

Numerical range and a conjugation on a Banach space

by

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Abstract

We introduce a conjugation on a Banach space $\mathcal{X}$ and show properties of a conjugation. After that we show properties of numerical ranges of operators concerning with a conjugation C . Next we introduce (m, C) -symmetric and (m, C) -isometric operators on a Banach space and show spectral properties of such operators.

1 Conjugation on a Banach space

First we explain a conjugation on a complex Hilbert space.

Definition 1.1 *Let $\mathcal{H}$ be a complex Hilbert space. An operator C on $\mathcal{H}$ is antilinear if it holds that, for all $x, y \in \mathcal{H}$ and $a, b \in \mathbb{C}$,*

$$C(ax + by) = \bar{a}Cx + \bar{b}Cy.$$

Antilinear operator C is said to be a conjugation if it holds that, for all $x, y \in \mathcal{H}$ and $a, b \in \mathbb{C}$,

$$C^2 = I \text{ and } \langle Cx, Cy \rangle = \langle y, x \rangle,$$

where I is the identity operator on $\mathcal{H}$.

If C is a conjugation, then $\|Cx\| = \|x\|$ for all $x \in \mathcal{H}$. For a bounded linear operator T on a complex Hilbert space $\mathcal{H}$, let $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_s(T)$, $\sigma_e(T)$ and $\sigma_w(T)$ denote the spectrum, the point spectrum, the approximate spectrum, the surjective spectrum, the essential spectrum and the Weyl spectrum of T , respectively. Then the following result is important.

Theorem 1.1 (S. Jung, E. Ko and J. E. Lee, [3]) *Let C be conjugation on $\mathcal{H}$. Then it holds the following statement hold:*

$$\begin{aligned} \sigma(CTC) &= \overline{\sigma(T)}, \quad \sigma_p(CTC) = \overline{\sigma_p(T)}, \quad \sigma_a(CTC) = \overline{\sigma_a(T)}, \\ \sigma_s(CTC) &= \overline{\sigma_s(T)}, \quad \sigma_e(CTC) = \overline{\sigma_e(T)} \text{ and } \sigma_w(CTC) = \overline{\sigma_w(T)}, \end{aligned}$$

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where $\overline{E} = \{\bar{z} : z \in E\} \subset \mathbb{C}$.

S. Jung, E. Ko and J. E. Lee, *On complex symmetric operator matrices*, J. Math. Anal. Appl., **406**(2013), 373-385. This case doesn't need $CTC = T^*$. Only relation between T and CTC . Next we explain a conjugation on a Banach space.

Definition 1.2 Let $\mathcal{X}$ be a complex Banach space with the norm $\|\cdot\|$ and C be an operator on $\mathcal{X}$. If C satisfies the following, then C is said to be a conjugation on a Banach space $\mathcal{X}$. For all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{C}$,

$$(*) \quad C^2 = I, \quad C(\alpha x + \beta y) = \bar{\alpha}Cx + \bar{\beta}Cy \text{ and } \|C\| \leq 1,$$

where I is the identity operator on $\mathcal{X}$.

Of course, from the definition it holds $\|Cx\| = \|x\|$ for all $x \in \mathcal{X}$.

Theorem 1.2 If C satisfies $(*)$ on a Hilbert space $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$, then $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$.

Proof. Let $x, y \in \mathcal{H}$, $\alpha \in \mathbb{R}$ and let $Cy = z$. Since

$$\|Cx + \alpha z\| = \|C(x + \alpha Cz)\| \leq \|x + \alpha Cz\| = \|C(Cx + \alpha z)\| \leq \|Cx + \alpha z\|,$$

we have $\|Cx + \alpha z\| = \|x + \alpha Cz\|$. By taking square, we have $\operatorname{Re}\langle Cx, z \rangle = \operatorname{Re}\langle Cz, x \rangle$ and

$$\operatorname{Re}\langle Cx, Cy \rangle = \operatorname{Re}\langle Cx, z \rangle = \operatorname{Re}\langle Cz, x \rangle = \operatorname{Re}\langle C^2y, x \rangle = \operatorname{Re}\langle y, x \rangle.$$

By taking ix instead of x , we have $\operatorname{Im}\langle Cx, Cy \rangle = \operatorname{Im}\langle y, x \rangle$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$. $\square$

Example 1.1 Let $\mathcal{H}$ be a Hilbert space and $\mathcal{X} = B(\mathcal{H})$. For conjugations C, J on $\mathcal{H}$, M_{CJ} on $\mathcal{X}$ is defined by

$$M_{CJ}(T) := CTJ \quad (T \in B(\mathcal{H}) = \mathcal{X}).$$

Then M_{CJ} is a conjugation on a Banach space $\mathcal{X}$.

