

Keller-Segel System and the Concentration Lemma

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1 Introduction

We consider time-global existence and blow-up of solutions of the following system related to chemotaxis

$$(P) \quad \begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \nabla v), & x \in \Omega, t > 0, \\ 0 = \Delta v - \gamma v + \alpha u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\ u(\cdot, 0) = u_0, & x \in \Omega. \end{cases}$$

Here Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, χ , γ and α are positive constants and u_0 is a non-negative smooth function on $\bar{\Omega}$.

There exists a unique solution (u, v) to (P) defined on a maximal interval of existence $[0, T_{max})$, which is smooth in $x \in \bar{\Omega}$ and $0 < t < T_{max}$. If $u_0 \not\equiv 0$ in Ω , the solution satisfies that $u(x, t) > 0$, $v(x, t) > 0$ for $(x, t) \in \Omega \times (0, T_{max})$. If $T_{max} < \infty$, we can observe the following.

Proposition 1 *If $T_{max} < \infty$, then the following relations hold.*

- (i) $\lim_{t \rightarrow T_{max}} \|u \log u\|_{L^1(\Omega)} = \infty$
- (ii) $\lim_{t \rightarrow T_{max}} \|\nabla v\|_{L^2(\Omega)} = \infty$.
- (iii) For $a > \chi/2$, then $\lim_{t \rightarrow T_{max}} \int_{\Omega} e^{av(x,t)} dx = \infty$.

Then, if $T_{max} < \infty$, we have that

$$\lim_{t \rightarrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \lim_{t \rightarrow T_{max}} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty,$$

which we mean that the solution blows up in finite time.

Let L be an arbitrary positive constant and let $D_L = \{x \in \mathbf{R}^2 | |x| < L\}$. We have the following results.

Theorem 1 *Suppose*

$$\Omega = D_L \quad \text{and} \quad \|u_0\|_{L^1(D_L)} < 8\pi/(\alpha\chi). \quad (1)$$

Let $u_0(x) = u_0(-x)$ on D_L . Then (P) admits a unique classical solution (u, v) on $\overline{D_L} \times (0, \infty)$ satisfying

$$\sup_{t \geq 0} \{ \|u(\cdot, t)\|_{L^\infty(D_L)} + \|v(\cdot, t)\|_{L^\infty(D_L)} \} < \infty.$$

Definition 1 We say that q is a blow-up point of u if there exists $\{t_k\}_{k=1}^\infty \subset [0, T_{\max})$ and $\{x_k\}_{k=1}^\infty \subset \overline{\Omega}$ satisfying $u(x_k, t_k) \rightarrow \infty$, $t_k \rightarrow T_{\max} < \infty$ and $x_k \rightarrow q \in \overline{\Omega}$ as $k \rightarrow \infty$. We denote the set of all blow-up points of u by $\mathcal{B}$.

Theorem 2 Let (1) hold. Let a_* be a root of $a_* - \chi/2 - \|u_0\|_{L^1(\Omega)} \alpha a_*^2 / 16\pi = 0$ such that $a_* < \chi$. If $T_{\max} < \infty$, then there exists a point $q \in \mathcal{B} \cap \partial\Omega$ satisfying

$$\limsup_{t \rightarrow T_{\max}} \int_{\Omega \cap B(q, \varepsilon)} u(x, t) dx \geq \frac{2\pi}{a_* \alpha} \quad \text{for any } \varepsilon > 0.$$

Definition 2 For $q \in \mathcal{B}$, we say that q is an isolated blow-up point if there exists $\delta > 0$ such that

$$\sup\{u(x, t) \mid 0 \leq t < T_{\max} \text{ and } x \in \overline{B(q, \delta) \setminus B(q, \varepsilon)} \cap \Omega\} < \infty \quad \text{for any } \varepsilon \in (0, \delta).$$

We denote the set of all isolated blow-up points of u by $\mathcal{B}_I$.

Theorem 3 Suppose $T_{\max} < \infty$. Then the following properties hold.

