

ON MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. The purpose of this paper is to derive some properties of certain meromorphically multivalent functions in annulus.

1. Introduction

Let Σ_p be the class of functions of the form

$$(1.1) \quad f(z) = 1/z^p + a_0/z^{p-1} + \dots + a_{k+p-1}z^k + \dots,$$

which are analytic in the annulus $D = \{Z : |z| < 1\}$, where $p \in \mathbb{N} = \{1, 2, 3, \dots\}$. For $f(z) \in \Sigma_p$, we define the operator $D^{n+p-1}f(z)$ by

$$(1.2) \quad \begin{aligned} D^{n+p-1}f(z) &= (z^{n+2p-1}f(z)/(n+p-1)!)^{(n+p-1)}/z^p \\ &= 1/z^p + (n+p)a_0/z^{p-1} + (n+p)(n+p+1)a_1/(2!z^{p-2}) + \dots \\ &\quad + (n+p)(n+p+1)\dots(n+k+2p-1)a_{k+p-1}z^k/(k+p)! + \dots, \end{aligned}$$

where n is an integer and $n > -p$.

Recently, Cho and Nunokawa [1] proved that

$$\operatorname{Re}\{z^{p+1}(D^{n+p}f(z))'\} < -\alpha \quad (0 \leq \alpha < p; |z| < 1)$$

$$\text{implies } \operatorname{Re}\{z^{p+1}(D^{n+p-1}f(z))'\} < -\beta \quad (|z| < 1)$$

where

$$\beta = (p+2\alpha(n+p))/(1+2(n+p)).$$

In the present paper, we show another properties of functions $f(z) \in \Sigma_p$ concerning with the operator $D^{n+p-1} f(z)$.

2. Main results

We need the following lemma due to Jack [2] (or, due to Miller and Mocanu[3]).

Lemma. Let $w(z)$ be non-constant analytic in $U = \{ Z: |z| < 1 \}$ with $w(0)=1$. If $|w(z)|$ attains its maximum value at a point z_0 on the circle $|z|=r<1$, then we have

$$z_0 w'(z_0) = kw(z_0)$$

where k is real and $k \geq 1$.

Theorem 1. If $f(z) \in \Sigma_p$ satisfies

$$(2.1) \quad \operatorname{Re}\{ z^{p+1} (D^{n+p} f(z))' \} > -\alpha \quad (z \in U)$$

for some α ($\alpha > p$), then

$$(2.2) \quad \operatorname{Re}\{ z^{p+1} (D^{n+p-1} f(z))' \} > -\beta \quad (z \in U),$$

where

$$\beta = (p + 2\alpha(n+p))/(1 + 2(n+p)).$$

Proof. Define the function $w(z)$ by

$$(2.3) \quad z^{p+1} (D^{n+p-1} f(z))' = ((p-2\beta)w(z) - p)/(1+w(z)),$$

$w(z) \neq -1$, with

$$\beta = (p + 2\alpha(n+p))/(1 + 2(n+p)).$$

Then $w(z)$ is analytic in U and $w(0)=0$. Note that

$$(2.4) \quad z(D^{n+p-1} f(z))' = (n+p)D^{n+p} f(z) - (n+2p)D^{n+p-1} f(z).$$

It follows from (2.3) that

$$(2.5) \quad \begin{aligned} (n+p) z^p D^{n+p} f(z) - (n+2p) z^p D^{n+p-1} f(z) \\ = ((p-2\beta)w(z) - p)/(1+w(z)). \end{aligned}$$

Taking the differentiations in both sides of (2.5), we have

$$(2.6) \quad z^{p+1} (D^{n+p} f(z))' = ((p-2\beta)w(z)-p)/(1+w(z)) \\ + 2(p-\beta)zw'(z)/((n+p)(1+w(z))^2).$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| \quad (w(z_0) \neq -1),$$

then, by Lemma, we have

$$z_0 w'(z_0) = kw(z_0) \quad (k \geq 1).$$

Therefore, letting $w(z_0) = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$), we see that

$$(2.7) \quad \operatorname{Re}\{z_0^{p+1} (D^{n+p} f(z_0))'\} + \alpha \\ = \alpha + \operatorname{Re}\{((p-2\beta)e^{i\theta}-p)/(1+e^{i\theta})\} + 2(p-\beta)k/(n+p) \operatorname{Re}\{e^{i\theta}(1+e^{i\theta})^{-2}\} \\ = \alpha - \beta + (p-\beta)k/((n+p)(1+\cos\theta)) \\ \leq \alpha - \beta + (p-\beta)/2(n+p) = 0$$

for $\alpha > p$ and $\beta = (p+2\alpha(n+p))/(1+2(n+p))$.

This contradicts our condition (2.1). Therefore, $|w(z)| < 1$ for all $z \in U$, or

$$\operatorname{Re}\{z^{p+1} (D^{n+p} f(z))'\} > -\beta \quad (z \in U).$$

Next, we prove

Theorem 2. Let

$$(2.8) \quad F_c(z) = cz^{-c-p} \int_0^z t^{c+p-1} f(t) dt \quad (c > 0)$$

for $f(z) \in \Sigma_p$. If $f(z)$ satisfies

$$(2.9) \quad \operatorname{Re}\{z^{p+1} (D^{n+p-1} f(z))'\} > -\alpha \quad (z \in U)$$

for some α ($\alpha > p$), then

$$(2.10) \quad \operatorname{Re}\{z^{p+1} (D^{n+p-1} F_c(z))'\} > -\beta, \quad (z \in U)$$

where $\beta = (p + 2\alpha c)/(1+2c)$.

Proof. We define the function $w(z)$ by

$$(2.11) \quad z^{p+1} (D^{n+p-1} F_c(z))' = ((p-2\beta)w(z)-p)/(1+w(z)) \quad (w(z) \neq -1).$$

Then $w(z)$ is analytic in U and $w(0)=0$. Noting that

$$(2.12) \quad z(D^{n+p-1} F_c(z))' = cD^{n+p-1} f(z) - (c+p) D^{n+p-1} F_c(z),$$

therefore we have

$$(2.13) \quad \begin{aligned} & z^{p+1} (D^{n+p-1} f(z))' \\ &= ((p-2\beta)w(z)-p)/(1+w(z)) + 2(p-\beta) zw'(z)/c(1+w(z))^2. \end{aligned}$$

Therefore, if we assume that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq -1)$$

then Lemma gives us that

$$(2.14) \quad \begin{aligned} \operatorname{Re}\{z_0^{p+1} (D^{n+p-1} f(z_0))'\} + \alpha \\ \leq \alpha - \beta + (p-\beta)/2c \\ = 0 \end{aligned}$$

which contradicts our condition (2.9). This completes the proof of Theorem 2.

References

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