

On the Weyl Quantized Relativistic Hamiltonian
- Kato's inequality and essential selfadjointness -

Takashi Ichinose (金沢大 理 一瀬 孝)
Department of Mathematics, Kanazawa University

1. Introduction.

The classical relativistic Hamiltonian of a spinless particle with mass $m \geq 0$ in an electromagnetic field is given by

$$(1.1) \quad h(p, x) = h_A(p, x) + \Phi(x) \equiv \sqrt{(p-A(x))^2 + m^2} + \Phi(x),$$

$(p, x) \in \mathbb{R}^d \times \mathbb{R}^d$.

Here measurable functions $A: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\Phi: \mathbb{R}^d \rightarrow \mathbb{R}$ are respectively the vector and scalar potentials of the field. For $A(x)$ and $\Phi(x)$ as general as possible, we want to define the Weyl quantized relativistic Hamiltonian $H = H_A + \Phi$ corresponding to (1.1). Φ may be defined as the multiplication operator $\Phi(x) \times$ by the function $\Phi(x)$. But how does one define H_A corresponding to the symbol $h_A(p, x)$? Indeed, if $A \in \mathcal{B}^\infty$, H_A may be defined as the Weyl pseudo-differential operator H_A^W :

$$(1.2) \quad (H_A^W u)(x) = (2\pi)^{-d} \iint e^{i(x-y)p} \sqrt{(p-A(\frac{x+y}{2}))^2 + m^2} u(y) dy dp,$$

$u \in \mathcal{S}(\mathbb{R}^d)$.

The right-hand side of (1.2) exists as an oscillatory integral,

so that H_A^W defines a symmetric operator in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$. It can be shown [4] with the general theory of Shubin [12] that H_A^W is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$. How about the case for a more general $A(x)$ which is not necessarily smooth and bounded? This question is motivated by an inspection of the path integral representation, obtained in [4], for the semi-group $\exp[-tH_A^W]$: for $g \in L^2(\mathbb{R}^d)$,

$$(1.3) \quad \left(\exp[-t(H_A^W - m)] g \right)(x) = \int e^{-iS(t,X)} g(X(t)) d\lambda_X(X),$$

$$\begin{aligned} \text{with} \quad S(t,X) = & \int_0^{t+} \int_{|y|>0} A(X(s-)+y/2) \cdot y \tilde{N}_X(dsdy) \\ & + \int_0^t \int_{|y|>0} [A(X(s)+y/2) - A(X(s))] \cdot y dsn(dy). \end{aligned}$$

Here $n(dy)$ is a σ -finite measure on $\mathbb{R}^d \setminus \{0\}$, called the Lévy measure, which behaves as $O(|y|^{-(d+1)})dy$ near $y=0$, and is, on $\{|y| \geq 1\}$, a bounded measure. Hence the right-hand side of (1.3) makes sense, at least, if $A(x)$ is locally Hölder continuous. This suggests that there may be an alternative definition of the Weyl quantized relativistic Hamiltonian H_A corresponding to the classical symbol $h_A(p,x)$ which is still valid for general $A(x)$. In the present lecture we shall give a survey of our recent results [2], [3] on this matter. Finally we quickly explain here the other notations in (1.3). λ_X is a probability measure on the space $D_X([0, \infty) \rightarrow \mathbb{R}^d)$ of the right-continuous paths $X : [0, \infty) \rightarrow \mathbb{R}^d$ having left-hand limits with $X(0)=x$. $\tilde{N}_X(dsdy)$ is a measure, depending on each path X , on $(0, \infty) \times (\mathbb{R}^d \setminus \{0\})$ defined by $\tilde{N}_X(dsdy) \equiv N_X(dsdy) - dsn(dy)$ with a counting measure

$$N_X((t, t'] \times B) = \#\{s \in (t, t']; X(s) - X(s-) \in B\},$$

where $0 < t < t'$ and B is a Borel set in $\mathbb{R}^d \setminus \{0\}$.

In Section 2 we give our definition of the Weyl quantized relativistic Hamiltonian H_A for general $A(x)$ and discuss the problem of its essential selfadjointness. The solution is reduced to establishing of an analogue of Kato's inequality between H_A and $\sqrt{-\Delta+m^2}$. Section 3 is devoted to an outline of proofs of the theorems. In Section 4 some remarks are given.

