

Statics of Powdered Material Under Compression

By

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The behavior of powdered material under compression seems not to obey the theories the statics of rigid or elastic and the hydrostatics develop. Attempts have been made to establish the theories suitable for powdered material, e. g., by Coulomb, Rankine, Krey and others, but these are applicable only for special case and they lack mathematical strictness.

We made experiments and found the general characteristics of powdered material under compression. Using these results we tried to derive the fundamental differential equation of stress equilibrium and to solve the equation for several cases.

1. Characteristics of Powdered Material under Compression.

For the convenience we described here notation used in the following.

$\epsilon_1, \epsilon_2, \epsilon_3$	principal strain
u, v, w	displacement in the direction of x, y, z -axes
$\sigma_x, \sigma_y, \sigma_z$	normal stress in the direction of x, y, z -axes
τ_x, τ_y, τ_z	shearing stress in the direction of x, y, z -axes
$\sigma_r, \sigma_t, \sigma_z$	normal stress in the direction of r, t, z -axes
τ_r, τ_t, τ_z	shearing stress in the direction of r, t, z -axes
P	load applied on powdered material
P_0	initial load on powdered material
E_1	constant determined by initial load
k, C	constant determined by the characteristics of powdered material
$l_1, m_1, n_1, l_2, m_2, n_2, l_3, m_3, n_3$	direction cosines
h	height of compressed powdered material after once compressed

We filled powdered material, e. g., in a rigid cylinder shown in Fig. 1, and pressed from the top. An example of the load-deformation diagram is shown

in Fig. 2. If we show this relation in a semi-logarithmic diagram it becomes a straight line shown in Fig. 3 and we get the next formula from the diagram,

$$P + P_0 = P_0 e^{k' \epsilon_s} \quad (1)$$

where k' is a certain constant. Before we fill the powder we divided those powder into ten equal parts and after filling every one part we pressed from the top under the load P_0 . The reason is that if we fill all powder at the same time there may be bridge action and some other phenomena which disturb the homogeneity of powder. We tested about powder of copper, aluminium, iron and others and get the similar result. It is obvious that the friction between

powder and wall of the cylinder influences upon the relation. So we made the same test changing the friction by painting graphite, wax and others on the wall. We got the similar results except the value of k' . From these results we can attribute that even if there is no friction between the wall and powder the formula (1) is correct. When there is