For $q \in \mathcal{B}_I$, there exist two positive constants δ , $m \geq m_*$ and a non-negative function $f \in L^1(E(q, \delta)) \cap C(E(q, \delta) \setminus \{q\})$ such that

$$w^* - \lim_{t \rightarrow T_{\max}} u(\cdot, t) = m\delta_q + f \quad \text{in } \mathcal{M}(E(q, \delta)),$$

where $E(q, \delta) = \overline{B(q, \delta)} \cap \Omega$,

$$m_* = \begin{cases} 4\pi/(\alpha\chi) & \text{if } q \in \partial\Omega, \\ 8\pi/(\alpha\chi) & \text{if } q \in \Omega \end{cases}$$

and $\mathcal{M}(S)$ is the Banach space consisting of all Radon measures on a compact Hausdorff space S with the usual norm.

For a set K , we denote the number of elements of K by $\#K$. The following corollary is an immediate consequence of Theorem 3.

Corollary 1 Suppose $T_{\max} < \infty$. Then $\mathcal{B}_I$ satisfies that

$$\#\{\mathcal{B}_I \cap \Omega\} + \frac{1}{2} \#\{\mathcal{B}_I \cap \partial\Omega\} \leq \frac{\alpha\chi}{8\pi} \|u_0\|_{L^1(\Omega)}.$$

The following corollary is an immediate consequence of Theorem 3 and [6].

Corollary 2 Suppose $\Omega = D_L$ and that u_0 be radially symmetric. If $\int_{D_L} u_0(x) dx > 8\pi/(\alpha\chi)$ and $\int_{D_L} u_0(x) |x|^2 dx$ is sufficiently small, then $T_{\max} < \infty$ and there exist a positive constant $m \geq 8\pi/(\alpha\chi)$ and a non-negative function $f \in L^1(D_L) \cap C(\overline{D_L} \setminus \{0\})$ such that

$$w^* - \lim_{t \rightarrow T_{\max}} u(\cdot, t) = m\delta_0 + f \quad \text{in } \mathcal{M}(\overline{D_L}).$$

2 Proof of Theorem 1

Lemma 1 *Let (u, v) be a solution to (P). Put*

$$W(t) = \int_{\Omega} \left\{ u \log u - \frac{\chi}{2\alpha} (|\nabla v|^2 + \gamma v^2) \right\} dx.$$

Then, it follows that

$$\frac{d}{dt} W(t) + \int_{\Omega} u |\nabla (\log u - \chi v)|^2 dx = 0 \quad \text{for } t \in (0, T_{max}).$$

Proof of Lemma 1: Multiplying $\log u - \chi v$ by the first equation of (P) and using the second equation of (P), then we have this lemma.

Lemma 2 *Suppose that (u, v) is a solution to (P). Let a be an arbitrary positive constant and let $M = \|u_0\|_{L^1}$. Then, the inequality*

$$a \int_{\Omega} uv dx \leq \int_{\Omega} u \log u dx + M \log \left(\int_{\Omega} e^{av} dx \right) - M \log M$$

holds for $0 \leq t < T_{max}$.

Proof of Lemma 2. Let

$$\mu = \int_{\Omega} e^{av} dx \text{ and } \psi = \frac{M}{\mu} e^{av}.$$

Then, we have that

$$\begin{aligned} 0 &= -\log \left(\int_{\Omega} \frac{\psi u}{u \mu} dx \right) \\ &\leq \int_{\Omega} \left(-\log \frac{\psi}{u} \right) \frac{u}{M} dx, \end{aligned}$$

by which together with Jensen's inequality we get this lemma.

Proposition 2 *If w is a function on $\overline{D_L}$ satisfies that $w \in C^1(D_L)$, $w(x) = w(-x)$ on ∂D_L and*

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial D_L,$$

then there exists absolute constants C and K such that

$$\log \left(\int_{\Omega} e^w dx \right) \leq \frac{1}{16\pi} \int_{\Omega} |\nabla w|^2 dx + C \|w\|_{L^1} + K.$$

Proof of Theorem 1. By Lemmas 1, 2 and the second equation of (P), we have that

$$\left(a - \frac{\chi}{2}\right) \frac{1}{\alpha} \int_{\Omega} (|\nabla v(x, t)|^2 + \gamma |v(x, t)|^2) dx \leq W(0) + M \log \left(\int_{\Omega} e^{av(x, t)} dx \right) - M \log M,$$

by which together with Proposition 2 it follows that

$$\left\{ \left(a - \frac{\chi}{2}\right) \frac{1}{\alpha} - \frac{Ma^2}{16\pi} \right\} \int_{\Omega} (|\nabla v(x, t)|^2 + \gamma |v(x, t)|^2) dx \leq W(0) + M \left(\frac{CaM}{|D_L|} + K - \log M \right).$$