Definition 1.3 Let C be a conjugation on a Banach space $\mathcal{X}$. The dual operator $C^* : \mathcal{X}^* \rightarrow \mathcal{X}^*$ of C is defined by

$$(C^*(f))(x) = \overline{f(Cx)} \quad (x \in \mathcal{X}, f \in \mathcal{X}^*),$$

where $\mathcal{X}^*$ is the dual space of $\mathcal{X}$ and $\overline{f(Cx)}$ is the complex conjugate of $f(Cx)$.

Theorem 1.3 If C is a conjugation on $\mathcal{X}$, then C^* is a conjugation on $\mathcal{X}^*$.

Proof. It is clear that $C^{*2} = I^*$ and

$$C^*(f + g) = C^*(f) + C^*(g) \text{ for all } f, g \in \mathcal{X}^*.$$

For $\lambda \in \mathbb{C}$ and $x \in \mathcal{X}$, it holds $(C^*(\lambda f))(x) = \bar{\lambda} \overline{f(Cx)} = \bar{\lambda} (C^*f)(x)$ and $C^*(\lambda f) = \bar{\lambda} C^*(f)$. Since, for all $f \in \mathcal{X}^*$, it holds

$$|(C^*f)(x)| = |\overline{f(Cx)}| \leq \|f\| \|Cx\| = \|f\| \|x\|,$$

we have $\|C^*f\| \leq \|f\|$ and $\|C^*\| \leq 1$. $\square$

The same results hold for spectral properties of an operator on a Banach space concerning with a conjugation.

Theorem 1.4 *Let $T \in B(\mathcal{X})$ and C be a conjugation on $\mathcal{X}$. Then it holds the following :*

$$\sigma(CTC) = \overline{\sigma(T)}, \sigma_a(CTC) = \overline{\sigma_a(T)}, \sigma_p(CTC) = \overline{\sigma_p(T)} \text{ and } \sigma_s(CTC) = \overline{\sigma_s(T)}.$$

2 Numerical range of Banach space operator

In this section, we explain definition of the numerical range $V(T)$ of T on a Banach space $\mathcal{X}$.

Definition 2.1 *Let Π be the set*

$$\Pi := \{(x, f) \in \mathcal{X} \times \mathcal{X}^* : \|f\| = f(x) = \|x\| = 1\}.$$

For an operator $T \in B(\mathcal{X})$, the numerical range $V(T)$ of T is given by

$$V(T) = \{f(Tx) : (x, f) \in \Pi\}.$$

Defitinion 2.2 *For $T \in B(\mathcal{X})$;*

- *T is Hermitian if $V(T) \subset \mathbb{R}$.*
- *T is positive if $V(T) \subset [0, \infty)$. In this case, we write $T \geq 0$.*
- *T is normal if there exist Hermitian operators H and K such that $T = H + iK$ and $HK = KH$.*
- *T is hyponormal if there exist Hermitian operators H and K such that $T = H + iK$ and $i(HK - KH) \geq 0$.*

Theorem 2.1 *If $(x, f) \in \Pi$, then $(Cx, C^*f) \in \Pi$.*

Proof. Let $(x, f) \in \Pi$. Then $\|f\| = f(x) = \|x\| = 1$.

$$(C^*f)(Cx) = \overline{f(C^2x)} = \overline{f(x)} = 1$$

Since $\|Cx\| = \|x\| = 1$, we have

$$\|C^*f\| = \sup_{\|x\|=1} |(C^*f)(x)| = \sup_{\|x\|=1} |f(Cx)| \leq \|f\| \|Cx\| = 1.$$

Therefore, we have $\|C^*f\| \leq 1$ and $\|C^*f\| = 1$ and so $(Cx, C^*f) \in \Pi$. $\square$

