

ASYMPTOTICS FOR HIGHER DERIVATIVES OF THE LERCH ZETA-FUNCTION: APPLICATIONS TO THE FORMULAE OF KUMMER, LERCH AND GAUSS

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ABSTRACT. Let s be a complex variables, z a complex parameter, and a and λ real parameters with $a > 0$, and write $e(s) = e^{2\pi is}$. The Lerch zeta-function $\phi(s, a, \lambda)$ is defined by the Dirichlet series $\sum_{l=0}^{\infty} e(\lambda l)(a+l)^{-s}$ ($\operatorname{Re} s > 1$), and its meromorphic continuation over the whole s -plane; this reduces to the Hurwitz zeta-function $\zeta(s, a)$ if λ is an integer, and further to the Riemann zeta-function $\zeta(s) = \zeta(s, 1)$. Note that the domain of the parameter a can be extended through the procedure in [13]. Let $\phi^{(m)}(s, z, \lambda) = (\partial/\partial s)^m \phi(s, z, \lambda)$ for $m = 0, 1, 2, \dots$ denote any derivative. The aim of this paper is to show that complete asymptotic expansions exist for $\phi^{(m)}(s, a+z, \lambda)$ ($m = 0, 1, \dots$) when both $z \rightarrow 0$ and $z \rightarrow \infty$ through $|\arg z| < \pi$ (Theorems 1 and 2), together with the explicit expressions of their remainders (Corollaries 1.1 and 2.2); these can be applied to deduce the classical Fourier series expansions of the log-gamma function $\log \Gamma(s)$ (Corollary 2.3) and the di-gamma function $\psi(s) = (\Gamma'/\Gamma)(s)$ (Corollary 2.4) both for $0 < s < 1$, due to Kummer and Lerch, respectively, as well as to deduce the celebrated closed form evaluation of $\psi(r)$ at any rational point r with $0 < r < 1$ (Corollary 2.5), due to Gauß. Our results in Theorems 1 and 2 further lead us to define and study a generalization of Deninger's $\mathcal{R}_m$ -function (Corollaries 1.4–1.6 and 2.6–2.9), which was first introduced by Deninger [3] for extending the log-gamma function into higher orders. The detailed proofs of our results in the present paper will appear, among other things, in the forthcoming article [21].

1. INTRODUCTION

Throughout the paper, the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, non-negative integers, all integers, real numbers, and complex numbers, respectively, and further $s = \sigma + it$ is a complex variable (with real coordinates σ and t), a and λ are real parameters with $a > 0$, and the notation $e(s) = e^{2\pi is}$ is frequently used. The Lerch zeta-function $\phi(s, a, \lambda)$ is defined by the Dirichlet series

$$(1.1) \quad \phi(s, a, \lambda) = \sum_{l=0}^{\infty} e(\lambda l)(a+l)^{-s} \quad (\operatorname{Re} s > 1),$$

and its meromorphic continuation over the whole s -plane (cf. [30][31]); this reduces if $\lambda \in \mathbb{Z}$ to the Hurwitz zeta-function $\zeta(s, a)$, to the exponential zeta-function $\zeta_{\lambda}(s) = e(\lambda)\phi(s, 1, \lambda)$ for $\lambda \in \mathbb{R}$, and hence to the Riemann zeta-function $\zeta(s) = \zeta(s, 1) = \zeta_{\lambda}(s)$ if $\lambda \in \mathbb{Z}$. We note that the domain of the parameter a can be extended to the whole sector $|\arg z| < \pi$ through the procedure in [13].

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It is the principal aim of the present paper to treat asymptotic aspects of the derivatives (of any order) $\phi^{(m)}(s, z, \lambda) = (\partial/\partial s)^m \phi(s, z, \lambda)$ for $m = 0, 1, 2, \dots$, when z becomes small and large through the sector $|\arg z| < \pi$. Let $\Gamma(s)$ denote the gamma function, and $\psi(s) = (\Gamma'/\Gamma)(s)$ the di-gamma function. We shall then show that complete asymptotic expansions exist for $\phi^{(m)}(s, a + z, \lambda)$ ($m = 0, 1, 2, \dots$) as both $z \rightarrow 0$ and $z \rightarrow \infty$ through $|\arg z| < \pi$ (Theorems 1 and 2), together with the explicit expressions of their remainders (Corollaries 1.1 and 2.2); these can further be applied to deduce the classical Fourier series expansions of $\log \Gamma(s)$ (Corollary 2.3) and of $\psi(s)$ (Corollary 2.4) both on the unit interval, due to Kummer and Lerch, respectively, as well as to deduce the celebrated closed form evaluation of $\psi(r)$ at any rational point r on the unit interval (Corollary 2.5), due to Gauß. Furthermore, our results in Theorems 1 and 2 lead us to define and study a generalization of Deninger's $\mathcal{R}_m$ -function (Corollaries 1.4–1.6 and 2.6–2.9), which was first introduced by Deninger [3] for extending $\log \Gamma(s)$ into higher orders. The detailed proofs of our results will appear, among other things, in the forthcoming article [21].

