

Invariants of Unitary Reflection Groups

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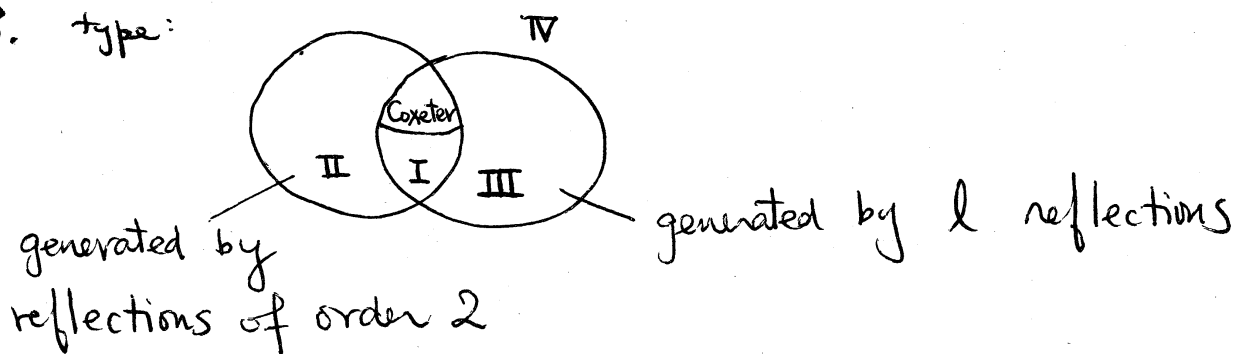
有限ユニタリ鏡映群は, Shephard-Todd [3] によ, 2
分類されてい, 1 ~ 37 までの名前が与えられている。本
稿では, それらの不変量の表を作り, そこから得られる
observation を紹介したい。

§ 1. Notations and Table

1. The names of finite unitary reflection groups are due to Shephard-Todd [3],
2. $m_1, \dots, m_\ell$ are the exponents of group (i.e., the degrees of basis of invariant polynomials -1); ($m_1 \leq m_2 \leq \dots \leq m_\ell$)
 $n_1, \dots, n_\ell$ are the exponents of arrangement (i.e., the degrees of basis of logarithmic derivations $+1$). (See [4]) ($n_1 \leq n_2 \leq \dots \leq n_\ell$)
3. $m = \sum_{i=1}^{\ell} m_i$, $n = \sum_{i=1}^{\ell} n_i$.

4. $r_i = \# \{ \text{reflections of order } i \},$
 $s_i = \# \{ \text{reflecting hyperplane } H \text{ such that}$
 $\quad \text{there exist exactly } i$
 $\quad \text{fixing } H \text{ pointwise} \}$
5. $(d_1, \dots, d_\ell)$ is self-dual $\stackrel{\text{def}}{\iff} d_i + d_{\ell+1-i} = \text{const.}$
 $(1 \leq i \leq \ell)$
6. $(m_1, \dots, m_\ell)$ and $(n_1, \dots, n_\ell)$ are codual
 $\stackrel{\text{def}}{\iff} m_i + n_{\ell+1-i} = \text{const.} \quad (1 \leq i \leq \ell)$
7. $t := \frac{2n}{\ell}, \quad H := \frac{2(m+n)}{2\ell + (m-n)}$

8. type:



9. Condition (A): Are (m_i) & (n_i) both self-dual?
 Condition (B): If yes, $m_1 + m_\ell = n_1 + n_\ell$?
 Condition (C): Are (m_i) & (n_i) codual?
 Condition (D): Is G generated by ℓ reflections?
 Condition (E): Is G generated by reflections of order 2?