Theorem 2.2 *Let $\mathcal{X}$ be a complex Banach space, $T \in B(\mathcal{X})$ and C be a conjugation on $\mathcal{X}$. Then $V(CTC) = \overline{V(T)}$.*

Proof. Let $z \in V(CTC)$. Then there exists $(x, f) \in \Pi$ such that $z = f(CTCx)$. We obtain $z = \overline{(C^*f)(TCx)}$. Since $(Cx, C^*f) \in \Pi$, we have $z \in \overline{V(T)}$ and $V(CTC) \subset \overline{V(T)}$. Therefore, we have $V(T) = V(C^2TC^2) \subset \overline{V(CTC)}$ and $V(CTC) = \overline{V(T)}$. $\square$

Theorem 2.3 *Let $T \in B(\mathcal{X})$ and C be a conjugation on $\mathcal{X}$. Then following results hold.*

- (1) *T is Hermitian if and only if CTC is Hermitian.*
- (2) *T is positive if and only if CTC is positive.*
- (3) *T is normal if and only if CTC is normal.*
- (4) *T is hyponormal if and only if CTC is hyponormal.*
- (5) *T is compact if and only if CTC is compact.*

Definition 2.3

- Denote by $V_\omega(T)$ the set of all $z \in \mathbb{C}$ such that there exists a sequence $(x_n, f_n) \in \Pi$ which satisfies $w\text{-}\lim x_n = 0$ and $\lim f_n(Tx_n) = z$. The set $V_\omega(T)$ is said to be the sequential essential numerical range of T .
- For an operator $T \in B(\mathcal{X})$, the essential numerical range $V_e(T)$ of T is given by

$$V_e(T) := \{\mathcal{F}(T) : \mathcal{F} \in B(\mathcal{X})^*, \|\mathcal{F}\| = \mathcal{F}(I) = 1, \mathcal{F}(\mathcal{C}(\mathcal{X})) = \{0\}\},$$

where $\mathcal{C}(\mathcal{X})$ is the set of all compact operators on $\mathcal{X}$.

- Denote by $W_e(T)$ the set of all $z \in \mathbb{C}$ with the property that there are nets $(x_\alpha) \subset \mathcal{X}$, $(f_\alpha) \subset \mathcal{X}^*$ such that $\|f_\alpha\| = f_\alpha(x_\alpha) = 1$ ($\forall \alpha$), $x_\alpha \rightarrow 0$ (weakly) and $f_\alpha(x_\alpha) \rightarrow z$. The set $W_e(T)$ is said to be the spatial essential numerical range of T .

Theorem 2.4 *For any conjugation C , $w\text{-}\lim x_n = 0$ if and only if $w\text{-}\lim Cx_n = 0$.*

Proof. Assume $w\text{-}\lim_{n \rightarrow \infty} x_n = 0$. Then, for any $f \in \mathcal{X}^*$, since $C^*f \in \mathcal{X}^*$, we have $f(Cx_n) = \overline{(C^*f)(x_n)} \rightarrow 0$. Hence $w\text{-}\lim_{n \rightarrow \infty} Cx_n = 0$. Since $x_n = C^2x_n$, the converse is clear. $\square$

Theorem 2.5 *For any conjugation C , $V_\omega(CTC) = \overline{V_\omega(T)}$ and $W_e(CTC) = \overline{W_e(T)}$.*

Proof. Let $z \in V_\omega(CTC)$. There exists a sequence $\{(x_n, f_n)\}_{n=1}^\infty$ of Π such that $w\text{-}\lim_{n \rightarrow \infty} x_n = 0$ and $\lim_{n \rightarrow \infty} f_n(CTCx_n) = z$. We have

$$\lim_{n \rightarrow \infty} (C^*f_n)(TCx_n) = \overline{\lim_{n \rightarrow \infty} f_n(CTCx_n)} = \bar{z}.$$

Since $(Cx_n, C^*f_n) \in \Pi$ and $w\text{-}\lim_{n \rightarrow \infty} Cx_n = 0$ by Theorem 2.4, we obtain $\bar{z} \in V_\omega(T)$ and $\overline{V_\omega(CTC)} \subset V_\omega(T)$. Hence we have $V_\omega(T) = V_\omega(C^2TC^2) \subset \overline{V_\omega(CTC)}$ and $V_\omega(CTC) = \overline{V_\omega(T)}$. The proof of $W_e(CTC) = \overline{W_e(T)}$ is almost the same. $\square$

Theorem 2.6 For any conjugation C , $V_e(CTC) = \overline{V_e(T)}$.