2. STATEMENT OF RESULTS: ASYMPTOTIC EXPANSIONS

We prepare for describing our results the shifted factorial $(s)_n = \Gamma(s+n)/\Gamma(s)$ with any $n \in \mathbb{Z}$, and the (modified) Stirling polynomial of the first kind, defined for any $j, k \in \mathbb{N}_0$ by

$$(2.1) \quad \mathfrak{s}_j^k(x) = \frac{1}{j!} \left(\frac{\partial}{\partial z} \right)^k (1-z)^{-x} \{ -\log(1-z) \}^j \Big|_{z=0}.$$

The following Theorems 1 and 2 assert complete asymptotic expansions as $z \rightarrow 0$ and as $z \rightarrow \infty$, respectively, through the sector $|\arg z| < \pi$.

Theorem 1. *Let $m \in \mathbb{N}_0$, and $a, \lambda \in \mathbb{R}$ be arbitrary with $a > 0$. Then for any integer $K \geq 0$, in the region $\sigma > 1 - K$ except at $s = 1$, we have*

$$(2.2) \quad \begin{aligned} \phi^{(m)}(s, a + z, \lambda) = m! \sum_{k=0}^{K-1} \frac{(-1)^k}{k!} \sum_{j=0}^m \frac{\mathfrak{s}_{m-j}^k(s)}{j!} \phi^{(j)}(s + k, a, \lambda) z^k \\ + (\rho_K^+)^{(m)}(s, a, \lambda; z) \end{aligned}$$

for $|\arg z| < \pi$. Here ρ_K^+ is expressed by the Mellin-Barnes type integral (4.2) below, and its m th derivative $(\rho_K^+)^{(m)} = (\partial/\partial s)^m \rho_K^+$ satisfies the estimate

$$(2.3) \quad (\rho_K^+)^{(m)}(s, a, \lambda; z) = O(|z|^K)$$

as $z \rightarrow 0$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the implied O -constant depends at most on s, a, λ, K and δ .

The following expression holds for the case $m = 0$ of the remainder in (2.2).

Corollary 1.1. *For any $K \geq 1$, in the region $\sigma > 1 - K$ and in the sector $|\arg z| < \pi$, the Mellin-Barnes type integral in (4.2) is transformed to*

$$\rho_K^+(s, a, \lambda; z) = \frac{(-1)^K (s)_K z^K}{\Gamma(K)} \int_0^1 \phi(s + K, a + z\tau, \lambda) (1 - \tau)^{K-1} d\tau.$$

Proof. To remove the poles of the integrand in (4.2) at $w = k$ ($k = 0, 1, \dots, K - 1$), the expression

$$\Gamma(-w) = \frac{(-1)^K \Gamma(-w + K)}{\Gamma(K)} \int_0^1 \tau^{w-K} (1 - \tau)^{K-1} d\tau,$$

being valid on the path $\operatorname{Re} w = u_K^+$, is inserted in the integrand on the right side of (4.2); this yields the assertion of the corollary upon changing the order of the w - and τ -integration, where the resulting inner w -integral can be evaluated by substituting the variable $w = w' + K$, and by noting the fact that $\Gamma(s) = \Gamma(s + K)/(s)_K$. $\square$

It can be seen from Corollary 1.1 that $\lim_{K \rightarrow +\infty} (\rho_K^+)^{(m)}(s, a, \lambda; z) = 0$ for $|z| < a$; Theorem 1 readily implies the following result.

Corollary 1.2. *Let m , a and λ be as in Theorem 1. Then we have for $|z| < a$ and for any $s \in \mathbb{C}$ except at $s = 1$ the Taylor series expansion*

$$(2.4) \quad \phi^{(m)}(s, a + z, \lambda) = m! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{j=0}^m \frac{\mathfrak{s}_{m-j}^k(s)}{j!} \phi^{(j)}(s + k, a, \lambda) z^k.$$

We remark here in connection with Corollary 1.2 that the monograph of Srivastava-Choi [35] gives a quite systematic presentation of various sums involving the values of zeta and allied functions.