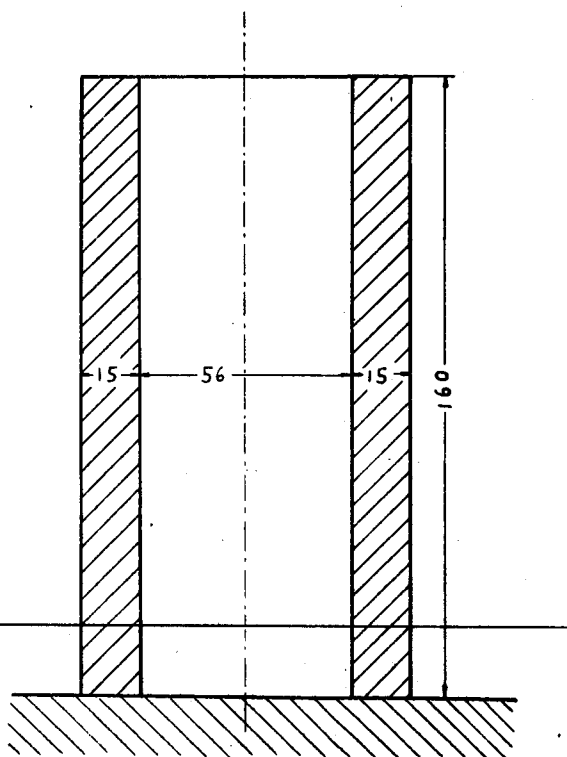


Fig. 1

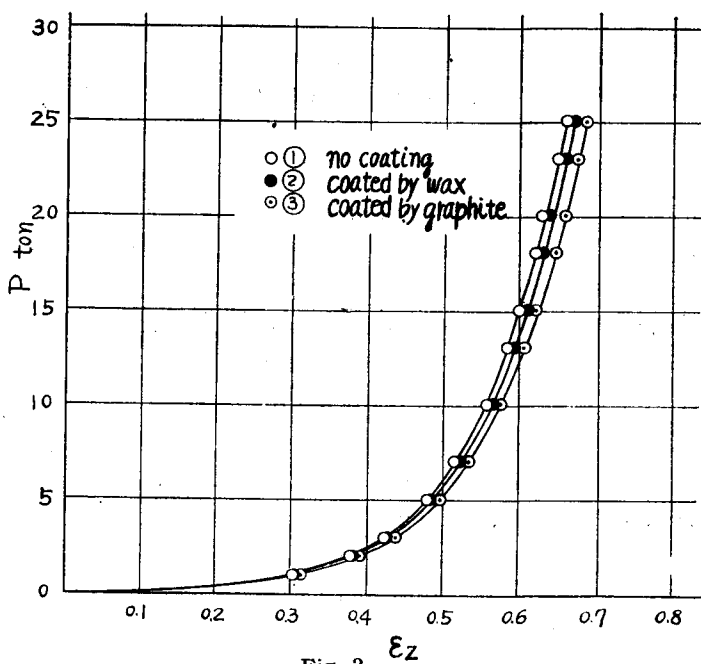


Fig. 2

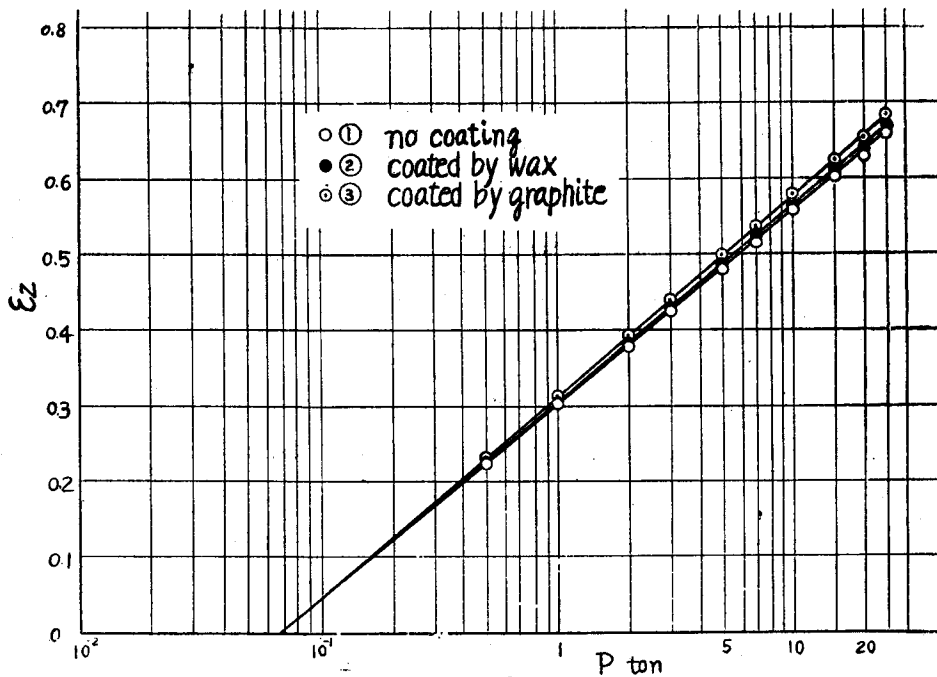


Fig. 3

no friction we use k instead of k' and we get

$$P + P_0 = P_0 e^{kz} \quad (2)$$

As there is no friction on the wall the stress distribution is same as in the part shown by the dotted line in Fig. 4, i. e. a part of semi-infinitely distributed powdered material subjected to uniform load at the boundary. So, in the cylinder, the stress in the same direction have the same magnitude. The stress in such a case was calculated by Rankine correctly. The normal stresses in r , t and z direction are principal stresses and they have the relation

$$\sigma_1 = \sigma_2 = C\sigma_3 = \frac{1 - \sin \varphi}{1 + \sin \varphi} \sigma_3 \quad (3)$$

where φ is angle of internal friction. As φ is constant and the formula (3) determines the relation between σ_1 , σ_2 and σ_3 uniquely. We divide (2) by the area F of the cross section of the cylinder and we get

$$\sigma_3 + E_1 = E_1 e^{kz} \quad (4)$$

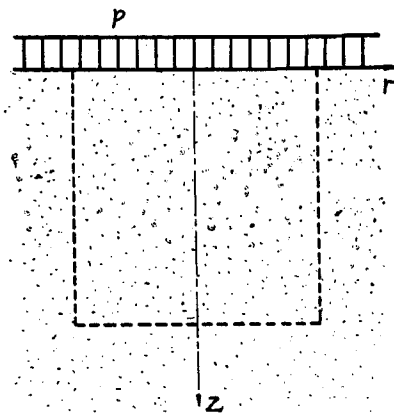


Fig. 4

where E_1 is initial stress corresponding to P_0 . From (4) we get

$$C(\sigma_3 + E_1) = \sigma_1 + CE_1 = \sigma_2 + CE_1 = CE_1 e^{k\varepsilon_3} \quad (5)$$

If we take E_1 constant, $\sigma_1 + CE_1$, $\sigma_2 + CE_1$, $\sigma_3 + E_1$ are proportional to $e^{k\varepsilon_3}$. So we may assume

$$\alpha(\sigma_3 + E_1) + \beta(\sigma_1 + CE_1) + \gamma(\sigma_2 + CE_1) = e^{k\varepsilon_3} \quad (6)$$

As we can consider the powder in the cylinder is isotropic and homogeneous before compressed, we can get the next formulae from the same idea as in the formula (6)

$$\left. \begin{aligned} \alpha(\sigma_1 + E_1) + \beta(\sigma_2 + CE_1) + \gamma(\sigma_3 + CE_1) &= e^{k\varepsilon_1} \\ \alpha(\sigma_2 + E_1) + \beta(\sigma_3 + CE_1) + \gamma(\sigma_1 + CE_1) &= e^{k\varepsilon_2} \end{aligned} \right\} \quad (7)$$