Proof. Let $\mathcal{F}(CTC) \in V_e(CTC)$. Then there exists $\mathcal{F} \in B(\mathcal{X})^*$ such that $\|\mathcal{F}\| = \mathcal{F}(I) = 1$, $\mathcal{F}(\mathcal{C}(\mathcal{X})) = \{0\}$. Since

$$|C^*\mathcal{F}(T)| = |\overline{\mathcal{F}(CTC)}| \leq \|\mathcal{F}\| \cdot \|CTC\| \leq \|T\|$$

and

$$C^*\mathcal{F}(I) = \overline{\mathcal{F}(CIC)} = \overline{\mathcal{F}(I)} = \overline{1} = 1,$$

we have $\|C^*\mathcal{F}\| = 1$. Moreover, by Theorem 2.3 (5), $C^*\mathcal{F}(\mathcal{C}(\mathcal{X})) = \{0\}$. Therefore, we obtain $\mathcal{F}(CTC) \in \overline{V_e(T)}$ and so $V_e(CTC) \subset \overline{V_e(T)}$. Hence we have $V_e(T) = V_e(C^2TC^2) \subset \overline{V_e(CTC)}$ and $V_e(CTC) = \overline{V_e(T)}$. $\square$

Theorem 2.7 Let $T \in B(\mathcal{X})$ and C be a conjugation on $\mathcal{X}$. Then following results hold.

- (1) $x \in \ker(T)$ if and only if $Cx \in \ker(CTC)$.
- (2) $x \in R(T)$ if and only if $Cx \in R(CTC)$.
- (3) $R(T)$ is closed if and only if $R(CTC)$ is closed.

Proof. (1) If $x \in \ker(T)$, then we have $(CTC)Cx = CTx = 0$ and hence $Cx \in \ker(CTC)$. Conversely, if $Cx \in \ker(CTC)$, then we obtain $x = C^2x \in \ker(C^2TC^2) = \ker(T)$.

(2) Let $x \in R(T)$. Since $\exists y \in \mathcal{X}$; $x = Ty$, it follows that $Cx = CTy = CTC(Cy)$ and hence $Cx \in R(CTC)$. Conversely, if $Cx \in R(CTC)$, then $x = C^2x \in R(C^2TC^2) = R(T)$.

(3) Let $R(T)$ be closed and $\{x_n\} \subset R(CTC)$ be a Cauchy sequence. By Theorem 2.7 (2), it follows $Cx_n \in R(C^2TC^2) = R(T)$. Since

$$\|Cx_m - Cx_n\| \leq \|C\| \|x_m - x_n\| \rightarrow 0 \text{ as } m, n \rightarrow \infty,$$

$\{Cx_n\} \subset R(T)$ is a Cauchy sequence. Since $R(T)$ is closed, $\exists x_0 \in R(T)$; $x_0 = \lim_{n \rightarrow \infty} Cx_n$.

Then $x_n = C^2x_n \rightarrow Cx_0$ and by Theorem 2.7 (2), we have $Cx_0 \in R(CTC)$. Therefore, $R(CTC)$ is closed. Conversely if $R(CTC)$ is closed, then $R(T) = R(C^2TC^2)$ is closed. $\square$

Definition 2.4 Let $\sigma_{\text{eap}}(T)$ denote the set of all $z \in \mathbb{C}$ such that there exists a sequence $\{x_n\}$ of unit vectors which satisfies $x_n \rightarrow 0$ (weakly) and $(T - z)x_n \rightarrow 0$. The set $\sigma_{\text{eap}}(T)$ is said to be the essential approximate point spectrum of T .

Theorem 2.8 For $T \in B(\mathcal{X})$ and any conjugation C , $\sigma_{\text{eap}}(CTC) = \overline{\sigma_{\text{eap}}(T)}$.