The case $a = 1$ of Theorem 1 with the relation

$$(2.5) \quad \phi(s, z, \lambda) - z^{-s} = e(\lambda) \phi(s, 1 + z, \lambda)$$

asserts the following asymptotic expansion as $z \rightarrow 0$.

Corollary 1.3. *Let m , a and λ be as in Theorem 1. Then for any integer $K \geq 0$, in the region $\sigma > 1 - K$ except at $s = 1$, we have*

$$(2.6) \quad \begin{aligned} \phi^{(m)}(s, z, \lambda) = & z^{-s} (-\log z)^m + m! \sum_{k=0}^{K-1} \frac{(-1)^k}{k!} \sum_{j=0}^m \frac{\mathfrak{s}_{m-j}^k(s)}{j!} \\ & \times \zeta_{\lambda}^{(j)}(s + k) z^k + e(\lambda) (\rho_K^+)^{(m)}(s, 1, \lambda; z), \end{aligned}$$

where the remainder $e(\lambda) (\rho_K^+)^{(m)}$ satisfies the same estimate as in (2.3) when $z \rightarrow 0$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.

Next let $\delta(\lambda)$ be the symbol which equals 1 or 0, according to $\lambda \in \mathbb{Z}$ or otherwise. Then Apostol [1] introduced the sequence of functions $B_k(x, y)$ ($k \in \mathbb{N}_0$), defined for any $x, y \in \mathbb{C}$ by the Taylor series expansion

$$\frac{ze^{xz}}{ye^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x, y)}{k!} z^k$$

centered at $z = 0$; note that

$$(2.7) \quad B_0(x, y) = \begin{cases} 1 & \text{if } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $B_k(x, y)$ reduces to the usual Bernoulli polynomial $B_k(x)$ (cf. [5, 1.13 (2)]) if $y = 1$.

Theorem 2. *Let m , a and λ be as in Theorem 1, and define the polynomials $p_m(s, w)$ and $q_{k,m}(s, w)$ for $m, k \in \mathbb{N}_0$ by*

$$(2.8) \quad p_m(s; w) = \sum_{j=0}^m \frac{\{(s-1)w\}^j}{j!},$$

$$(2.9) \quad q_{m,k}(s; w) = \sum_{j=0}^m \frac{\mathfrak{s}_{m-j}^k(s)}{j!} (-w)^j.$$

Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s = 1$, we have the formula

$$(2.10) \quad \begin{aligned} \phi^{(m)}(s, a + z, \lambda) &= \frac{\delta(\lambda)(-1)^m m!}{(s-1)^{m+1}} z^{1-s} p_m(s; \log z) \\ &+ m! \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a, e(\lambda))}{(k+1)!} z^{-s-k} q_{m,k}(s; \log z) \\ &+ (\rho_K^-)^{(m)}(s, a, \lambda; z), \end{aligned}$$

where ρ_K^- is expressed by the Mellin-Barnes type integral (4.3) below, and its m th derivative $(\rho_K^-)^{(m)} = (\partial/\partial s)^m \rho_K^-$ satisfies the estimate

$$(2.11) \quad (\rho_K^-)^{(m)}(s, a, \lambda; z) = O(|z|^{-\sigma-K} \log^m |z|)$$

as $z \rightarrow \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the implied O -constant depends at most on m, s, a, λ, K and δ .

The case $a = 1$ of Theorem 2, together with the relations (2.5) and

$$y B_j(1, y) = (-1)^j B_j(0, 1/y) = \begin{cases} B_j(0, y) & \text{if } j \neq 1, \\ B_1(0, y) + 1 & \text{if } j = 1 \end{cases}$$

for any $y \in \mathbb{C} \setminus \{0\}$ (cf. [14, (7.1) and (7.2)]), asserts the following formula.

Corollary 2.1. *Let $m, \lambda, p_m(s; w)$ and $q_{m,k}(s; w)$ be as in Theorem 2. Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s = 1$, we have*

$$(2.12) \quad \begin{aligned} \phi^{(m)}(s, z, \lambda) &= \frac{\delta(\lambda)e(\lambda)(-1)^m m!}{(s-1)^{m+1}} z^{1-s} p_m(s; \log z) \\ &+ m! \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(0, e(\lambda))}{(k+1)!} z^{-s-k} q_{m,k}(s; \log z) \\ &+ e(\lambda)(\rho_K^-)^{(m)}(s, 1, \lambda; z), \end{aligned}$$

where the remainder $e(\lambda)(\rho_K^-)^{(m)}$ satisfies the same estimate as in (2.11) when $z \rightarrow \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.