2. Definition of the Weyl Quantized Relativistic Hamiltonian and Theorems.

Unless otherwise specified, we assume that $A: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable and satisfies that

$$(2.1) \quad A(x) \text{ and } \int_{0 < |y| < 1} |A(x-y/2) - A(x)| |y|^{-d} dy$$

are locally bounded.

In particular, a locally Hölder continuous $A(x)$ satisfies (2.1).

We shall define the *Weyl quantized relativistic Hamiltonian* H_A corresponding to the classical symbol $h_A(p, x)$ as follows.

Definition.

$$(2.2) \quad (H_A u)(x) = mu(x) - \int_{|y| > 0} [e^{-iyA(x+y/2)} u(x+y) - u(x) - I_{\{|y| < 1\}} y (\partial_x - iA(x)) u(x)] n(dy),$$

$u \in \mathcal{G}(\mathbb{R}^d).$

Here $I_{\{|y| < 1\}}$ is the characteristic function of the set $\{|y| < 1\}$.

The Lévy measure $n(dy)$ is given by

$$(2.3) \quad n(dy) = \begin{cases} C(d) m^{(d+1)/2} |y|^{-(d+1)/2} K_{(d+1)/2}(m|y|) dy, & m > 0, \\ C'(d) |y|^{-(d+1)} dy, & m = 0, \end{cases}$$

where $C(d)$ and $C'(d)$ are constants depending on the dimension d , and $K_\nu(z)$ is the modified Bessel function of the third kind of

order ν . One can directly calculate (2.3), using the fact [7] that $t^{-1}k_0(t,y)dy \rightarrow n(dy)$ as $t \downarrow 0$, where $k_0(t,x-y)$ is the kernel of the operator $\exp[-t(\sqrt{-\Delta+m^2}-m)]$.

Lemma 1. H_A is a symmetric operator in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^d)$, and write

$$\begin{aligned} (2.4) \quad (H_A u)(x) &= mu(x) - \int_{|y| \geq 1} [e^{-iyA(x+y/2)} u(x+y) - u(x)] n(dy) \\ &\quad - \int_{0 < |y| < 1} [e^{-iyA(x+y/2)} u(x+y) - u(x) - y(\partial_x^{-1} A(x)) u(x)] n(dy) \\ &\equiv mu + I_1 u + I_2 u. \end{aligned}$$

Noting (2.3), we can show that I_1 is a bounded linear operator on $L^2(\mathbb{R}^d)$ and that $I_2 u$ is a continuous function with compact support, and for every compact $K_4 \subseteq \mathbb{R}^d$ there exists a constant C_K such that, for $u \in C_0^\infty(\mathbb{R}^d)$ with $\text{supp } u \subseteq K$,

$$(2.5) \quad \|I_2 u\|_2 \leq C_K [\|u\|_\infty + \|\partial u\|_\infty + \|\partial \partial u\|_\infty].$$

To show H_A is symmetric we have to show that for I_1 and I_2 . It is seen that $(I_1 u, v) = (u, I_1 v)$, $u, v \in C_0^\infty(\mathbb{R}^d)$, by change of variables and by invariance of $n(dy)$ under the transformation $y \rightarrow -y$. Similarly, I_2 is symmetric, if we note

$$(I_2 u)(x) = - \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \leq |y| < 1} [e^{-iyA(x+y/2)} u(x+y) - u(x)] n(dy). \quad \square$$

Next, we shall explain where the definition (2.2) of H_A comes from and see that H_A coincides with the Weyl pseudo-differential operator H_A^W , (1.2), if $A(x)$ satisfies, for instance,

$$(2.6) \quad A \in C^\infty, \quad |\partial^\alpha A(x)| \leq C_\alpha, \quad |\alpha| \geq 1.$$

Notice that the condition (2.6), which is a little more general than $A \in B^\infty$, includes the physically important case of constant magnetic fields : $A(x) = A \cdot x$ with A a real constant matrix.