Group	$m_1, \dots, m_\ell$	m	$r_2, r_3, \dots$	t	$m_{\ell+1}$	(A)	(B)	(C)	(D)	(E)	type
$\frac{1}{(A_\ell)}$	$m_1, \dots, m_\ell$ $1, 2, \dots, \ell$	n $\frac{\ell}{2}(\ell+1)$	$S_2, S_3, \dots$ $\frac{\ell}{2}(\ell+1), 0, 0, \dots$	H $\ell+1$	$m_{\ell+1}$ $\ell+1$						
	$1, 2, \dots, \ell$	$\frac{\ell}{2}(\ell+1)$	$\frac{\ell}{2}(\ell+1), 0, 0, \dots$	$\ell+1$	$\ell+1$						Coxeter

2											
$G(r, p, \ell)$											
$r = p = \ell$											
$r = \ell = 2$	$1, 3, 5, \dots, 2\ell-1$ $1, 3, 5, \dots, 2\ell-1$	ℓ^2 ℓ^2	$\ell^2, 0, 0, \dots$ $\ell^2, 0, 0, \dots$	2ℓ 2ℓ	2ℓ 2ℓ						Coxeter
$r = 4$	$3, 7, \dots, 4\ell-5 (=m_2), 2\ell-1$	$\ell(2\ell-1)$	$\ell(2\ell-1), 0, 0, \dots$	$4\ell-2$	$4\ell-4$						II
$p = \ell = 2$	$1, 5, \dots, 4\ell-3$	$\ell(2\ell-1)$	$\ell(2\ell-1), 0, 0, \dots$	$4\ell-2$	$4\ell-2$						III
$r = \ell > 2$	$r-1, 2r-1, \dots, \ell r-1$ $1, r+1, \dots, (\ell-1)r+1$	$\ell(\frac{(\ell+1)r}{2}-1)$ $\ell(\frac{(\ell-1)r}{2}+1)$	$(\ell-1)r+2$ 2ℓ	2ℓ $(\ell-1)r+2$	2ℓ $(\ell-1)r+2$						IV
$r = 2\ell > 4$	$r-1, 2r-1, \dots, (\ell-1)r-1, \frac{\ell r}{2}-1$ $1, r+1, \dots, (\ell-1)r+1$	$\ell(\frac{(\ell-1)r}{2}-1)$ $\ell(\frac{(\ell-1)r}{2}+1)$	$(\ell-1)r+2$ $4\ell-2$	$(\ell-1)r$ $(\ell-1)r+2$	$(\ell-1)r$ $(\ell-1)r+2$						IV
$\ell \neq r \neq 2\ell$	$r-1, 2r-1, \dots, (\ell-1)r-1, \ell\ell-1$ $1, r+1, \dots, (\ell-1)r+1$	$\ell(\frac{(\ell-1)r}{2}+g-1)$ $\ell(\frac{(\ell-1)r}{2}+1)$	$(\ell-1)r+2$ $2(\ell-1)r+2$	$(\ell-1)r$ $(\ell-1)r+2$	$(\ell-1)r$ $(\ell-1)r+2$						IV
$\ell \neq r \neq 2\ell$	$r-1, 2r-1, \dots, (\ell-1)r-1, 2\ell-1$ $1, r+1, \dots, (\ell-1)r+1$	$\ell(\frac{(\ell-1)r}{2}+1)$ $\ell(\frac{(\ell-1)r}{2}+1)$	$(\ell-1)r+2$ $2(\ell-1)r+2$	$(\ell-1)r$ $(\ell-1)r+2$	$(\ell-1)r$ $(\ell-1)r+2$						II
$\ell \neq r \neq 2\ell$	$r-1, 2r-1, \dots, (\ell-1)r-1, \ell-1$ $1, r+1, \dots, (\ell-2)r+1, (\ell-1)r-1$	$\frac{\ell(\ell-1)r}{2}$ $\frac{\ell(\ell-1)r}{2}$	$\frac{\ell(\ell-1)r}{2}, 0, 0, \dots$ $\frac{\ell(\ell-1)r}{2}, 0, 0, \dots$	$(\ell-1)r$ $(\ell-1)r$	$(\ell-1)r$ $(\ell-1)r$						I
$p = r = 2$	$1, 3, 5, \dots, 2\ell-3, \ell-1$ $1, 3, 5, \dots, 2\ell-3, \ell-1$	$\ell(\ell-1)$ $\ell(\ell-1)$	$\ell(\ell-1), 0, 0, \dots$ $\ell(\ell-1), 0, 0, \dots$	$2(\ell-1)$ $2(\ell-1)$	$2(\ell-1)$ $2(\ell-1)$						Coxeter