β should be equal to γ as even if we change the co-ordinate axes the formulae (6) and (7) are true. That is

$$\alpha(\sigma_1 + E_1) + \beta\{(\sigma_2 + CE_1) + (\sigma_3 + CE_1)\} = e^{k\varepsilon_1} \quad (8)$$

$$\alpha(\sigma_2 + E_1) + \beta\{(\sigma_3 + CE_1) + (\sigma_1 + CE_1)\} = e^{k\varepsilon_2} \quad (9)$$

$$\alpha(\sigma_3 + E_1) + \beta\{(\sigma_1 + CE_1) + (\sigma_2 + CE_1)\} = e^{k\varepsilon_3} \quad (10)$$

To simplify these relations we put the formulae (3) into (8) and (10) and we get

$$\{(a + \beta)C + \beta\}\sigma_3 + (a + 2\beta C)E_1 = e^{k\varepsilon_1} \quad (11)$$

$$(a + 2\beta C)(\sigma_3 + E_1) = e^{k\varepsilon_3} \quad (12)$$

In this case as $\varepsilon_1 = \varepsilon_2 = 0$, we get the next relation from (11)

$$\{(a + \beta)C + \beta\}\sigma_3 + \{(a + 2\beta C)E_1 - 1\} = 0 \quad (13)$$

In order to content (13) for any magnitude of σ_3 , we obtain

$$(a + \beta)C + \beta = 0 \quad (14)$$

$$(a + 2\beta C)E_1 = 1 \quad (15)$$

From (15)

$$a + 2\beta C = \frac{1}{E_1} \quad (16)$$

and from (14), (16) we obtain

$$a = -\frac{1}{E_1} \frac{C + 1}{(2C + 1)(C - 1)} \quad (17)$$

$$\beta = -\frac{1}{E_1} \frac{C}{(2C + 1)(C - 1)} \quad (18)$$

That is, a, β are function of E_1 and C , and if E_1 is constant a, β are constant according to the kind of powdered material. Substituting (15) in (8), (9) and

(10), we get

$$a\sigma_1 + \beta(\sigma_2 + \sigma_3) + 1 = e^{k\varepsilon_1} \quad (19)$$

$$a\sigma_2 + \beta(\sigma_3 + \sigma_1) + 1 = e^{k\varepsilon_2} \quad (20)$$

$$a\sigma_3 + \beta(\sigma_1 + \sigma_2) + 1 = e^{k\varepsilon_3} \quad (21)$$

or

$$\varepsilon_1 = \frac{1}{k} \log \{a\sigma_1 + \beta(\sigma_2 + \sigma_3) + 1\} \quad (22)$$

$$\varepsilon_2 = \frac{1}{k} \log \{a\sigma_2 + \beta(\sigma_3 + \sigma_1) + 1\} \quad (23)$$

$$\varepsilon_3 = \frac{1}{k} \log \{a\sigma_3 + \beta(\sigma_1 + \sigma_2) + 1\} \quad (24)$$

Solving these equations concerning σ we get

$$\sigma_1 = \frac{(a + \beta)e^{k\varepsilon_1} - \beta(e^{k\varepsilon_2} + e^{k\varepsilon_3}) - (a - \beta)}{(a - \beta)(a + 2\beta)} \quad (25)$$

and by the relation

$$\frac{a + \beta}{(a - \beta)(a + 2\beta)} = E_1, \quad \frac{\beta}{a + \beta} = C$$

we obtain finally

$$\sigma_1 = E_1 \{e^{k\varepsilon_1} + C(e^{k\varepsilon_2} + e^{k\varepsilon_3} - 2) - 1\} \quad (26)$$

$$\sigma_2 = E_1 \{e^{k\varepsilon_2} + C(e^{k\varepsilon_3} + e^{k\varepsilon_1} - 2) - 1\} \quad (27)$$

$$\sigma_3 = E_1 \{e^{k\varepsilon_3} + C(e^{k\varepsilon_1} + e^{k\varepsilon_2} - 2) - 1\} \quad (28)$$

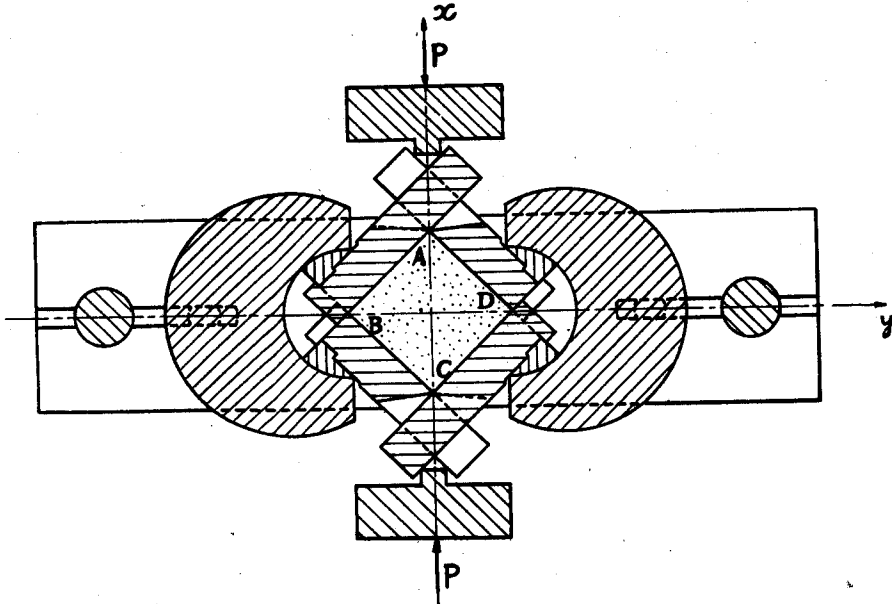


Fig. 5

These relations are reduced from the results of the experiment made for simple compression, where $\epsilon_1, \epsilon_2, \epsilon_3$ are positive or zero. Positive means here compressive strain. To ascertain that the relations are also correct when one or two of $\epsilon_1, \epsilon_2, \epsilon_3$ are negative, we made a further experiment. We used apparatus shown in Fig. 5. We filled powdered material in the square part $ABCD$. As the side plates AB, BC, CD, DA can turn very easily around A, B, C, D , $ABCD$ can change