Our starting point is the Lévy-Khinchin formula ([6],[11])

$$(2.7) \quad \sqrt{p^2 + m^2} = m - \int_{|y|>0} [e^{ipy} - 1 - I_{\{|y|<1\}} ipy] n(dy).$$

Let $u \in \mathcal{G}(\mathbb{R}^d)$. Multiply both sides of (2.7) by the Fourier transform $\hat{u}(p)$ of u and make the inverse Fourier transform. Then

$$(2.8) \quad \left(\sqrt{-\Delta + m^2} u \right)(x) = mu(x) - \int_{|y|>0} [u(x+y) - u(x) - I_{\{|y|<1\}} y \partial_x u(x)] n(dy).$$

First note with $H_0 \equiv \sqrt{-\Delta + m^2}$ that when $A(x) \equiv 0$, (2.8) is consistent with (2.2). On the other hand, if $A(x)$ satisfies (2.6), we can rewrite (1.2) as oscillatory integrals, by changing the variables $p - A\left(\frac{x+y}{2}\right) = p'$ (writing p again instead of p'), to get

$$\begin{aligned} (H_A^W u)(x) &= (2\pi)^{-d} \iint \exp \left[i(x-y) \cdot \left(p + A\left(\frac{x+y}{2}\right) \right) \right] \sqrt{p^2 + m^2} u(y) dy dp \\ &= \left(\sqrt{-\Delta + m^2} \left(\exp \left[i(x-\cdot) \cdot A\left(\frac{x+\cdot}{2}\right) \right] u(\cdot) \right) \right)(x). \end{aligned}$$

Since, for x fixed, the function $y \rightarrow \exp \left[i(x-y) \cdot A\left(\frac{x+y}{2}\right) \right] u(y)$ belongs to $\mathcal{G}(\mathbb{R}^d)$, we see in virtue of (2.8) that the above last formula equals $H_A u$, concluding that $H_A^W = H_A$ on $\mathcal{G}(\mathbb{R}^d)$. Thus we have shown

Lemma 2. If $A(x)$ satisfies (2.6), then, for $u \in \mathcal{G}(\mathbb{R}^d)$,

$$(2.9) \quad (H_A u)(x) = (H_A^W u)(x) = \left(\sqrt{-\Delta + m^2} \left(\exp \left[i(x-\cdot) \cdot A\left(\frac{x+\cdot}{2}\right) \right] u(\cdot) \right) \right)(x).$$

Remarks 1^o. The relation (2.9) says that apply H_A or H_A^W to

u amounts to the same thing as apply the free quantum Hamiltonian $\sqrt{-\Delta+m^2}$ to the appropriately "gauge transformed" u . Of course, the same is valid for the Schrödinger operator with magnetic fields:

$$(-i\partial-A(x))^2 u(x) = \left(-\Delta \left(\exp \left[i(x-\cdot) \cdot A \left(\frac{x+\cdot}{2} \right) \right] u(\cdot) \right) \right)(x), \quad u \in \mathcal{G}(\mathbb{R}^d).$$

2°. The expression (2.2) of H_A can also be obtained by calculating, through Itô's formula (e.g. [6]), the generator of the semigroup represented by path integral (1.3).

The main results are the following two theorems.

Theorem 1. Suppose that $A(x)$ satisfies (2.1) and $\Phi \in L^2_{loc}(\mathbb{R}^d)$, $\Phi(x) \geq 0$ a.e. Then

- (i) $H_A + \Phi$ is essentially selfadjoint on $C^\infty_0(\mathbb{R}^d)$.
- (ii) The selfadjoint extension of H_A , denoted again by the same H_A , is bounded from below: $H_A \geq m$.

Remark. Nagase-Umeda [10] have shown that if $A(x)$ satisfies (2.6), the Weyl pseudo-differential operator H_A^W is essentially selfadjoint.

Theorem 1-(i) can be shown in just the same way as in Kato [8], if an analogue of Kato's inequality (as in Theorem 2 below) is established. Notice that $\left(\sqrt{-\Delta+m^2} + 1 \right)^{-1}$ is positivity preserving. The proof of Theorem 2-(ii) follows from the proof of Theorem 2.