Proof. Let $z \in \sigma_{\text{eap}}(CTC)$. Take a sequence $\{x_n\}$ of unit vectors such that $x_n \rightarrow 0$ (weakly) and $(CTC - z)x_n \rightarrow 0$ as $n \rightarrow \infty$. We have

$$C(T - \bar{z})Cx_n = (CTC - z)x_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus we obtain $(T - \bar{z})Cx_n \rightarrow 0$ as $n \rightarrow \infty$. Since $\|Cx_n\| = \|x_n\| = 1$ and $Cx_n \rightarrow 0$ (weakly), we have $\bar{z} \in \sigma_{\text{eap}}(T)$ and hence $\overline{\sigma_{\text{eap}}(CTC)} \subset \sigma_{\text{eap}}(T)$. Therefore, we obtain $\sigma_{\text{eap}}(T) = \sigma_{\text{eap}}(C^2TC^2) \subset \overline{\sigma_{\text{eap}}(CTC)}$ and $\sigma_{\text{eap}}(CTC) = \overline{\sigma_{\text{eap}}(T)}$. $\square$

Definition 2.5 An operator $T \in B(\mathcal{X})$ is Fredholm if and only if there exists operators $S_1, S_2 \in B(\mathcal{X})$ such that $TS_1 - I$ and $S_2T - I$ are compact operators. The essential spectrum σ_e of T is the set of all $z \in \mathbb{C}$ such that $T - z$ is not Fredholm.

We have the following results.

Theorem 2.9 For $T \in B(\mathcal{X})$ and a conjugation C on $\mathcal{X}$, T is Fredholm if and only if CTC is Fredholm.

Theorem 2.10 For $T \in B(\mathcal{X})$, $\sigma_e(CTC) = \overline{\sigma_e(T)}$.

These definitions (numerical range, Fredholm, essential spectrum and others) are from the following paper: Barraa and Müller; On the essential numerical range, Acta Sci. Math. (Szeged) **71** (2005), 285-298.

3 (m, C) -Symmetric Operators on a Banach space

We introduce and show some properties of (m, C) -symmetric operators on a Banach space.

Definition 3.1 For an operator $T \in B(\mathcal{X})$ and a conjugation C on $\mathcal{X}$, we define an operator $\alpha_m(T; C)$ by

$$\alpha_m(T; C) := \sum_{j=0}^m (-1)^j \binom{m}{j} CT^{m-j}CT^j.$$

An operator T is said to be (m, C) -symmetry if $\alpha_m(T; C) = 0$.

It hold that

$$CTC \alpha_m(T; C) - \alpha_m(T; C)T = \alpha_{m+1}(T; C).$$

Hence if T is (m, C) -symmetry, then T is (n, C) -symmetry for every $n \geq m$.

Example 3.1 If Q is an n -nilpotent operator on $\mathcal{X}$, then Q is $(2n-1, C)$ -symmetry for any conjugation C .

Proof. By the definition, we have

$$\alpha_{2n-1}(Q; C) := \sum_{j=0}^{2n-1} (-1)^j \binom{2n-1}{j} CQ^{2n-1-j}CQ^j.$$

When $0 \leq j \leq n-1$, we have $Q^{2n-1-j} = 0$. When $j \geq n$, we obtain $Q^j = 0$. Therefore, we conclude $\alpha_{2n-1}(Q; C) = 0$. $\square$

Example 3.2 Let $T \in B(\mathcal{H})$ satisfy $\sum_{j=0}^m (-1)^j \binom{m}{j} CT^{m-j}CT^j = 0$ for some conjugation C on a Hilbert space $\mathcal{H}$. We define a conjugation $M_C(S)$ on $\mathcal{H}$ by $M_C(S) := CSC$ ($S \in B(\mathcal{H})$). Let an operator $L_T(S)$ be $L_T(S) := TS$ ($S \in B(\mathcal{H})$). Then L_T is an (m, M_C) -symmetric operator on a Banach space $B(\mathcal{H})$.

Definition 3.2 A pair (T, S) of operators $T, S \in B(\mathcal{H})$ is said to be C -doubly commuting if $TS = ST$ and $S \cdot CTC = CTC \cdot S$.