its shape into rhombic shape corresponding to the magnitude of a load as shown in Fig. 6. The length of this apparatus is 14.5cm and $AB=BC=CD=DA=3\text{cm}$ and we have plates at the ends of the apparatus to prevent pouring out of the powder. As we can consider that the state of deformation at any point in the

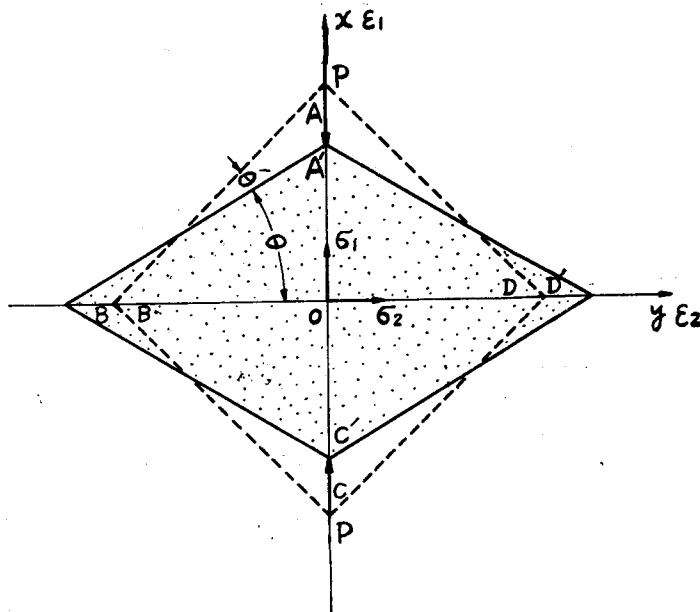


Fig. 6

rhombus is similar, the distribution of strain and stress may be assumed to be uniform in $ABCD$. Then the direction of x, y, z axes coincides with that of principal stresses. If we denote $\angle A'B'C' = \theta$ in Fig. 6, we get

$$\left. \begin{aligned} \epsilon_1 &= 1 - \sqrt{2} \sin \theta \\ \epsilon_2 &= 1 - \sqrt{2} \cos \theta \end{aligned} \right\} \quad (29)$$

From (26), (27), (29) we get

$$\left. \begin{aligned} \sigma_1 &= E_1 \left\{ e^{k(1-\sqrt{2} \sin \theta)} + C(e^{k(1-\sqrt{2} \cos \theta)} - 1) - 1 \right\} \\ \sigma_2 &= E_1 \left\{ e^{k(1-\sqrt{2} \cos \theta)} + C(e^{k(1-\sqrt{2} \sin \theta)} - 1) - 1 \right\} \end{aligned} \right\} \quad (30)$$

We obtain, after brief consideration, the load to be applied

$$P = \sigma f \cos \theta + 2\tau f \sin \theta - \sigma f \sin \theta \tan \theta \quad (31)$$

where σ and τ are normal and shearing stress at the four side plates $AB, BC,$

CD, DA , and f is the area of one plate. From the relation between σ, τ and σ_1, σ_2 we get

$$P = \frac{1}{2}f \left[\{(\sigma_1 + \sigma_2) + (\sigma_1 - \sigma_2) \cos 2\theta\} \left(\cos \theta - \sin 2\theta \frac{1}{\cos \theta} \right) + \frac{1}{2}(\sigma_1 - \sigma_2) \sin 2\theta \sin \theta \right] \quad (32)$$

Substituting (30) in (32) we can get the relation between P and θ . Using electrolytic copper powder we measured θ for each step of the load P and compared it to the result of calculation and found they coincide very closely as shown in Fig. 7. So if the relation

$$(1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) < 1 \quad (33)$$

is satisfied (26), (27), (28) are true even though one or two of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are negative.

The methods of obtaining k, C and E_1 are easy. We measured the relation between k' and the resistance to push out the powdered material from the cylinder after compression and drew diagram changing the magnitude of friction in the cylinder. The method of changing the friction of the cylinder wall is the same stated above. We extrapolated the curve and determined the value of k corresponding to k' for no friction. The value of C is

$$C = \frac{1 - \sin \varphi}{1 + \sin \varphi}$$

and E_1 can easily be found by the initial load.