Now, for $u \in L^2(\mathbb{R}^d)$, define a distribution $H_A u \in \mathcal{D}'(\mathbb{R}^d)$ by

$$(2.10) \quad (H_A u, \varphi) = (u, H_A \varphi), \quad \varphi \in C^\infty_0(\mathbb{R}^d).$$

Here note with (2.4) that $\|I_1 \phi\|_2 \leq C \|\phi\|_2$ and (2.5).

Theorem 2 (Kato's inequality). Suppose $A(x)$ satisfies (2.1).

If $u \in L^2$ and $H_A u \in L^1_{loc}$, then the following distributional inequality holds:

$$(2.11) \quad \operatorname{Re}[(\operatorname{sgn} u) H_A u] \geq \sqrt{-\Delta + m^2} |u|.$$

$$\text{with} \quad (\operatorname{sgn} u)(x) = \begin{cases} \overline{u(x)} / |u(x)|, & u(x) \neq 0 \\ 0, & u(x) = 0. \end{cases}$$

3. Outline of Proofs of Theorem 2 and Theorem 1-(ii).

In the proof it is crucial that H_A is represented as an integral operator (2.2).

(First Step) Let $u \in C^\infty \cap L^2$, and put $u_\varepsilon(x) = \sqrt{|u(x)|^2 + \varepsilon^2}$, $\varepsilon > 0$. Then u_ε is C^∞ . Since $-|v(x)||v(x+y)| + |v(x)|^2 \geq -v_\varepsilon(x)v_\varepsilon(x+y) + v_\varepsilon(x)^2$, and $\partial|u(x)|^2 = \partial u_\varepsilon(x)^2$, we have (writing, for simplicity, $((H_A - m)u)(x)$ and $((H_O - m)u_\varepsilon)(x)$ as $(H_A - m)u(x)$, and $(H_O - m)u_\varepsilon(x)$, respectively)

$$\begin{aligned} (3.1) \quad \operatorname{Re}[\overline{u(x)}(H_A - m)u(x)] &= 2^{-1} \{ \overline{u(x)}(H_A - m)u(x) + u(x)\overline{(H_A - m)u(x)} \} \\ &= \frac{1}{2} \int_{|y|>0} - \left(\overline{u(x)} [e^{-iyA(x+y/2)} u(x+y) - u(x) - I_{\{|y|<1\}} y (\partial_x - iA(x)) u(x)] \right. \\ &\quad \left. + u(x) [e^{iyA(x+y/2)} \overline{u(x+y)} - \overline{u(x)} - I_{\{|y|<1\}} y (\partial_x + iA(x)) \overline{u(x)}] \right) n(dy) \\ &\geq \int_{|y|>0} [-|u(x)||u(x+y)| + |u(x)|^2 + 2^{-1} I_{\{|y|<1\}} y \partial |u(x)|^2] n(dy). \\ &\geq \int_{|y|>0} [-u_\varepsilon(x)u_\varepsilon(x+y) + u_\varepsilon(x)^2 + 2^{-1} I_{\{|y|<1\}} y \partial u_\varepsilon(x)^2] n(dy) \end{aligned}$$

$$= u_\varepsilon(x)(H_0 - m)u_\varepsilon(x),$$

pointwise. Integrating the first and last members of (3.1) yields $\operatorname{Re}((H_A - m)u, u) \geq ((H_0 - m)u_\varepsilon, u_\varepsilon) \geq 0$. This proves Theorem 1-(ii), since H_A is symmetric by Lemma 1.

On the other hand, dividing the first and last members of (3.1) by u_ε yields

$$(3.2) \quad \operatorname{Re}[(\overline{u(x)}/u_\varepsilon(x))(H_A - m)u] \geq (H_0 - m)u_\varepsilon,$$

pointwise and so in the distribution sense.