• If (T, S) is C -doubly commuting, then it holds that

$$\alpha_n(T + S; C) = \sum_{j=0}^n \binom{n}{j} \alpha_{n-j}(T; C) \alpha_j(S; C)$$

and

$$\alpha_n(TS; C) = \sum_{j=0}^n \binom{n}{j} CT^jC \cdot \alpha_{n-j}(T; C) \alpha_j(S; C) \cdot S^{n-j}.$$

Theorem 3.1 Let T be (m, C) -symmetry and S be (n, C) -symmetry. If (T, S) is C -doubly commuting, then $T + S$ is $(m + n - 1, C)$ -symmetry.

Proof. We have

$$\alpha_{m+n-1}(T + S; C) = \sum_{j=0}^{m+n-1} (-1)^j \binom{m+n-1}{j} \alpha_{m+n-1-j}(T; C) \cdot \alpha_j(S; C).$$

When $j \geq n$, we have $\alpha_j(S; C) = 0$. When $j \leq n - 1$, we obtain $\alpha_{m+n-1-j}(T; C) = 0$ since $m + n - 1 - j \geq m + n - 1 - (n - 1) = m$. Therefore, we conclude $\alpha_{m+n-1}(T + S; C) = 0$. $\square$

By Example 3.1 and Theorem 3.1, we have the following Theorem 3.2.

Theorem 3.2 Let T be (m, C) -symmetry and Q be n -nilpotent. If (T, Q) is C -doubly commuting, then $T + Q$ is $(m + 2n - 2, C)$ -symmetry.

Theorem 3.3 Let T be (m, C) -symmetry. Then

- (1) T^n is (m, C) -symmetry for any $n \in \mathbb{N}$.
- (2) If T is invertible, then T^{-1} is (m, C) -symmetry.

Proof. (1) Since $\alpha_m(T; C) = 0$ and

$$\begin{aligned} (a^n - b^n)^m &= (a - b)^m (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})^m \\ &= (a - b)^m (\xi_0 a^{m(n-1)} + \xi_1 a^{m(n-1)-1}b + \cdots + \xi_{m(n-1)} b^{m(n-1)}) \end{aligned}$$

where ξ_i are coefficients ($i = 0, \dots, m(n-1)$), it follows that

$$\alpha_m(T^n; C) = \sum_{j=0}^{m(n-1)} \xi_j CT^{m(n-1)-j}C \cdot \alpha_m(T; C) \cdot T^j = 0.$$

Hence the operator T^n is (m, C) -symmetry.

(2) Suppose that T is invertible and (m, C) -symmetry. Since $\alpha_m(T; C) = 0$, we have

$$\begin{aligned} 0 &= CT^{-m}C \left(\alpha_m(T; C) \right) T^{-m} \\ &= CT^{-m}C \left(\sum_{j=0}^m (-1)^j \binom{m}{j} CT^{m-j}CT^j \right) T^{-m} \\ &= \sum_{j=0}^m (-1)^j \binom{m}{j} C(T^{-1})^jC \cdot (T^{-1})^{m-j}. \end{aligned}$$

Therefore, the operator T^{-1} is (m, C) -symmetry. $\square$

Next we show spectral properties of (m, C) -symmetric operators. It needs the following result.

Theorem 3.4 (C. Schmoege, [5]) *Let $T \in B(\mathcal{X})$ and f be a polynomial. Then*

$$(1) \sigma_a(f(T)) = f(\sigma_a(T)) \quad \text{and} \quad (2) \sigma_{\text{eap}}(f(T)) \subset f(\sigma_{\text{eap}}(T)).$$

Theorem 3.5 *Let $T \in B(\mathcal{X})$ be (m, C) -symmetry.*

(1) *If $z \in \sigma_a(T)$ ($\sigma_p(T)$), then $\bar{z} \in \sigma_a(T)$ ($\sigma_p(T)$).*

(2) *If $z \in \sigma_{\text{eap}}(T)$, then $\bar{z} \in \sigma_{\text{eap}}(T)$.*

Proof. (1) Let $z \in \sigma_a(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T - z)x_n \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\alpha_m(T; C) = \sum_{j=0}^m (-1)^j \binom{m}{j} (CTC - z)^{m-j} (T - z)^j,$$

we have

$$0 = \lim_{n \rightarrow \infty} \alpha_m(T; C)x_n = \lim_{n \rightarrow \infty} (CTC - z)^m x_n.$$