Group	$m_1, \dots, m_r$	m	$k_2, k_3, \dots$ $s_2, s_3, \dots$	t	m_{r+1} m_{r+1}	(A)	(B)	(C)	(D)	(E)	type
3	$r-1$	$r-1$	$0, 0, \dots, 0, 1, 0, \dots$	2	2	y	n	y	y	n	III
$r \geq 2$	$r-1$	$r-1$	$0, 0, \dots, 0, 1, 0, \dots$	2	2	y	n	y	y	n	III
4	$3, 5$ $1, 3$	8	$0, 8, 0, \dots$ $0, 4, 0, \dots$	4	6	y	n	y	y	n	III
5	$5, 11$ $1, 7$	16	$0, 16, 0, \dots$ $0, 8, 0, \dots$	8	12	y	n	y	y	n	III
6	$3, 11$ $1, 9$	14	$6, 8, 0, \dots$ $6, 4, 0, \dots$	10	12	y	n	y	y	n	III
7	$11, 11$ $1, 13$	22	$6, 16, 0, \dots$ $6, 8, 0, \dots$	14	12	y	n	n	n	n	IV
8	$7, 11$ $1, 5$	18	$6, 0, 12, \dots$ $0, 0, 6, \dots$	6	12	y	n	y	y	n	III
9	$7, 23$ $1, 17$	30	$18, 0, 12, \dots$ $12, 0, 6, \dots$	18	24	y	n	y	y	n	III
10	$11, 23$ $1, 13$	34	$6, 16, 12, \dots$ $0, 8, 6, \dots$	14	24	y	n	y	y	n	III

Group	$m_1, \dots, m_r$	m	$K_2, K_3, \dots$ $S_2, S_3, \dots$	$\dagger$	m_{e+1} m_{e+1}	(A)	(B)	(C)	(D)	(E)	type
11	23, 23 1, 25	46 26	18, 16, 12, ... 12, 8, 6, ...	26 6	24 26	γ	n	n	n	n	IV
12	5, 7 1, 11	12 12	12, 0, ... 12, 0, ...	12 12	8 12	γ	γ	n	n	γ	II
13	7, 11 1, 17	18 18	18, 0, ... 18, 0, ...	18 18	12 18	γ	γ	n	n	γ	II
14	5, 23 1, 19	28 20	12, 16, ... 12, 8, ...	20 8	24 20	γ	n	γ	γ	n	III
15	11, 23 1, 25	34 26	18, 16, ... 18, 8, ...	26 10	24 26	γ	n	n	n	n	IV
16	19, 29 1, 11	48 12	0, 0, 0, 48 0, 0, 0, 12	12 3	30 12	γ	n	γ	γ	n	III
17	19, 59 1, 41	78 42	30, 0, 0, 48 30, 0, 0, 12	42 6	60 42	γ	n	γ	γ	n	III
18	29, 59 1, 31	88 32	0, 40, 0, 48 0, 20, 0, 12	32 4	60 32	γ	n	γ	γ	n	III
19	59, 59 1, 61	118 62	30, 40, 0, 48 30, 20, 0, 12	62 6	60 62	γ	n	n	n	n	IV
20	11, 29 1, 19	40 20	0, 40, ... 0, 20, ...	20 5	30 20	γ	n	γ	γ	n	III