It is to be remarked that the case $m = 0$ of Corollary 1.2 and Theorem 2 were first proved (in a unified manner) in terms of Mellin-Barnes type integrals by the author [13, (1.6) and Theorem 1], where the expression (2.14) below for the remainder in (2.10) (with $m = 0$) has been shown at the same time.

Let $U(\alpha; \gamma; Z)$ denote Kummer's confluent hypergeometric function of the second kind, defined by the integral

$$U(\alpha; \gamma; Z) = \frac{1}{\Gamma(\alpha)\{e(\alpha) - 1\}} \int_{\infty e^{i\varphi}}^{(0+)} e^{-Zw} w^{\alpha-1} (1+w)^{\gamma-\alpha-1} dw$$

for any $\alpha, \gamma \in \mathbb{C}$ and for $|\arg Z + \varphi| < \pi/2$ with any fixed $\varphi \in]-\pi, \pi[$. Here the path of integration is the loop cranked with an angle φ around the origin, which starts from $\infty e^{i\varphi}$, proceeds along the ray from $\infty e^{i\varphi}$ to $\delta e^{i\varphi}$ with a small $\delta > 0$, encircles the origin counter-clockwise, and returns to $\infty e^{i\varphi}$ along the ray, where $\arg w$ varies from φ to $\varphi + 2\pi$

along the loop; this allows to prepare the analytic continuation of $U(\alpha; \gamma; Z)$ to the whole sector $|\arg Z| < 3\pi/2$ by rotating appropriately the path of integration. We now set

$$(2.13) \quad \begin{aligned} f_{s,K}(Z) &= U(1; 2-s-K; Z), \\ g_{s,K}(Z) &= U(s+K; s+K; Z) \end{aligned}$$

both for $|\arg Z| < 3\pi/2$. Then the following expressions are valid for the case $m = 0$ of the remainder in (2.10).

Corollary 2.2. *For any $a, \lambda \in [0, 1]$, and in the region $\sigma > -K$ with $K \geq 1$, we have for $|\arg z| < \pi$,*

$$(2.14) \quad \begin{aligned} \rho_K^-(s, a, \lambda; z) &= \frac{(s)_K z^{1-s-K}}{(2\pi i)^K} \left\{ \sum_{l=0}^{\infty} \frac{e(-a(\lambda+l))}{(\lambda+l)^K} f_{s,K}(2\pi(\lambda+l)e^{-\pi i/2}z) \right. \\ &\quad \left. + (-1)^K \sum_{l=0}^{\infty} \frac{e(a(1-\lambda+l))}{(1-\lambda+l)^K} f_{s,K}(2\pi(1-\lambda+l)e^{\pi i/2}z) \right\}, \end{aligned}$$

which is transformed through (2.17) below into

$$(2.15) \quad \begin{aligned} \rho_K^-(s, a, \lambda; z) &= (-1)^K (2\pi)^{s-1} (s)_K \left\{ e^{\pi i(1-s)/2} \sum_{l=0}^{\infty} \frac{e(-a(\lambda+l))}{(\lambda+l)^{1-s}} g_{s,K}(2\pi(\lambda+l)e^{-\pi i/2}z) \right. \\ &\quad \left. + e^{-\pi i(1-s)/2} \sum_{l=0}^{\infty} \frac{e(a(1-\lambda+l))}{(1-\lambda+l)^{1-s}} g_{s,K}(2\pi(1-\lambda+l)e^{\pi i/2}z) \right\} \end{aligned}$$

for the same σ , K and z as above.

Proof. We can apply the (slightly extended) functional equation, for any $a, \lambda \in [0, 1]$,

$$(2.16) \quad \begin{aligned} \phi(r, a, \lambda) &= \frac{\Gamma(1-r)}{(2\pi)^{1-r}} \left\{ e^{\pi i(1-r)/2} \sum_{l=0}^{\infty} \frac{e(-a(\lambda+l))}{(\lambda+l)^{1-r}} \right. \\ &\quad \left. + e^{-\pi i(1-r)/2} \sum_{l=0}^{\infty} \frac{e(a(1-\lambda+l))}{(1-\lambda+l)^{1-r}} \right\} \quad (\operatorname{Re} r < 0), \end{aligned}$$

in the argument of [13, Proof of Theorem 1] to deduce (2.14). Next the relation

$$(2.17) \quad U(\alpha; \gamma; Z) = Z^{1-\gamma} U(\alpha - \gamma + 1; 2 - \gamma; Z)$$