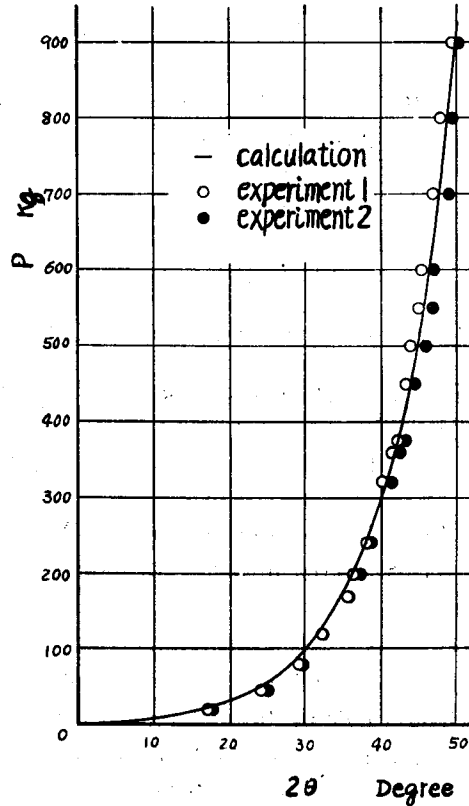


Fig. 7

2. Fundamental Differential Equation of Statics of Powdered Material

Generally the differential equations of stress equilibrium are

$$\left. \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial y} + \frac{\partial \tau_{xy}}{\partial z} &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \\ \frac{\partial \tau_{yz}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= 0 \end{aligned} \right\} \quad (35)$$

We denote the direction of 1, 2, 3 axes for principal stresses concerning x, y, z axes as follows.

	1	2	3
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

From the relation between principal stress and stress in any direction and the relations (26), (27), (28) we get

$$\sigma_x = E_1(1-C)\{l_1^2 e^{k\varepsilon_1} + m_1^2 e^{k\varepsilon_2} + n_1^2 e^{k\varepsilon_3}\} + CE_1(e^{k\varepsilon_1} + e^{k\varepsilon_2} + e^{k\varepsilon_3} - 2) - E_1$$

$$\tau_x = E_1(1-C)\{l_2 l_3 e^{k\varepsilon_1} + m_2 m_3 e^{k\varepsilon_2} + n_2 n_3 e^{k\varepsilon_3}\}$$

and similar four other formulae. Putting these formulae into (32) we get

$$\frac{\partial}{\partial x} \left[l_1^2 e^{k\varepsilon_1} + m_1^2 e^{k\varepsilon_2} + n_1^2 e^{k\varepsilon_3} + \frac{C}{1-C}(e^{k\varepsilon_1} + e^{k\varepsilon_2} + e^{k\varepsilon_3}) \right]$$

$$+ \frac{\partial}{\partial y} \left[l_1 l_2 e^{k\varepsilon_1} + m_1 m_2 e^{k\varepsilon_2} + n_1 n_2 e^{k\varepsilon_3} \right] + \frac{\partial}{\partial z} (l_3 l_1 e^{k\varepsilon_1} + m_3 m_1 e^{k\varepsilon_2} + n_3 n_1 e^{k\varepsilon_3}) = 0$$

and two other similar equations. We can easily transform these equations into the forms referred to polar, cylindrical co-ordinates and the form of two dimension.

3. Solution of the Differential Equations of Powdered Material

The fundamental differential equations are very difficult to solve and more over we can not use the boundary condition by displacement because we can not get any simple relation between strain and displacement. And as we can not prove the uniqueness of the solution we must ascertain that the solution obtained fits for our case or not. We show an example of solution in the case of electrolytic copper powder filled in a rigid cylinder and pressed from the top by the load P .

As it is convenient, in this case, to avail cylindrical co-ordinate we used it. By the condition of axial symmetry we can simplify the equation and we get

$$\frac{\partial}{\partial r} \left\{ (l_1'^2 e^{k\varepsilon_1} + n_1'^2 e^{k\varepsilon_3}) + \frac{C}{1-C}(e^{k\varepsilon_1} + e^{k\varepsilon_2} + e^{k\varepsilon_3}) \right\}$$

$$+ \frac{\partial}{\partial z} l_3' l_1' (e^{k\varepsilon_1} - e^{k\varepsilon_3}) + \frac{1}{r} \left\{ l_1'^2 e^{k\varepsilon_1} - e^{k\varepsilon_2} + (1 - l_1'^2) e^{k\varepsilon_3} \right\} = 0$$

$$\frac{\partial}{\partial r} l_3' l_1' (e^{k\epsilon_1} - e^{k\epsilon_3}) + \frac{\partial}{\partial z} \left[- \left\{ l_1'^2 e^{k\epsilon_1} + (1 - l_1'^2) e^{k\epsilon_3} \right\} \right. \\ \left. + \frac{1}{1-C} (e^{k\epsilon_1} + e^{k\epsilon_2}) + \frac{C}{1-C} e^{k\epsilon_2} \right] + \frac{1}{r} l_3' l_1' (e^{k\epsilon_1} - e^{k\epsilon_3}) = 0$$

where l_1', m_1' and n_1' are direction cosines of 1, 2, 3 axes for principal stresses referred to r, t, z axes.