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Group	$m_1, \dots, m_e$	m	$k_2, k_3, \dots$ $s_2, s_3, \dots$	T	m_e+1 m_{e+1}	(A)	(B)	(C)	(D)	(E)	type
21	1, 5, 9 1, 4, 9	70 50	30, 40, ... 30, 20, ...	50 10	60 50	Y	m	Y	Y	m	III
22	1, 1, 19 1, 2, 9	30 30	30, 0, ... 30, 0, ...	30 30	20 30	Y	Y	m	m	Y	II
23 (H ₃)	1, 5, 9 1, 5, 9	15 15	15, 0, ... 15, 0, ...	10 10	10 10	Y	Y	Y	Y	Y	I
24	3, 5, 13 1, 9, 11	21 21	21, 0, ... 21, 0, ...	14 14	14 12	m	-	Y	Y	Y	I
25	5, 8, 11 1, 4, 7	24 12	0, 24, ... 0, 12, ...	8 4	12 8	Y	m	Y	Y	m	III
26	5, 11, 17 1, 7, 13	33 21	9, 24, ... 9, 12, ...	14 6	18 14	Y	m	Y	Y	m	III
27	5, 11, 29 1, 19, 25	45 45	45, 0, ... 45, 0, ...	30 30	30 26	m	-	Y	Y	Y	I
28 (F ₄)	1, 5, 7, 11 1, 5, 7, 11	24 24	24, 0, ... 24, 0, ...	12 12	12 12	Y	Y	Y	Y	Y	Core
29	3, 7, 11, 19 1, 9, 13, 17	40 40	40, 0, ... 40, 0, ...	20 20	20 18	m	-	Y	Y	Y	I
30 (H ₄)	1, 11, 19, 29 1, 11, 19, 29	60 60	60, 0, ... 60, 0, ...	30 30	30 30	Y	Y	Y	Y	Y	Core

Groups	$m_1, \dots, m_e$	n	$v_2, v_3, \dots$	t	$m_e + 1$	(A)	(B)	(C)	(D)	(E)	type
31	7, 11, 19, 23 1, 13, 17, 29	60	60, 0, ... 60, 0, ...	30 30	24 30	Y	Y	n	n	Y	II
32	11, 17, 23, 29 1, 7, 13, 19	80 40	0, 80, ... 0, 40, ...	20 5	30 20	Y	n	Y	Y	n	III
33	3, 5, 9, 11, 17 1, 7, 9, 13, 15	45 45	45, 0, ... 45, 0, ...	18 18	18 16	n	-	Y	Y	Y	I
34	5, 11, 17, 23, 29, 41 1, 13, 19, 25, 31, 37	126 126	126, 0, ... 126, 0, ...	42 42	42 38	n	-	Y	Y	Y	I
35 (E ₆)	1, 4, 5, 7, 8, 11 1, 4, 5, 7, 8, 11	36 36	36, 0, ... 36, 0, ...	12 12	12 12	Y	Y	Y	Y	Y	Correct
36 (E ₇)	1, 5, 7, 9, 11, 13, 17 1, 5, 7, 9, 11, 13, 17	63 63	63, 0, ... 63, 0, ...	18 18	18 18	Y	Y	Y	Y	Y	Correct
37 (E ₈)	1, 7, 11, 13, 17, 19, 23, 29 1, 7, 11, 13, 17, 19, 23, 29	120 120	120, 0, ... 120, 0, ...	30 30	30 30	Y	Y	Y	Y	Y	Correct

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§ 2. Observations

1. (Orlik-Solomon [1])

(m_i) & (n_i) are codual (condition (C))

$\Updownarrow$
 G is generated by l reflections (condition (D))

$\Updownarrow$
 $m_e \geq n_e$

2. H is always a natural number. (This invariant was introduced by T. Yano.)
 t has been proved to be a natural number [5].

3. There are five cases:

$t=H=m_e+1=n_e+1 \Leftrightarrow$ (A) (B) (C) (D) (E) $\Leftrightarrow$ Coxeter

$t=H=m_e+1 > n_e+1 \Leftrightarrow$ n - y y y $\Leftrightarrow$ I

$t=H=n_e+1 > m_e+1 \Leftrightarrow$ y y n n y $\Leftrightarrow$ II

$m_e+1 > n_e+1 = t > H \Leftrightarrow$ y n y y n $\Leftrightarrow$ III

$n_e+1 = t > m_e+1 > H \Leftrightarrow$ y n n n n $\Leftrightarrow$ IV

(with only exception: 2)

4. III $\Leftrightarrow$ Shephard group (Orlik-Solomon [2])

References

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