(Second Step) For general u , let $u^\delta = \rho_\delta * u$, where ρ_δ is Friedrichs' mollifier. We obtain from (3.2)

$$(3.3) \quad \operatorname{Re}\left[\left(\overline{u^\delta}/(u^\delta)_\varepsilon\right)(H_A - m)u^\delta\right] \geq (H_0 - m)(u^\delta)_\varepsilon,$$

where $(u^\delta)_\varepsilon = (|u^\delta|^2 + \varepsilon^2)^{1/2}$, $\varepsilon > 0$. We let $\delta \downarrow 0$ first and then $\varepsilon \downarrow 0$. As $\delta \downarrow 0$, we have (by taking a subsequence if necessary) $u^\delta \rightarrow u$ in L^2 and a.e. so that $(u^\delta)_\varepsilon \rightarrow u_\varepsilon$ in L^2 and a.e. It follows that

$\{\overline{u^\delta}/(u^\delta)_\varepsilon\}$ is bounded and converges to $\overline{u}/u_\varepsilon$ a.e. and $H_0(u^\delta)_\varepsilon \rightarrow H_0 u_\varepsilon$ in $\mathcal{D}'$. For the moment, suppose that

$$(3.4) \quad H_A u^\delta \rightarrow H_A u \quad \text{in } L^1_{\text{loc}}, \quad \delta \downarrow 0.$$

Then the left-hand side of (3.3) converges in L^1_{loc} . Thus we get

$$(3.5) \quad \operatorname{Re}[(\operatorname{sgn} u)(H_A - m)u] \geq (H_0 - m)|u|,$$

in the distribution sense. Finally let $\varepsilon \downarrow 0$. The left-hand side of (3.5) converges to $\operatorname{Re}[(\operatorname{sgn} u)(H_A - m)u]$ a.e., while the right-hand side to $(H_0 - m)|u|$ in $\mathcal{D}'$. $\square$

To prove the remaining assertion (3.4), we need regularity of a function $u \in L^2$ with $H_A u \in L^1_{\text{loc}}$ as in the following lemma.

Lemma 3. If $u \in L^2$ and $H_A u \in L^1_{loc}$, then u has a decomposition $u = u_1 + u_2$ such that, for every $\psi \in C^\infty_0(\mathbb{R}^d)$,

$$\psi u_1, H_O \psi u_1 \in L^1, \text{ and } \psi u_2, H_O \psi u_2 \in L^2.$$

First we prove (3.4). By (2.4), $H_A = m + I_1 + I_2$. Let $u \in L^2$ and $H_A u \in L^1_{loc}$. Since I_1 is a bounded operator on $L^2(\mathbb{R}^d)$, we have $I_1 u \in L^2$, and hence $I_2 u \in L^1_{loc}$. Since $I_1 u^\delta \rightarrow I_1 u$ in L^2 as $\delta \downarrow 0$, we have only to show $I_2 u^\delta \rightarrow I_2 u$ in L^1_{loc} . It is

clear that $I_2 u^\delta \rightarrow I_2 u$ in $\mathcal{D}'$. Therefore it suffices to show

$$(3.6) \quad I_2 u^\delta - I_2 u^{\delta'} \rightarrow 0 \quad \text{in } L^1_{loc}, \quad \delta, \delta' \downarrow 0.$$

To see (3.6), first note that for every compact $K \subseteq \mathbb{R}^d$ there is a constant C_K such that, for $\varphi \in C^\infty_0(\mathbb{R}^d)$ with $\text{supp } \varphi \subseteq K_4$,

$$(3.7) \quad \|I_2 \varphi\|_{1,K} \equiv \int_K |I_2 \varphi| dx \leq C_K [\|H_O \varphi\|_1 + \|\varphi\|_2], \quad i=1,2,$$

where $K_r = \{x; \text{dist}(x, K) \leq r\}$. Next let $\psi \in C^\infty_0(\mathbb{R}^d)$, $0 \leq \psi(x) \leq 1$, with $\psi(x)=1$ on K_2 and $\text{supp } \psi \subseteq K_3$. By Lemma 3, $u = u_1 + u_2$. ψu and ψu_2 are L^2 , and so is ψu_1 . If $0 < \delta < 1$, the $(\psi u_i)^\delta$ satisfy the condition for φ in (3.7). We have, for $i=1,2$, $H_O(\psi u_i)^\delta = (H_O \psi u_i)^\delta \in L^1$, and $I_2 u_i^\delta = I_2(\psi u_i)^\delta$ on K . Then, by (3.7),

$$\begin{aligned} & \|I_2 u^\delta - I_2 u^{\delta'}\|_{1,K} \\ & \leq C_K \sum_{i=1}^2 \left[\|(H_O \psi u_i)^\delta - (H_O \psi u_i)^{\delta'}\|_1 + \|(\psi u_i)^\delta - (\psi u_i)^{\delta'}\|_2 \right], \end{aligned}$$

whence follows (3.6).

