $|\text{LIrr}_P(H)|$ distinct characters of the form $\chi\phi$, $\phi \in \text{LIrr}_P(H)$, and each of them has $|P : [P, P]|$ extensions. Thus the assertion holds. $\square$

The converse of Corollary 3.2 is true for groups of small order, for example, for 3-nilpotent groups of order $2^n \cdot 3$, $n \leq 7$. But there exists a trivial counter example of it, consider a simple group of p' -order with the trivial action of an arbitrary p -group.

4. p -LENGTH 1 CASE

In this section, we use module theoretic descriptions.

Theorem 4.1. *Let G be a finite group of p -length 1 with a Sylow p -subgroup P . The following are equivalent.*

- (1) $N_G(P)/P$ is abelian.
- (2) $N_G(P) \cap O_{p'}(G)$ is abelian, every block of kG is quasi-primary, and the restriction S to $O_{p'}(G)$ is irreducible for every irreducible kG -module S with $p \nmid \dim_k S$.

Proof. Put $N = N_G(P)$, $E = O_{p'}(G)$, and $M = N \cap E$.

Assume (2). We can define the restriction map $R : \text{IRR}^0(kG) \rightarrow \text{IRR}_P(kE)$. First we shall show that R is surjective. Let $X \in \text{IRR}_P(kE)$. Then X can be extended to PE . Let $S \in \text{IRR}(kG)$ such that S_E has X as a direct summand. Since G is p -solvable, by [3, Corollary 11.29] and Fong-Swan's theorem, we have $p \nmid \dim_k S$. Thus $S_E = X$, and R is surjective. Also R is a $|G : PE[G, G]|$ to 1 map.

Let $\pi : \text{IRR}_P(kE) \rightarrow \text{IRR}(kM)$ be the Glauberman correspondence. Let $X \in \text{IRR}_P(kE)$. By Lemma 2.3(1) and [8, Theorem 4.9 (2)], X is extendible to G if and only if $\pi(X)$ is extendible to N . Since every $X \in \text{IRR}_P(kE)$ is extendible to G , so is every $Y \in \text{IRR}(kM)$ to N , and the number of extensions of Y to N is $|N : PM[N, N]|$. But $|G : PE[G, G]| = |N : PM[N, N]|$ since $G/E \cong N/M$. By [8, Theorem 4.1], $|\text{IRR}^0(kG)| = |\text{IRR}(kN)|$. This yields that every irreducible kN -module restricts irreducibly to M . Since M is abelian, every irreducible kN -module is of dimension one, and thus N/P is abelian.

Assume (1). By Lemma 2.3(2), G/PE and M are both abelian. Let $X \in \text{IRR}_P(kE)$. Since N/P is abelian, $\pi(X)$ is extendible to N , and so is X to G . Similar argument as the above yields (2). $\square$

Corollary 4.2. *Let G be a finite group of p -length 1 with a Sylow p -subgroup P . Assume $N_G(P)/P$ is abelian. Then the number of irreducible kG -modules of k -dimension one divides the number of irreducible kG -modules of k -dimension d for any positive integer d with $p \nmid d$.*

Proof. Put $E = O_{p'}(G)$. Let S be an irreducible kG -module with $p \nmid \dim_k S$. Then S_E is irreducible by Theorem 4.1. So we can define the restriction map $R : \text{IRR}^0(kG) \rightarrow \text{Irr}_P(E)$. As in the proof of Theorem 4.1, R is surjective and for any element $\chi \in \text{Irr}_P(E)$ there are exactly $|G : PE|$ distinct elements in $\text{IRR}^0(kG)$ which are sent to χ through R , and clearly R preserves the degrees. Now Corollary 3.2 yields the result. $\square$

Theorem 4.3. *Let G be a finite group of p -length 1 with a Sylow p -subgroup P . Then the following are equivalent.*

- (1) $N_G(P)/P$ is abelian.

- (2) $N_G(P) \cap O_{p'}(G)$ is abelian, every block of kG is quasi-primary, and all full defect blocks of kG have the same numbers of irreducible modules.

Proof. Put $N = N_G(P)$ and $M = N \cap O_{p'}(G)$.

Let B be a block of kG of full defect, and let b be the block of kN which is the Brauer correspondent of B . By [5], all irreducible kG -modules in B have the same degrees, and by [8, Theorem 4.9], we have $|\text{IRR}(B)| = |\text{IRR}(b)|$.

Assume (1). Let β be a block of kM . Since M is central in N , only one block b of kN covers β . By the assumption that $N_G(P)/P$ is abelian, we have $|\text{IRR}(b)| = |N : PM|$. Thus (2) holds.

Assume (2). Let b_0 is the principal block of kN . Since N/PM is abelian, $|\text{IRR}(b_0)| = |N : PM|$. Thus $|\text{IRR}(b)| = |N : PM|$ for any kN -block b . We know that N -conjugacy classes of $\text{Irr}(M)$ correspond to blocks kN . Let $\xi \in \text{Irr}(M)$, let b be a block of kN which covers blocks $\{\xi\}$ of kM , and let T be the inertial group of ξ in N . If $T \subsetneq N$ then $|\text{IRR}(b)| \leq |T : PM| \subsetneq |\text{IRR}(b_0)|$. So ξ is N -invariant. Since $|\text{IRR}(b)| = |N : PM|$, ξ must be extendible to N and any irreducible Brauer character in b is a extension of ξ . Since M is abelian, ξ is of degree 1, and so is any irreducible Brauer character in b . Now the proof is complete. $\square$

In Theorem 4.3(2), the conditions except $N_G(P) \cap O_{p'}(G)$ being abelian are characterized by the structure of kG as a k -algebra. So it seems for us that the problem for groups of p -length 1 can be reduced to one for p -nilpotent groups.

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