(cf. [5, 6.5 (6)]) shows that $f_{s,K}(Z) = Z^{s+K-1} g_{s,K}(Z)$, which is substituted into the right side of (2.14) to imply the assertion (2.15). $\square$

We mention here several results relevant to Theorem 2. Meijer's G -function (cf. [5, 5.3 (1)]) theoretic interpretation of the formula (2.10) with $m = 0$, as well as of the author's result [16, Theorem 1] on complete asymptotic expansions for Epstein zeta-function, were made by Kuzumaki [29]. Also, the proof of (2.10) with $m = 0$ in [13] is reproduced in the monograph of Chakraborty-Kanemitsu-Tsukada [2, Chap.5.3], in which various alternative proofs of (3.5) below are given. A complete asymptotic expansion, whose shape differs far from that of (2.10) with $\lambda \in \mathbb{Z}$, for $\zeta^{(m)}(s, z)$ ($m = 0, 1, 2, \dots$) as $z \rightarrow \infty$ through $|\arg z| < \pi$ was obtained more recently by Seri [34] (see also the references therein for various related articles). Matsumoto [33], on the other hand, established complete asymptotic expansions for the extensions of $\zeta(s, z)$ to several variable cases.

3. APPLICATIONS

We proceed in this section to present several applications of Theorems 1 and 2. For this, let $\text{si } x$ and $\text{Ci } x$ denote the sine and cosine integrals, defined respectively by

$$(3.1) \quad \text{si } x = \int_{+\infty}^x \frac{\sin u}{u} du \quad \text{and} \quad \text{Ci } x = \int_{+\infty}^x \frac{\cos u}{u} du$$

for any $x \in]0, +\infty[$ (cf. [6, 9.8 (1) and (3)]). It is classically known that the evaluations

$$(3.2) \quad \left. \frac{\partial}{\partial s} \zeta(s, z) \right|_{s=0} = \log \left\{ \frac{\Gamma(z)}{\sqrt{2\pi}} \right\},$$

$$(3.3) \quad \left\{ \zeta(s, z) - \frac{1}{s-1} \right\} \Big|_{s=1} = -\frac{\Gamma'}{\Gamma}(z) = -\psi(z)$$

hold both for $|\arg z| < \pi$ (cf. [5, 1.10 (9) and (10)]). Then in view of the relation (2.5) with $\lambda \in \mathbb{Z}$, a particular case of the formula (2.10), combined with (2.14) or (2.15), in fact yields the Fourier series expansions (3.5) and (3.7) below, due to Kummer and to Lerch (cf. [5, 1.9.1 (14) and (15)]), respectively. Let $\gamma_0 = -\Gamma'(1)$ denote the 0th Euler constant.

Corollary 2.3. *For any $a \in]0, 1[$ and $\lambda \in \{0, 1\}$, we have the Fourier series expansion*

$$(3.4) \quad (\rho_1^-)'(0, a, \lambda; 1) = \sum_{l=1}^{\infty} \frac{1}{\pi l} \{ -\text{si}(2\pi l) \cos(2\pi a l) + \text{Ci}(2\pi l) \sin(2\pi a l) \},$$

which with (2.10) and (3.2) implies that

$$(3.5) \quad \log \left\{ \frac{\Gamma(a)}{\sqrt{2\pi}} \right\} = \sum_{l=1}^{\infty} \frac{\cos(2\pi a l)}{2l} + \sum_{l=1}^{\infty} \frac{1}{\pi l} \{ \gamma_0 + \log(2\pi l) \} \sin(2\pi a l).$$

Corollary 2.4. *For any $a \in]0, 1[$ and $\lambda \in \{0, 1\}$, we have the Fourier series expansion*

$$(3.6) \quad \rho_1^-(1, a, \lambda; 1) = B_1(a) - 2 \sum_{l=1}^{\infty} \{ \text{Ci}(2\pi l) \cos(2\pi a l) + \text{si}(2\pi l) \sin(2\pi a l) \},$$

which with (2.10) and (3.3) implies that

$$(3.7) \quad \begin{aligned} \psi(a) \sin(\pi a) &= \sum_{l=1}^{\infty} \log \left(\frac{l}{l+1} \right) \sin \{ (2l+1)\pi a \} - \{ \gamma_0 + \log(2\pi) \} \sin(\pi a) \\ &\quad - \frac{\pi}{2} \cos(\pi a). \end{aligned}$$

The case $(m, K) = (1, 2)$ of Theorem 2 can be applied to (3.7), upon yielding the following celebrated closed form evaluation due to Gauß (cf. [5, 1.7.3 (29)]).