By Theorem 3.5 for a polynomial $f(x) = z^m$, we obtain $0 \in \sigma_a(CTC - z)$ and hence $z \in \sigma_a(CTC)$. By Theorem 1.7, it holds $\bar{z} \in \sigma_a(T)$. $\square$

Theorem 3.6 *If T is (m, C) -symmetry, then $\ker(T) \subset C(\ker(T^m))$.*

Proof. If $x \in \ker(T)$, then we obtain

$$CT^mCx = \sum_{j=1}^m (-1)^{j+1} \binom{m}{j} CT^{m-j}CT^jx = 0$$

and $T^mCx = 0$. Hence we have $Cx \in \ker(T^m)$ and $x \in C(\ker(T^m))$. $\square$

4 (m, C) -Isometric Operators on a Banach space

We introduce and show some properties of an (m, C) -isometric operators on a Banach space.

Definition 4.1 For an operator $T \in B(\mathcal{X})$ and a conjugation C on $\mathcal{X}$, we define an operator $\beta_m(T; C)$ by

$$\beta_m(T; C) := \sum_{j=0}^m (-1)^j \binom{m}{j} C T^{m-j} C T^{m-j}$$

An operator T is said to be (m, C) -isometry if $\beta_m(T; C) = 0$.

It hold that

$$CTC \beta_m(T; C) T - \beta_m(T; C) = \beta_{m+1}(T; C).$$

Hence if T is a (m, C) -isometry, then T is a (n, C) -isometry for every $n \geq m$. It holds similar results.

Example 4.1 Let $T \in B(\mathcal{H})$ satisfy $\sum_{j=0}^m (-1)^j \binom{m}{j} C T^{m-j} C T^{m-j} = 0$ for some conjugation C on a Hilbert space $\mathcal{H}$. We define a conjugation M_C on $\mathcal{H}$ by $M_C(S) := CSC$ ($S \in B(\mathcal{H})$). Let an operator L_T be $L_T(S) := TS$ ($S \in B(\mathcal{H})$). Then L_T is an (m, M_C) -isometric operator on a Banach space $B(\mathcal{H})$.

Theorem 4.1 Let T is (m, C) -isometry. Then

- (1) $0 \notin \sigma_a(T)$.
- (2) If $z \in \sigma_a(T)$, then $\bar{z}^{-1} \in \sigma_a(T)$.

The statement (2) holds for $\sigma_p(T)$ and $\sigma_{\text{cap}}(T)$. Therefore, if T is (m, C) -isometry, then $\|T\| \geq 1$.

Proof. (1) Assume that there exists a sequence $\{x_n\}$ of unit vectors such that $Tx_n \rightarrow 0$ as $n \rightarrow \infty$. Since it holds

$$0 = \beta_m(T; C) = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} C T^{m-j} C \cdot T^{m-j} + (-1)^m I,$$

we have $\lim_{n \rightarrow \infty} Ix_n = 0$. Hence, it's a contradiction and $0 \notin \sigma_a(T)$.

(2) Let $z \in \sigma_a(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T-z)x_n \rightarrow 0$. Since it holds

$$(zCTC - 1)^m x_n = \left(\sum_{j=0}^m (-1)^{j+1} \binom{m}{j} C T^{m-j} C (T^{m-j} - z^{m-j}) \right) x_n \rightarrow 0,$$

we have $0 \in \sigma_a((zCTC - 1)^m)$. By Theorem 3.5 for a polynomial $f(x) = z^m$, we obtain $0 \in \sigma_a(zCTC - 1)$. By (1), since $z \neq 0$, we have $z^{-1} \in \sigma_a(CTC)$ and hence, by Theorem 1.7, it holds $\bar{z}^{-1} \in \sigma_a(T)$. $\square$

We have the following results.

Theorem 4.2 *Let T be (m, C) -isometry and Q be n -nilpotent. If (T, Q) is C -doubly commuting, then $T + Q$ is $(m + 2n - 2, C)$ -isometry.*

Theorem 4.3 *Let T be (m, C) -isometry. Then*

- (1) T^n is (m, C) -isometry for any $n \in \mathbb{N}$.
- (2) If T is invertible, then T^{-1} is (m, C) -isometry.

Please see following references for details.

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