We transform these equations into difference form and if we know the value of ϵ_2, l_1' and ϵ_1, ϵ_3 at the boundary we can solve the equations by trial. In this case we know that $\epsilon_2 = \frac{u}{r}$. In order to get the radial displacement u we insert pieces of black paper in the powder at equidistance before compression. They are cut in a shape shown in Fig. 8 and concentric circles are drawn by white paint on them. After pressing the powder we separate the consolidated powder along the surface of powder where the piece of paper inserted, and could see concentric circles shown in Fig. 9. Comparing these circles to the concentric circles on the paper before pressing, we got the approximate value of u , Table 1, and found it very small. Therefore ϵ_2 is so

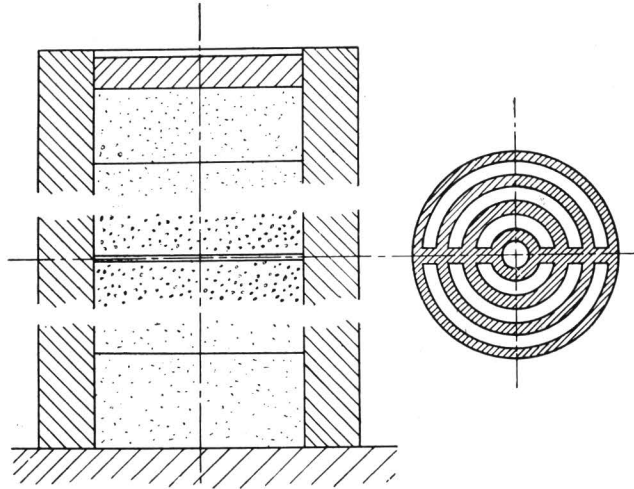


Fig. 8

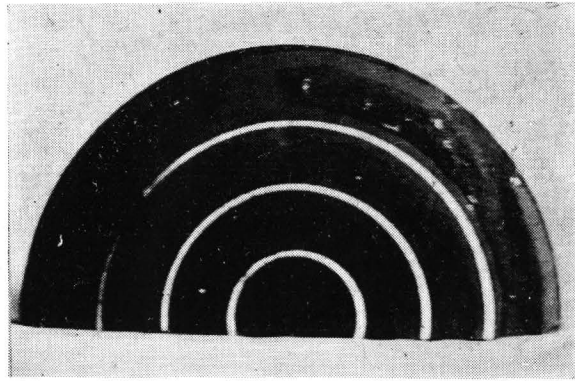


Fig. 9

$\begin{matrix} \backslash & r \\ z & \end{matrix}$	0	$\frac{a}{4}$	$\frac{a}{2}$	$\frac{3}{4}a$	a
0	0	0	0	0	0
$\frac{h}{4}$	0	- 0.35	- 0.45	- 0.45	0
$\frac{h}{2}$	0	- 0.30	- 0.40	- 0.50	0
$\frac{3}{4}h$	0	0	+ 0.30	0	0
h	0	0	0	0	0

Table 1 u mm

small that we can neglect it when compared with ε_3 . In order to determine direction cosine l_1' we insert several very thin white powder layers in the powder at equidistance before pressing. After compression we took out the consolidated powder and cut it by a diametral plane and found white bent lines as shown in Fig. 10. The tangent of these lines at any point shows one of the directions of principal strain. To verify this fact we inserted semi-spheres of white powder before pressing. After compression they deform into semi-ellipsoids and their longest axes are found to coincide with the direction of the tangent to the white lines, Fig. 10 and 11. Then the tangent to the bent lines may be regarded as the direction of principal strain and thus we have found the direction of l_1' . We assume that $\varepsilon_1=0$ on the axis of

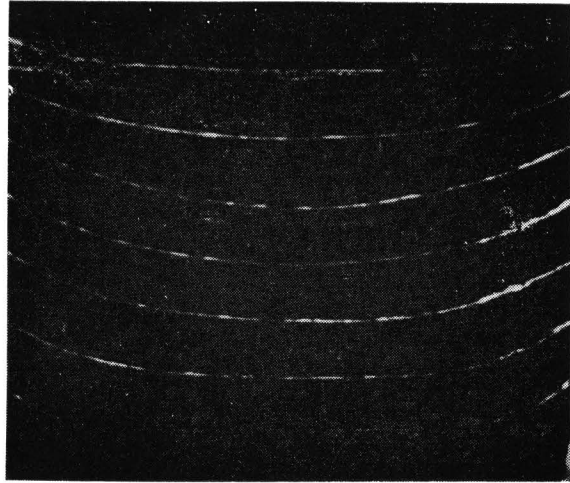


Fig. 10

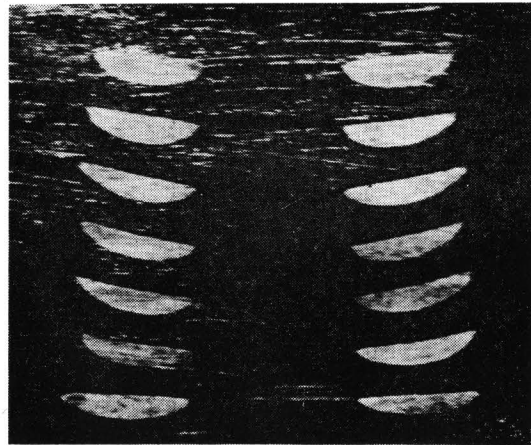


Fig. 11

z and ε_1 vanishes evidently on the planes $z=0, z=h$. On the points $z=0, z=h$ and $r=0$ we determined ε_3 from the distance of the bent lines shown in Fig. 10. Another boundary condition can be found on the line $r=r$ by the same idea stated above. Now we have determined necessary elements to solve the difference equation and we obtained as the result of calculation the distribution of $\sigma_r, \sigma_t, \sigma_z$ and τ shown in Fig. 12, 13, 14, 15. We may solve any other similar problem.