Corollary 2.5. *For any $p, q \in \mathbb{Z}$ with $0 < p < q$, we have*

$$(3.8) \quad \psi\left(\frac{p}{q}\right) = -\gamma_0 - \log q - \frac{\pi}{2} \cot\left(\frac{\pi p}{q}\right) + \sum_{r=1}^{\lfloor q/2 \rfloor} \cos\left(\frac{2\pi p r}{q}\right) \log \left\{ 2 - 2 \cos\left(\frac{2\pi r}{q}\right) \right\},$$

where the primed summation symbol on the right side indicates that the last term is to be halved if q is even.

We proceed to state the last assertions. The function

$$\mathcal{R}_{m,0}(z) = (-1)^{m+1} \left(\frac{\partial}{\partial s} \right)^m \zeta(s, z) \Big|_{s=0} \quad (m = 1, 2, \dots)$$

was first introduced and studied in detail by Deninger [3], for the purpose of obtaining a better understanding of the Kronecker limit formula for real quadratic fields. We introduce in this respect the generalized Deninger function $\mathcal{R}_{m,n}(z, \lambda)$ for any $m \in \mathbb{N}_0$, $n \in \{1\} \cup (-\mathbb{N}_0)$ and $\lambda \in \mathbb{R}$, defined by

$$(3.9) \quad \mathcal{R}_{m,1}(z, \lambda) = (-1)^{m+1} \left(\frac{\partial}{\partial s} \right)^m \left\{ \phi(s, z, \lambda) - \frac{\delta(\lambda)}{s-1} \right\} \Big|_{s=1}$$

for $n = 1$, and for any $n \in -\mathbb{N}_0$,

$$(3.10) \quad \mathcal{R}_{m,n}(z, \lambda) = (-1)^{m+1} \left(\frac{\partial}{\partial s} \right)^m \phi(s, z, \lambda) \Big|_{s=n},$$

both in $|\arg z| < \pi$. Then Theorems 1 and 2 readily imply the following Corollaries 1.4–1.6 and 2.6–2.9, respectively.

Corollary 1.4. *For any $m \in \mathbb{N}_0$ and for any $a, \lambda \in \mathbb{R}$ with $a > 0$, we have the formulae:*

i) *for any integer $K \geq 1$,*

$$(3.11) \quad \begin{aligned} \mathcal{R}_{m,1}(a+z, \lambda) &= \mathcal{R}_{m,1}(a, \lambda) + (-1)^{m+1} m! \sum_{k=1}^{K-1} \frac{(-z)^k}{k!} \\ &\quad \times \sum_{j=0}^m \frac{\mathfrak{s}_{m-j}^k(1)}{j!} \phi^{(j)}(k+1, a, \lambda) + O(|z|^K); \end{aligned}$$

ii) *for any $n \in \mathbb{N}_0$ and for any integer $K \geq n+2$,*

$$(3.12) \quad \begin{aligned} \mathcal{R}_{m,-n}(a+z, \lambda) &= (-1)^{m+1} m! \left[\sum_{k=0}^{K-1} \frac{(-z)^k}{k!} \sum_{j=0}^m \frac{(-1)^{j+1} \mathfrak{s}_{m-j}^k(-n)}{j!} \mathcal{R}_{j,k-n}(a, \lambda) \right. \\ &\quad \left. + \frac{(-z)^{n+1}}{(n+1)!} \left\{ \delta(\lambda) \mathfrak{s}_m^n(-n) + \sum_{j=1}^m \frac{(-1)^j \mathfrak{s}_{m-j}^n(-n)}{(j-1)!} \mathcal{R}_{j-1,1}(a, \lambda) \right\} \right. \\ &\quad \left. + \sum_{k=n+2}^{K-1} \frac{(-z)^k}{k!} \sum_{j=0}^m \frac{\mathfrak{s}_{m-j}^k(-n)}{j!} \phi^{(j)}(k-n, a, \lambda) \right] + O(|z|^K), \end{aligned}$$

both as $z \rightarrow 0$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the implied O -constants depend at most on a, λ, K, m, n and δ .

The limit case $K \rightarrow +\infty$ of (3.11) with $m = 1$ implies the following Taylor series expansion, which is a slight extension of [5, 1.17(5)], since $\psi(z) = \mathcal{R}_{1,1}(z, \lambda)$ holds, by (3.3) and (3.9), for $|\arg z| < \pi$ if $\lambda \in \mathbb{Z}$.