The proof of Lemma 3 needs task. We establish a kind of integral representation for $u \in L^2$ with $H_A u \in L^1_{loc}$ (cf. [5, Appendix]). We get from (2.9)

$$((H_A+1)u, \varphi) = (u, (H_A+1)\varphi), \quad \varphi \in C_0^\infty(\mathbb{R}^d).$$

Take $\varphi(y) = G_\varepsilon(x-y)$ with

$$G_\varepsilon(x) = (2\pi)^{d/2} \chi(x/R) \mathcal{F}^{-1} \left(\frac{\exp[-\varepsilon(\sqrt{p^2+m^2}+1)]}{\sqrt{p^2+m^2}+1} \right)(x), \quad \varepsilon \geq 0,$$

where $\chi \in C_0^\infty(\mathbb{R}^d)$ and $R > 0$ ($\mathcal{F}^{-1}$ denotes the inverse Fourier transform). Then

$$(3.8) \quad ((H_A+1)u, G_\varepsilon(x-\cdot)) = (u, (H_A+1)G_\varepsilon(x-\cdot)).$$

Write

$$((H_A+1)G_\varepsilon(x-\cdot))(y) = ((H_0+1)G_\varepsilon(x-\cdot))(y) - \overline{E_\varepsilon(x,y)} - \overline{F_\varepsilon(x,y)}$$

and let $\varepsilon \downarrow 0$. Then the right-hand side of (3.8) converges to $u - Qu - Eu - Fu$, while the left-hand side of (3.8) converges to $G[H_A+1]u$, both in L_{loc}^1 . Thus $u = G[H_A+1]u + Qu + Eu + Fu$. Here Q , E and F are certain integral operators, and G is the one with kernel $G_0(x-y)$. Then Lemma 3 follows by studying the kernels of G , Q , E and F with the aid of the theory of singular integrals. $\square$

4. Concluding Remarks.

1°. Our Weyl quantized relativistic Hamiltonian H_A generally differs from the square root of the nonnegative selfadjoint operator $(-i\partial - A(x))^2 + m^2$:

$$H_A \neq \sqrt{(-i\partial - A(x))^2 + m^2}.$$

They coincide for $A(x) = A \cdot x$, with A a real symmetric constant matrix. This can be seen with the composition formula for Weyl pseudo-differential operators (e.g. [1, p.151-2]).

However, we shall not discuss which is physically more appropriate as a relativistic quantum Hamiltonian of a spinless

particle. H_A suits better from the path integral point of view, because H_A has an exact classical symbol $h_A(p, x)$ as a Weyl pseudo-differential operator (cf. [9]). But H_A is not gauge-invariant, though $\sqrt{(-i\partial - A(x))^2 + m^2}$ is.

2°. When $A(x) \equiv 0$, Theorem 2 turns out: If $u \in L^2$ and $\sqrt{-\Delta + m^2} u \in L^1_{loc}$, then

$$(4.1) \quad \operatorname{Re}[(\operatorname{sgn} u) \sqrt{-\Delta + m^2} u] \geq \sqrt{-\Delta + m^2} |u|,$$

in the distribution sense. It appears that Theorem 2 should follow immediately from (4.1) and (2.9) by substituting the function $\exp\left[i(x-\cdot)A\left(\frac{x+\cdot}{2}\right)\right]u(\cdot)$ into u in (4.1). However, it is a problem whether (2.9) is true for $A(x)$ not satisfying (2.6) or $u(x)$ not belonging to $\mathcal{G}(\mathbb{R}^d)$.

3°. An analogue of Kato's inequality will be shown for the operator L corresponding to the Lévy process (e.g. [13]):

$$\begin{aligned} (Lu)(x) = & - \left[\sum_{j,k=1}^d \partial_j a_{jk}(x) \partial_k + \sum_{j=1}^d b_j(x) \partial_j + c(x) \right] u(x) \\ & - \int_{|y|>0} [u(x+y) - u(x) - I_{\{|y|<1\}} y \partial u(x)] n(x, dy). \end{aligned}$$

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