4. Discussion on the Results Obtained.

If we measured the hardness on the surface of the pressed powder the distri-

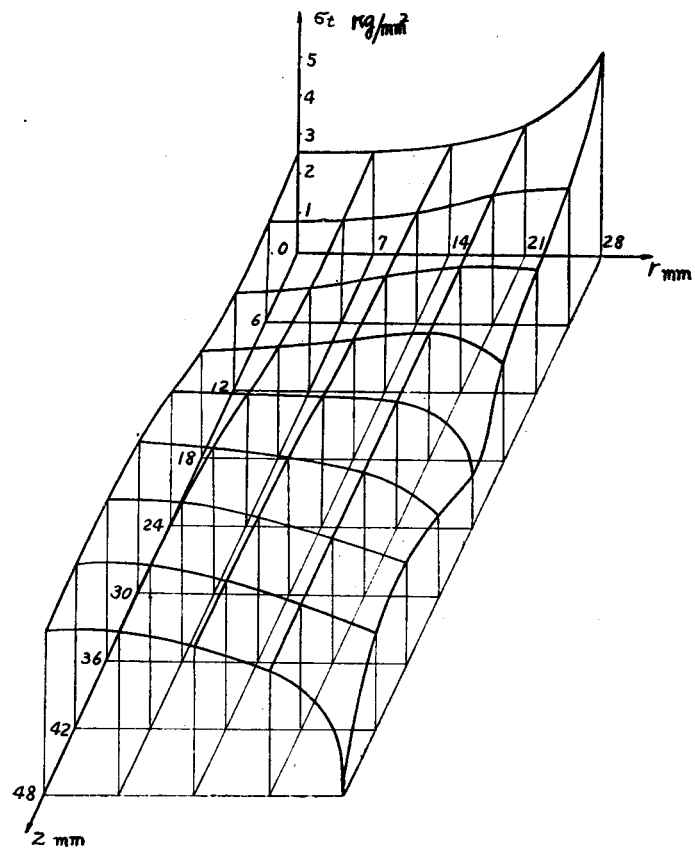


Fig. 13

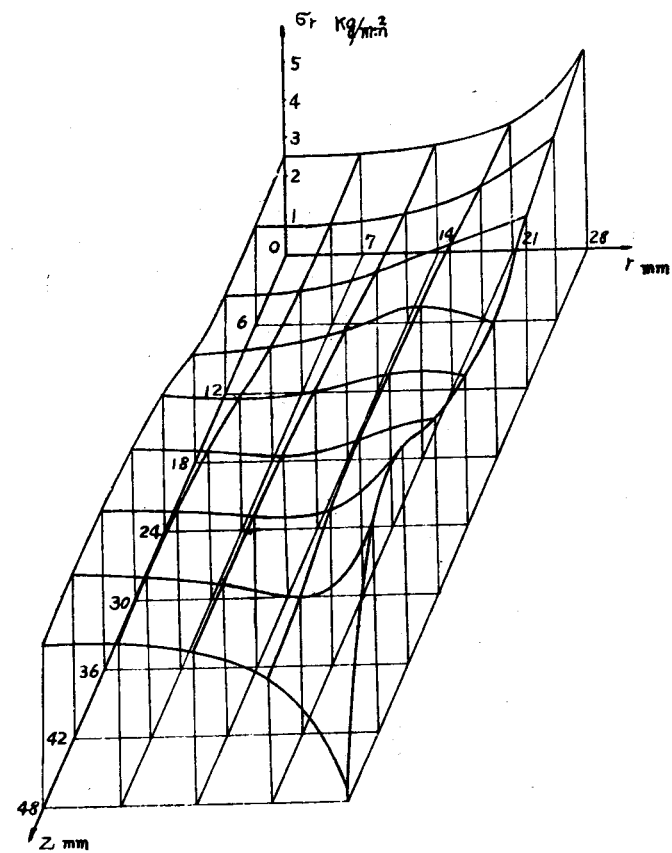


Fig. 12

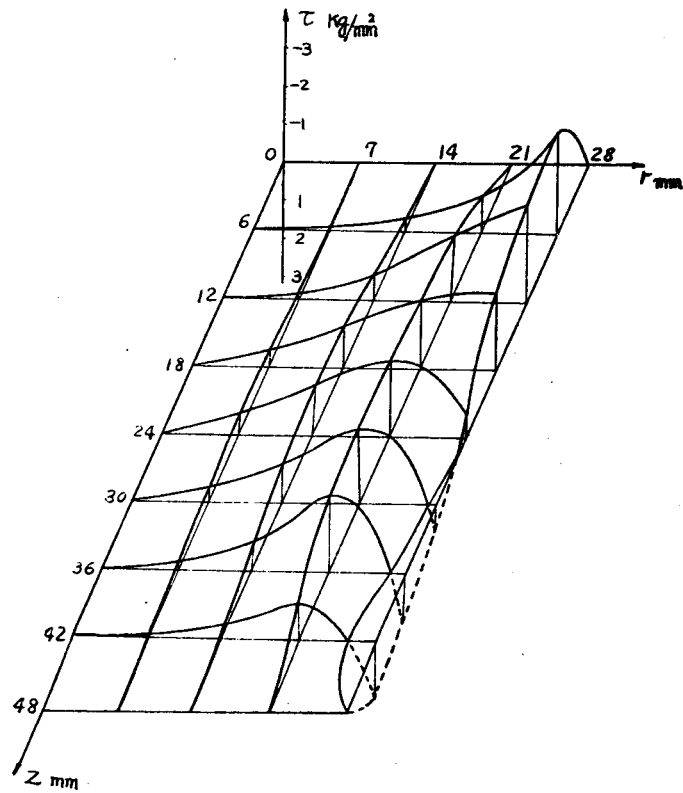


Fig. 15

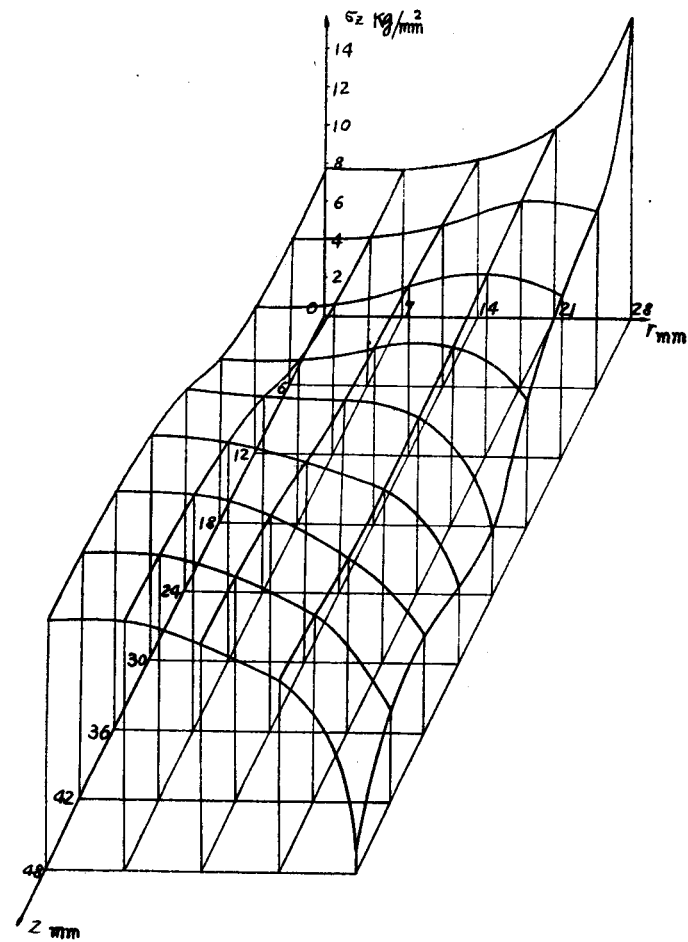


Fig. 14

bution of the hardness should show the similar tendency as those of $\sigma_r, \sigma_t, \sigma_z$. So we measured the hardness on its surface with Shore's hardness tester and the results obtained are shown in Fig. 16. Comparing the tendency of their distribution we may conclude that it is one of the clear evidence of showing the correctness of the calculation. We understand by this research that the distribution of stress in