Corollary 1.5. *For any real $a > 0$, in the disk $|z| < a$, we have*

$$(3.13) \quad \psi(a+z) = \psi(a) + \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(k+1, a) z^k.$$

The generalized Euler-Stieltjes constants $\gamma_m(z)$ ($m \in \mathbb{N}_0$) are defined by the Laurent series expansion

$$(3.14) \quad \zeta(s, z) = \frac{1}{s-1} + \sum_{m=0}^{\infty} \gamma_m(z)(s-1)^m \quad (0 < |s-1| < 1)$$

(cf. [7, 1.8(1.122)]), which shows with (3.9) that

$$(3.15) \quad \gamma_m(z) = \frac{(-1)^{m+1}}{m!} \mathcal{R}_{m,1}(z, \lambda) \quad (m = 0, 1, \dots)$$

for $|\arg z| < \pi$ if $\lambda \in \mathbb{Z}$; this asserts upon (3.11) the following asymptotic expansion as $z \rightarrow 0$.

Corollary 1.6. *Let a and m be as in Theorem 1. Then for any integer $K \geq 0$, we have*

$$(3.16) \quad \gamma_m(a+z) = \gamma_m(a) + \sum_{k=1}^{K-1} \frac{(-z)^k}{k!} \sum_{j=0}^m \frac{\mathfrak{s}_{m-j}^k(1)}{j!} \zeta^{(j)}(k+1, a) + O(|z|^K)$$

as $z \rightarrow 0$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.

Theorem 2 yields the following asymptotic expansion as $z \rightarrow \infty$.

Corollary 2.6. *Let a , λ , m and n be as in Corollary 1.4. Then for any integer $K \geq 0$, we have the formulae:*

i) *for any integer $K \geq 0$,*

$$(3.17) \quad \begin{aligned} \mathcal{R}_{m,1}(a+z, \lambda) &= \frac{\delta(\lambda) \log^{m+1} z}{m+1} - (-1)^m m! \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a, e(\lambda))}{(k+1)!} z^{-k-1} q_{m,k}(1; \log z) \\ &\quad + O(|z|^{-K-1} \log^m |z|); \end{aligned}$$

ii) *for any $n \in \mathbb{N}_0$ and for any integer $K \geq n+1$,*

$$(3.18) \quad \begin{aligned} \mathcal{R}_{m,-n}(a+z, \lambda) &= -\frac{\delta(\lambda)(-1)^m m!}{(n+1)^{m+1}} z^{n+1} p_m(-n; \log z) \\ &\quad - (-1)^m m! \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a, e(\lambda))}{(k+1)!} z^{n-k} q_{m,k}(-n; \log z) \\ &\quad + O(|z|^{n-K} \log^m |z|), \end{aligned}$$

both as $z \rightarrow \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the implied O -constants depend at most on a , λ , m , n , K and δ .

We obtain from (3.17) the following asymptotic expansion as $z \rightarrow \infty$, in view of (3.15).

Corollary 2.7. *Let a and m be as in Theorem 2. Then for any integer $K \geq 0$, we have*

$$(3.19) \quad \begin{aligned} \gamma_m(a+z) &= \frac{(-\log z)^{m+1}}{(m+1)!} + \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a)}{(k+1)!} z^{-k-1} q_{m,k}(1; \log z) \\ &\quad + O(|z|^{-K-1} \log^m |z|) \end{aligned}$$

as $z \rightarrow \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.

The case $(m, n) = (1, 0)$ of (3.18) further implies upon (3.2) the following (shifted) variant of Stirling's formula (cf. [5, 1.18(12)]).

Corollary 2.8. *For any integer $K \geq 0$, we have*

$$\log \Gamma(a + z) = \left(a + z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a)}{k(k+1)} z^{-k} \\ + O(|z|^{-K} \log |z|)$$

as $z \rightarrow \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.

The final corollary asserts the limit formulae for $\mathcal{R}_{m,n}(z, \lambda)$, which are also the consequences of Theorem 2.

Corollary 2.9. *For any $m \in \mathbb{N}_0$ and in $|\arg z| < \pi$, we have*

$$(3.20) \quad \mathcal{R}_{m,1}(z, \lambda) = \lim_{L \rightarrow +\infty} \left\{ \frac{\delta(\lambda) e(\lambda L) \log^{m+1} L}{m+1} - \sum_{l=0}^{L-1} \frac{e(\lambda l) \log^m(z+l)}{z+l} \right\}$$

for $n = 1$, and for any $n \in \mathbb{N}_0$,

$$(3.21) \quad \mathcal{R}_{m,-n}(z, \lambda) = \lim_{L \rightarrow +\infty} \left[(-1)^m m! e(\lambda L) \left\{ \frac{L^{m+1}}{(n+1)^{m+1}} p_m(-n; \log L) \right. \right. \\ \left. \left. - \sum_{k=0}^n \frac{(-1)^{k+1} B_{k+1}(z, e(\lambda))}{(k+1)!} L^{n-k} q_{m,k}(-n; \log L) \right\} \right. \\ \left. - \sum_{l=0}^{L-1} e(\lambda l) (z+l)^n \log^m(z+l) \right].$$

Remark. The case $\lambda \in \mathbb{Z}$ of (3.20) readily implies upon (3.15) the classical limit formula for $\gamma_m(z)$ with $m = 0, 1, 2, \dots$ (cf. [7, 1.8 (1.123)]).