Because of $\|u_0\|_{L^1(\Omega)} = M < 8\pi/\alpha\chi$, we can take a constant a satisfying

$$\left(a - \frac{\chi}{2}\right) \frac{1}{\alpha} - \frac{Ma^2}{16\pi} > 0.$$

Then gives

$$\sup_{0 \leq t < T_{max}} \int_{D_L} (|\nabla v|^2 + \gamma |v|^2) dx < \infty.$$

and hence $T_{max} = \infty$ by the case (ii) of Proposition 1. $\square$

3 Proof of Theorem 2

Proposition 3 *Let $h > 0$ and $\mathcal{E} = \{w \in C^1(\overline{D_L}) | \frac{\partial w}{\partial n} = 0 \text{ on } \partial D_L \text{ and } \|w\|_{L^1} \leq h\}$. Then, for any $\mathcal{F} \subset \mathcal{E}$ satisfy 1 or 2:*

(i) *For any $\varepsilon > 0$, there exists a positive constant C_ε s.t.*

$$\log \left(\int_{\Omega} e^w dx \right) \leq \frac{1 + \varepsilon}{16\pi} \int_{\Omega} |\nabla w|^2 dx + C_\varepsilon \text{ for } w \in \mathcal{F}.$$

(ii) *There exist a sequence $\{w_k\} \subset \mathcal{F}$, a point $q \in \partial D_L$, a constant $m \in [1/2, 1)$ and a regular measure ψ s.t.*

$$w^* - \lim_{k \rightarrow \infty} \frac{\exp(w_k) p_*}{\int_{D_L} \exp(w_k) p_* dx} = m \delta_q + \psi \quad \text{in } \mathcal{M}(\overline{D}),$$

where $p_* = \frac{8}{L^2(1+(x/L)^2)^2}$.

Next we consider the following elliptic problem related to the second equation of (P).

$$(EE) \quad \begin{cases} 0 = \Delta w - \gamma w + f, & x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial \Omega \end{cases}$$

with $f \in L^1(\Omega), \geq 0$.

Proposition 4 *Let $q \in \Omega$. Then there exists a positive constant η_0 such that for $\eta \in (0, \eta_0)$ there exists a positive constant C_{η_0} satisfying*

$$\int_{\Omega \cap B(q, \eta)} e^w dx \leq \exp \left(\frac{C_{\eta_0}}{\eta_0 - \eta} \|f\|_{L^1(\Omega)} \right) \int_{|x| < 2\eta_0} \frac{dx}{|x|^\theta},$$

where

$$\theta = \begin{cases} \frac{1}{2\pi} \int_{B(q, \eta_0)} |f| dx & \text{if } q \in \Omega, \\ \frac{1 + O(\eta_0)}{\pi} \int_{\Omega \cap D(q, \eta_0)} |f| dx & \text{if } q \in \partial\Omega. \end{cases}$$

Proof of Theorem 2: Let a be a positive constant satisfying

$$\left(a - \frac{\chi}{2}\right) \frac{1}{\alpha} - \frac{Ma^2}{16\pi} > 0.$$

We assume that $\{av\}$ satisfies 1 of Proposition 3, by which together with the arguments of proof of Theorem 1 it follows that $T_{max} = \infty$. It is the contradiction. Then, for any positive constant a with

$$\left(a - \frac{\chi}{2}\right) \frac{1}{\alpha} - \frac{Ma^2}{16\pi} > 0,$$

we observe that

$$\begin{aligned} \infty &= \limsup_{t \rightarrow T_{max}} \int_{\Omega} e^{av} dx \\ &\leq \frac{1}{m} \limsup_{t \rightarrow T_{max}} \int_{\Omega \cap B(q, \varepsilon)} e^{av} dx \quad \text{for any } \varepsilon > 0, \end{aligned} \quad (2)$$

by which together with Proposition 4 we have this theorem. $\square$

4 Proof of Theorem 3

By using Proposition 1 and Lemmas 1 and 2, we have (2) for any $q \in \mathcal{B}_I$ and any $a > \chi/2$, by which together with Proposition 4 we have this theorem.

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