pressed powdered material is very complicated. To avoid this state and to obtain uniform distribution in the pressed material is a matter of significance and further investigation is necessary to solve the problem. The reduction of the friction on the wall of the casing and the compression from both sides, top and bottom, are available method at hand.

5. Conclusion

By present investigation we established the fundamental differential equations to solve the stress distribution in homogeneous and isotropic powdered material under compression. And then we solved these

equations in an example and found out the stress distribution. The method will be available to solve many other similar problems.

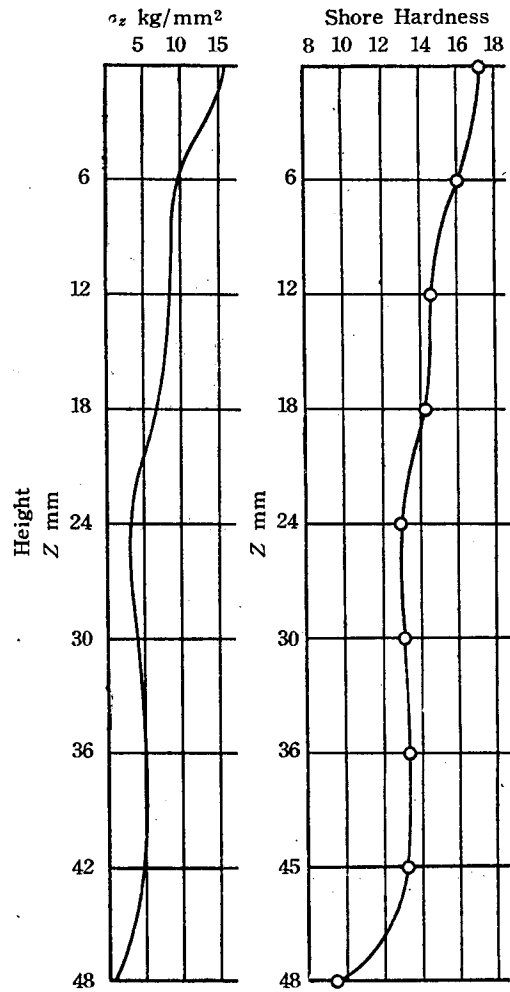


Fig. 16