4. OUTLINE OF THE PROOFS

We shall show in this section the outline of the proofs of Theorems 1 and 2.

The common starting point of the proofs of Theorems 1 and 2 is the Mellin-Barnes type integral formula

$$(4.1) \quad \phi(s, a + z, \lambda) = \frac{1}{2\pi i} \int_{(u)} \frac{\Gamma(s+w) \Gamma(-w)}{\Gamma(s)} \phi(s+w, a, \lambda) z^w dw$$

for $\sigma > 1$ in the sector $|\arg z| < \pi$, where u is a constant satisfying $1 - \sigma < u < 0$; this was first shown by the author [13, (2.6)].

Outline of the proof of Theorem 1. Suppose temporarily that $\sigma > 1$. Let u_K^+ for any integer $K \geq 0$ be a constant satisfying $K - 1 < u_K^+ < K$. Then the path in (4.1) can be moved from (u) to (u_K^+) , upon passing over the poles of the integrand at $w = k$ ($k = 0, 1, \dots, K - 1$); this yields the case $m = 0$ of (2.2) with

$$(4.2) \quad \rho_K^+(s, a, \lambda; z) = \frac{1}{2\pi i} \int_{(u_K^+)} \frac{\Gamma(s+w) \Gamma(-w)}{\Gamma(s)} \phi(s+w, a, \lambda) z^w dw.$$

The temporary restriction on σ can be relaxed at this stage to $\sigma > 1 - K$, under which u_K^+ can be taken as $\max(K - 1, 1 - \sigma) < u_K^+ < K$, and the path (u_K^+) separates the poles of the integrand at $w = 1 - s - k$ ($k = 0, 1, \dots$) and at $w = k$ ($k = 0, 1, \dots, K - 1$), from those at $w = k$ ($k = K, K + 1, \dots$). We now differentiate m -times the resulting formula, to obtain the expression in (2.2).

The remaining estimate (2.3) is derived by moving further the path in (4.2) from (u_K^+) to (u_{K+1}^+) , and then by the m -times differentiation of the resulting equality. $\square$

Outline of the proof of Theorem 2. Let u_K^- for any integer $K \geq 0$ be a constant satisfying $-\sigma - K < u_K^- < -\sigma - K + 1$. Then the path of integration in (4.1) can be moved from (u) to (u_K^-) , upon passing over the poles of the integrand at $w = -s - k$ ($k = -1, 0, 1, \dots, K - 1$). Collecting the residues of the relevant poles, we obtain the case $m = 0$ of (2.10) with

$$(4.3) \quad \rho_K^-(s, a, \lambda; z) = \frac{1}{2\pi i} \int_{(u_K^-)} \frac{\Gamma(s+w)\Gamma(-w)}{\Gamma(s)} \phi(s+w, a, \lambda) z^w dw,$$

where the residues are computed by

$$\begin{aligned} \operatorname{Res}_{s=1} \phi(s, a, \lambda) &= B_0(a, e(\lambda)) = \delta(\lambda), \\ \phi(-k, a, \lambda) &= -\frac{B_{k+1}(a, e(\lambda))}{k+1} \quad (k \in \mathbb{N}_0) \end{aligned}$$

(cf. [1][13]). Here the temporary restriction on σ can be relaxed at this stage into $\sigma > -K$, under which u_K^- is taken as $-\sigma - K < u_K^- < \min(-\sigma - K + 1, 0)$, and the path (u_K^-) separates the poles of the integrand at $w = k$ ($k = 0, 1, \dots$) and at $w = -s - k$ ($k = -1, 0, 1, \dots, K - 1$), from those at $w = -s - k$ ($k = K, K + 1, \dots$). The m -times differentiation of the resulting formula therefore gives the expression in (2.10).

The remaining estimate (2.11) is derived by moving further the path in (4.3) from (u_K^-) to (u_{K+1}^-) , and then by the m -times differentiation of the resulting equality. $\square$

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